Integral geometry in complex space forms

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Introduction

Classically, integral geometry in Euclidean space deals with two basic questions: the expression of the measure of planes meeting a convex domain, the so-called Crofton formulas; and the study of the measure of movements taking one convex domain over another fixed convex domain, the so-called kinematic formula.

In the Euclidean space $\mathbb{R}^n$, we denote by L_r a totally geodesic submanifold of dimension r, and we call it r-plane. We denote the space of r-planes by $\mathcal{L}_r$. This space has a unique (up to a constant factor) density invariant under the isometry group of $\mathbb{R}^n$, denoted by dL_r. Then, given a convex domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, the expression of the measure of r-planes meeting a convex domain is given by

$$\int_{\mathcal{L}_r} \chi(\Omega \cap L_r) dL_r = c_{n,r} M_{r-1}(\partial\Omega), \tag{1}$$

where $c_{n,r}$ only depends on the dimensions n, r, and $M_{r-1}(\partial\Omega)$ denotes the integral over $\partial\Omega$ of the $(r-1)$-th mean curvature integral.

Thus, the mean curvature integrals appear naturally in the Crofton formula. A classical known property of mean curvature integrals is the following

$$\int_{\mathcal{L}_r} M_i^{(r)}(\partial\Omega \cap L_r) dL_r = c'_{n,r,i} M_i(\partial\Omega) \tag{2}$$

where $c'_{n,r,i}$ only depends on the dimensions n, r, i, and $M_i^{(r)}(\partial\Omega \cap L_r)$ denotes the i-th mean curvature integral of $\partial\Omega \cap L_r$ as a hypersurface in $L_r \cong \mathbb{R}^r$. From (2), it is said that mean curvature integrals satisfy a *reproductive property*.

On the other hand, the kinematic formula in $\mathbb{R}^n$ is expressed as follows. Let Ω_1 and Ω_2 be two convex domains with smooth boundary, let $\overline{O(n)} := O(n) \ltimes \mathbb{R}^n$ denote the isometry group of $\mathbb{R}^n$, and let dg be an invariant density of $\overline{O(n)}$. Then,

$$\int_{\overline{O(n)}} \chi(\Omega_1 \cap g\Omega_2) dg = \sum_{i=0}^{n} c_{n,i} M_i(\partial\Omega_1) M_{n-i}(\partial\Omega_2). \tag{3}$$

The previous three formulas were extended to projective and hyperbolic spaces (cf. [San04]), i.e. they are known in the spaces of constant sectional curvature k. The generalization of integral (2) does not depend on k but in the expression (1) for projective and hyperbolic space appear other terms, depending on k. Moreover, its expression depends on the parity of the dimension of the planes. If r is even, then

$$\int_{\mathcal{L}_r} \chi(\Omega \cap L_r) dL_r = c_{n,r-1} M_{r-1}(\partial\Omega) + c_{n,r-3} M_{r-3}(\partial\Omega) + \cdots + c_{n,1} M_1(\partial\Omega) + c_n \mathrm{vol}(\Omega), \tag{4}$$

and if r is odd

$$\int_{\mathcal{L}_r} \chi(\Omega \cap L_r) dL_r = c_{n,r-1} M_{r-1}(\partial\Omega) + c_{n,r-3} M_{r-3}(\partial\Omega) + \cdots + c_{n,2} M_2(\partial\Omega) + c_n \mathrm{vol}(\partial\Omega), \tag{5}$$

where $c_{n,j}$ depends on the dimensions n and j and are multiples of k^{n-j}.

The facts that the expression depends on the parity, and that we study an integral of the Euler characteristic, remain us the Gauss-Bonnet formula in spaces of constant sectional curvature, which also depends on the parity of the ambient space. We recall here this formula in a space of constant sectional curvature k and dimension n.

If n is even, then

$$M_{n-1}(\partial\Omega) + c_{n-3}M_{n-3}(\partial\Omega) + \cdots + c_1 M_1(\partial\Omega) + k^{n/2}\mathrm{vol}(\Omega) = \mathrm{vol}(S^{n-1})\chi(\Omega),$$

and if n is odd,

$$M_{n-1}(\partial\Omega) + c_{n-3}M_{n-3}(\partial\Omega) + \cdots + c_2 M_2(\partial\Omega) + k^{(n-1)/2}\mathrm{vol}(\partial\Omega) = \frac{\mathrm{vol}(S^{n-1})}{2}\chi(\Omega)$$

where c_i depends only on the dimensions n, i and are multiples of the sectional curvature k.

Now, using the expression (2) and the Gauss-Bonnet formula, we get (4) and (5).

The goal of this work is generalize formulas (1) and (2) in the standard Hermitian space $\mathbb{C}^n$, in the complex projective space and in the complex hyperbolic space, denoted by $\mathbb{CK}^n(\epsilon)$ with 4ϵ the holomorphic curvature of the manifold (see Section 1.1).

In order to achieve this goal, we use the notion of valuation in a vector space V, a real-valued functional ϕ from the space of convex compact domains $\mathcal{K}(V)$ in V to $\mathbb{R}$ satisfying the following additive property

$$\phi(A \cup B) = \phi(A) + \phi(B) - \phi(A \cap B)$$

whenever A, B, $A \cup B \in \mathcal{K}(V)$.

The first examples of valuations are the volume of the convex domain, the area of the boundary, and the Euler characteristic. Other classical examples of valuations are the so-called *intrinsic volumes*. They are defined from the Steiner formula: given a convex domain $\Omega \subset \mathbb{R}^n$, if we denote by Ω_r the parallel domain at a distance r, the Steiner formula relates the volume of Ω_r with the so-called intrinsic volumes $V_i(\Omega)$ by

$$\mathrm{vol}(\Omega_r) = \sum_{i=0}^{n} r^{n-i}\omega_{n-i}V_i(\Omega)$$

where ω_{n-i} denotes the volume of the $(n-i)$-dimensional Euclidean ball with radius 1 (cf. Proposition 2.1.3).

If $\Omega \subset \mathbb{R}^n$ is a convex domain with smooth boundary, then intrinsic volumes satisfy

$$V_i(\Omega) = cM_{n-i-1}(\partial\Omega),$$

and they are the natural generalization of the mean curvature integrals for non-smooth convex domains.

Hadwiger in [Had57] proved that all continuous valuations in $\mathbb{R}^n$ invariant under the isometry group of $\mathbb{R}^n$ are linear combination of the volume of the convex domain, the area of the boundary, and the intrinsic volumes (see Section 2.2.1). This result has as immediate consequence formulas (1), (2) and (3).

Alesker in [Ale03] proved that the dimension of the space of continuous valuations in $\mathbb{C}^n$ invariant under the holomorphic isometry group of $\mathbb{C}^n$ is $\binom{n+2}{2}$ and gave a basis of this space. In the recent paper of Bernig and Fu, [BF08], there are given other basis of valuations in $\mathbb{C}^n$. In particular, the *Hermitian intrinsic volumes* are defined (see Section 2.4.2). These are the valuations we will use to work with. The fact that the dimension of the space of continuous valuations invariant under the isometry group of $\mathbb{C}^n$ is bigger than the one of $\mathbb{R}^{2n}$ is not surprising if we recall that the holomorphic isometry group of $\mathbb{C}^n$, $U(n)$, is smaller than the isometry group of $\mathbb{R}^{2n}$, $O(2n)$.

Hermitian intrinsic volumes are a kind of generalization of mean curvature integrals, but taking into account that $\mathbb{C}^n$ has a complex structure which defines a canonical vector field on hypersurfaces. Indeed, at each point x of a hypersurface, if we consider the normal vector, and we apply the complex structure, then we get a distinguished vector JN in the tangent space of the hypersurface at x. Moreover, the orthogonal space to JN in the tangent space defines a complex space of maximum dimension, $n-1$.

So, if S is a smooth hypersurface in $\mathbb{C}^n$, we can consider the integral

$$\int_S k_n(JN)dx$$

where $k_n(JN)$ denotes the normal curvature in the direction JN, and this is a valuation in $\mathbb{C}^n$. Other valuations related to normal curvature of the direction JN appear as elements in of Hermitian intrinsic volumes basis.

The notion of valuation can be also defined in a differentiable manifold (see Definition 2.4.1). In real space forms the volume of a convex domain and the area of its boundary are valuations. But, it is not known an analogous result to Hadwiger Theorem in these spaces.

The definition of the Hermitian intrinsic volumes can be extended to other space of constant holomorphic curvature. We denote by $\{\mu_{k,q}\}$ the Hermitian intrinsic volumes. The subscript k denotes the *degree* of the valuation (see Section 2.4.2).

In order to give a similar expression of (1) and (2) in the spaces of constant holomorphic curvature, we need to describe the integration space. Note that in spaces of constant sectional curvature we integrate over the space of r-planes, i.e. totally geodesic submanifold of fixed dimension. In spaces of constant holomorphic curvature, complete totally geodesic submanifolds are classified. If $\epsilon \neq 0$ they are complex submanifolds isometric to $\mathbb{CK}^r(\epsilon) \subset \mathbb{CK}^n(\epsilon)$, with $1 \leq r < n$ or totally real submanifolds isometric to $\mathbb{RK}^q(\epsilon) \subset \mathbb{CK}^n(\epsilon)$ with $1 \leq q \leq n$, where $\mathbb{RK}^q(\epsilon)$ denotes the space of constant sectional curvature ϵ. For $\epsilon = 0$ there are other totally geodesic submanifolds. We denote the space of complex planes with complex dimension r, $1 \leq r < n$, by $\mathcal{L}_r^{\mathbb{C}}$, and the space of totally real planes of maximum dimension n by $\mathcal{L}_n^{\mathbb{R}}$, the so-called Lagrangian manifolds.

In this work, we obtain a Crofton formula for complex r-planes and Lagrangian planes

$$\begin{aligned}\int_{\mathcal{L}_r^{\mathbb{C}}} \chi(\Omega \cap L_r) dL_r = \operatorname{vol}(G_{n-1,r}^{\mathbb{C}}) \binom{n-1}{r}^{-1} \cdot \qquad\qquad\qquad\qquad\qquad\qquad (6)\\ \cdot \Big(\sum_{k=n-r}^{n-1} \epsilon^{k-(n-r)} \omega_{2n-2k} \binom{n}{k}^{-1} \left(\sum_{q=\max\{0,2k-n\}}^{k-1} \frac{\binom{2k-2q}{k-q}}{4^{k-q}} \mu_{2k,q}(\Omega) + (k+r-n+1)\mu_{2k,k}(\Omega) \right) \\ + \epsilon^r (r+1) \operatorname{vol}(\Omega) \Big),\end{aligned}$$

$$\int_{\mathcal{L}_n^{\mathbb{R}}} \chi(\Omega \cap L) dL = \frac{\operatorname{vol}(G_{2n,n})\omega_n}{n!} \sum_{q=0}^{\frac{n-1}{2}} \binom{2q-1}{q-1}^{-1} \frac{4^{q-n}}{2q+1} \mu_{n,q}(\Omega) \quad \text{if } n \text{ is odd}, \qquad (7)$$

and

$$\begin{aligned}\int_{\mathcal{L}_n^{\mathbb{R}}} \chi(\Omega \cap L) dL = \frac{\operatorname{vol}(G_{2n,n})}{n!} \cdot \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (8)\\ \cdot \left(\sum_{q=0}^{\frac{n}{2}} \binom{2q-1}{q-1}^{-1} \frac{4^{q-n}\omega_n}{2q+1} \mu_{n,q}(\Omega) + \sum_{i=1}^{\frac{n}{2}} \epsilon^i \binom{n}{\frac{n}{2}+i}^{-1} \frac{2^{-n+1}\omega_{n-2i}}{n+1} \mu_{n+2i,\frac{n}{2}+i}(\Omega) \right) \quad \text{if } n \text{ is even},\end{aligned}$$

where ω_i denotes the volume of the i-dimensional Euclidean unit ball.

Previous formulas have more addends that the corresponding ones in the spaces of constant sectional curvature, but they are similar. If $\epsilon = 0$, i.e. in $\mathbb{C}^n$ also appear all the valuations with the corresponding degree. If $\epsilon \neq 0$ the notion of degree of a valuation has no sense but there is a similitude with the expression in spaces of constant sectional curvature comparing the subscripts of the valuations.

In order to get these expressions we use a variational method. That is, we take a smooth vector field X defined on the manifold and we consider its flow ϕ_t. We prove the following formula of first variation

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\mathcal{L}_r^{\mathbb{C}}} \chi(\phi_t(\Omega) \cap L_r) dL_r = \int_{\partial\Omega} \langle X, N\rangle \int_{G^{\mathbb{C}}_{n-1,r}(\mathcal{D}_p)} \sigma_{2r}(\mathrm{II}|_V) dV dp$$

where N is the exterior normal field, $\mathcal{D}$ is the distribution in the tangent space at $\partial\Omega$ orthogonal to JN, and $\sigma_{2r}(\mathrm{II}|_V)$ denotes the $2r$-th symmetric elementary function of the second fundamental form II restricted to $V \in G^{\mathbb{C}}_{n-1,r}(\mathcal{D}_p)$, the Grassmannian of complex planes with complex dimension r inside $\mathcal{D}_p$.

On the other hand, we get an expression of the variation of valuations $\mu_{k,q}$ using the method in [BF08].

Comparing both variations and solving a system of linear equations we obtain the result.

Using the same variational method we also obtain a Gauss-Bonnet formula for the spaces of constant holomorphic curvature. It is known that the variation of the Euler characteristic is zero. Thus, we can express it as a sum of Hermitian intrinsic volumes such that its variation vanishes. The obtained Gauss-Bonnet formula is the following

$$\omega_{2n}\chi(\Omega) = (n+1)\epsilon^n \mathrm{vol}(\Omega) + \sum_{c=0}^{n-1} \frac{\epsilon^c \omega_{2n-2c}(n-c)}{n\binom{n-1}{c}} \left(\sum_{q=\max\{0,2c-n\}}^{c-1} \frac{\binom{2c-2q}{c-q}}{4^{c-q}} \mu_{2c,q} + (c+1)\mu_{2c,c} \right). \tag{9}$$

In spaces of constant sectional curvature k, Solanes in [Sol06] related the measure of planes meeting a domain with the Euler characteristic of the domain

$$\omega_n \chi(\Omega) = \frac{1}{n} M_{n-1}(\partial\Omega) + \frac{2k}{n\omega_{n-1}} \int_{\mathcal{L}_{n-2}} \chi(\Omega \cap L_{n-2}) dL_{n-2}.$$

In $\mathbb{CK}^n(\epsilon)$, we get

$$\omega_{2n}\chi(\Omega) = \frac{1}{2n} M_{2n-1}(\partial\Omega) + \epsilon \int_{\mathcal{L}^{\mathbb{C}}_{n-1}} \chi(\Omega \cap L_{n-1}) dL_{n-1} + \sum_{j=1}^{n} \frac{\epsilon^j \omega_{2n}}{\omega_{2j}} \mu_{2j,j}(\Omega).$$

The analogous expression to (2), it is given when we integrate the mean curvature integral over complex r-planes. The obtained expression for a compact oriented (possible with boundary) hypersurface S of class $\mathcal{C}^2$ is

$$\int_{\mathcal{L}_r^{\mathbb{C}}} M_1^{(r)}(S \cap L_r) dL_r = \frac{\omega_{2n-2}\mathrm{vol}(G^{\mathbb{C}}_{n-2,r-1})}{2r(2r-1)} \binom{n}{r}^{-1} \left((2n-1)\frac{2nr-n-r}{n-r} M_1(S) + \int_S k_n(JN) \right) \tag{10}$$

where $k_n(JN)$ denotes the normal curvature in the direction $JN \in TS$.

To get this result, first we obtain, using moving frames, the following intermediate expression (for any mean curvature integral). If $r, i \in \mathbb{N}$ such that $1 \le r \le n$ and $0 \le i \le 2r-1$, then

$$\int_{\mathcal{L}_r^{\mathbb{C}}} M_i^{(r)}(S \cap L_r) dL_r \tag{11}$$
$$= \binom{2r-1}{i}^{-1} \int_S \int_{\mathbb{RP}^{2n-2}} \int_{G^{\mathbb{C}}_{n-2,r-1}} \frac{|\langle JN, e_r \rangle|^{2r-i}}{(1-\langle JN, e_r\rangle^2)^{r-1}} \sigma_i(p; e_r \oplus V) dV de_r dp,$$

where $e_r \in T_pS$ unit vector, V denotes a complex $(r-1)$-plane containing p and contained in $\{N, JN, e_r, Je_r\}^{\perp}$, $\sigma_i(p; e_r \oplus V)$ denotes the i-th symmetric elementary function of the second fundamental form of S restricted to the real subspace $e_r \oplus V$ and the integration over $\mathbb{RP}^{2n-2}$ denotes the projective space of the unit tangent space of the hypersurface.

In order to complete the generalization of equation (1) in $\mathbb{CK}^n(\epsilon)$, it remains to study the measure of (non-maximal) totally real planes. These are the other totally geodesic submanifolds of $\mathbb{CK}^n(\epsilon)$, $\epsilon \neq 0$. Using the same techniques as in the rest of this work, it does not seem possible to solve this case since we cannot obtain enough information of the variational properties of the measures of totally real planes meeting a domain in $\mathbb{CK}^n(\epsilon)$.

On the other hand, it would be interesting to extend formula (10) to $i \in \{2, \ldots, 2r-1\}$.

Next we explain the organization of the text.

Chapter 1 contains a description of the spaces of constant holomorphic curvature. We review its definition and describe some of the most important submanifolds, i.e. the totally geodesic submanifolds, the geodesic spheres, and the complex planes. In this chapter we also recall the method of moving frames, which will be used along this text. Using moving frames, we give an expression for the density of the space of complex planes. Finally, we prove that integral $\int_{\mathcal{L}_r^{\mathbb{C}}} \chi(\Omega \cap L_r) dL_r$ satisfies a reproductive property.

Chapter 2 is devoted to the study of valuations in $\mathbb{C}^n$ and in the spaces of constant holomorphic curvature. First of all, we review the concept and the main properties of the valuations on $\mathbb{R}^n$ together with the Hadwiger Theorem (which characterize all continuous valuations in $\mathbb{R}^n$ invariants under the isometry group). An analogous Hadwiger Theorem in $\mathbb{C}^n$ is stated. Finally, we define the used valuations in this work in spaces of constant holomorphic curvature, and we give new properties and relations with other valuations also important in the next chapters.

Chapter 3 gives a proof of (10). First of all, we prove some geometric lemmas and we obtain the expression for the mean curvature integrals over the space of complex planes in terms of an integral over the boundary of the domain given in (11). This expression will be fundamental to attain the goal of this chapter. As a corollary of (10) we characterize the valuations of degree $2n-2$ satisfying a reproductive property in $\mathbb{C}^n$, and we give the relation among different valuations defined by Alesker (already reviewed in Chapter 2). The results of this chapter are contained in [Aba].

In Chapter 4 we obtain the measure of complex planes intersecting a domain in the spaces of constant holomorphic curvature in terms of the Hermitian intrinsic volumes defined at Chapter 2. We also give an expression of the Gauss-Bonnet formula in terms of these valuations. In order to get these expressions we use a variational method. First, we obtain an expression for the variation of the measure of complex planes and for the Hermitian intrinsic volumes. In this chapter, we verify the certainty of (6). A constructive proof, where we find the constants in the expression is given in the appendix. As a corollary, we express the total mean Gauss curvature in $\mathbb{C}^n$ also in terms of the Hermitic intrinsic volumes. Finally, we relate Chapters 3 and 4 obtaining another method to compute the measure of complex lines meeting a domain. The results of this chapter are contained in [AGS09].

Chapter 5 studies the measure of another type of planes meeting a domain in $\mathbb{C}^n$, the so-called coisotropic planes. These planes are the orthogonal direct sum of a complex subspace of complex dimension $n-p$ and a totally real subspace of dimension p. Totally real planes of maximum dimension and real hyperplanes are particular cases of this type of planes. Using similar techniques as in Chapter 4 we give an expression for the measure of planes of this type meeting a domain. For the spaces of constant holomorphic curvature we prove (7) and (8), which give the measure of totally real planes of maximum dimension, the so-called Lagrangian planes.

The appendix contains the constructive proof of (6) and (9). That is, we give the method that allowed us to obtain the constants appearing in these expressions. This proof consists, at a final instance, to solve a linear system obtained from the study of the variation of both sides of the expressions, as it is detailed in Chapter 4.

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Chapter 1

Spaces of constant holomorphic curvature

1.1 First definitions

In this section we introduce the spaces of constant holomorphic curvature, also called complex space forms, and give the properties we shall use along this work. First of all, we recall some basic definitions.

Definition 1.1.1. Let M be a differentiable manifold. M is a *complex manifold* if it has an atlas such that the change of coordinates are holomorphic, that is, $\{(U_\alpha, \phi_\alpha)\}$ is an atlas with $\{U_\alpha\}$ an open covering of M and $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ homeomorphisms such that $\phi_\beta \circ \phi_\alpha^{-1}$ are holomorphic in its domain of definition.

Examples

(i) The vector space $\mathbb{C}^n$ is a complex manifold of complex dimension n.

(ii) The complex projective space $\mathbb{CP}^n$ is a complex manifold of complex dimension n.

The complex projective space can be defined analogously to the real projective space. Let us consider in $\mathbb{C}^{n+1}\backslash\{0\}$ the equivalence relation which identifies the points differing by a complex multiple. Then, we take as an atlas the open sets $\{U_0, ..., U_n\}$ such that

$$U_j = \{(z_0, ..., z_n) \in \mathbb{C}^{n+1} \mid z_j \neq 0\}$$

and for every U_j we take the map $\phi_j(z_0, ..., z_n) = (z_0/z_j, ..., z_{j-1}/z_j, z_{j+1}/z_j, ..., z_n/z_j)$ which is a homeomorphism. It can be proved that the change of coordinates are holomorphics.

Definition 1.1.2. Let M be a complex manifold. A linear map $J : T_xM \to T_xM$ is an *almost complex structure of* M if for each $x \in M$, the restriction of J at T_xM satisfies $J_x : T_xM \to T_xM$, $J^2 = -\mathrm{Id}$, and J varies differentially on M.

Note that any complex manifold admits an almost complex structure. Indeed, the tangent space of a complex manifold has a complex vector space structure, so the map "multiply by i" is well-defined and satisfies that applied twice is the map $-\mathrm{Id}$. We call this canonical almost complex structure *complex structure.*

Definition 1.1.3. Let V be a complex vector space and let $u, v \in V$. It is said that $h : V \times V \to \mathbb{C}$ is an *Hermitian product on* V if

1. it is $\mathbb{C}$-linear with respect to the first component,
2. $h(u,v) = \overline{h(u,v)}$.

Remark 1.1.4. From the properties of a Hermitian product, it follows that if $\lambda \in \mathbb{C}$ then $h(u,v) = \overline{\lambda} h(u,v)$. Indeed, by definition we get the following equalities

$$h(u,\lambda v) = \overline{h(\lambda v, u)} = \overline{\lambda}\overline{h(v,u)} = \overline{\lambda} h(u,v).$$

Definition 1.1.5. Let M be a differentiable manifold with complex structure J and a Riemannian metric g. Then, g is called a *Hermitian metric* if it is compatible with the complex structure, i.e. it satisfies $g_x(Ju, Jv) = g_x(u,v)$ for every $x \in M$ and $u, v \in T_xM$.

Definition 1.1.6. Let M be a complex manifold with complex structure J and Hermitian metric g. The 2-form ω defined by

$$\omega(u,v) = g(u, Jv), \ \ \forall\ u, v \in T_xM$$

is the *Kähler form.*

Remark 1.1.7. Given a complex manifold M with complex structure J and a Hermitian product defined on TM, we get a Hermitian metric on M from the real part of the Hermitian product, and a Kähler form on M from the imaginary part of the Hermitian product.

Definition 1.1.8. A complex manifold M is called a *Kähler manifold* if it has a Hermitian metric such that the Kähler form associated to this metric is closed.

Proposition 1.1.9 ([O'N83] page 326)**.** *Let M be a Kähler manifold with connection ∇. Then,*

$$\nabla JX = J\nabla X, \quad \forall X \in \mathfrak{X}(M).$$

Definition 1.1.10. A subspace $W \subset T_xM$ of complex dimension 1 is a *complex direction* or a *holomorphic section* of the tangent space if W is invariant under J, i.e. $JW = W$.

If $w \neq 0 \in W$, then the vectors $\{w, Jw\}$ constitute a basis of W, as a real subspace.

Definition 1.1.11. The *holomorphic curvature* is the sectional curvature of holomorphic sections.

Definition 1.1.12. A *space of constant holomorphic curvature* 4ϵ of dimension n is a complete, simply connected Kähler manifold of complex dimension n, such that the holomorphic curvature is constant and equal to 4ϵ for every point and every complex direction.

Theorem 1.1.13 ([KN69] Theorem 7.9 page 170)**.** *Two complete, simply connected Kähler manifolds with constant holomorphic curvature equal to 4ϵ are holomorphically isometric.*

Definition 1.1.14. We denote by $\mathbb{CK}^n(\epsilon)$ any space of constant holomorphic curvature of dimension n. If $\epsilon > 0$, then it corresponds to the complex projective space $\mathbb{CP}^n$, if $\epsilon < 0$, to the complex hyperbolic space $\mathbb{CH}^n$, and if $\epsilon = 0$, to the Hermitian standard space $\mathbb{C}^n$.

Spaces of constant holomorphic curvature are also called *complex space forms.*

Remark 1.1.15. Complex space forms are, in some sense, a generalization of real space forms (spaces of constant sectional curvature), i.e. the complete simply connected Riemannian manifolds with constant sectional curvature. Real space forms are (up to isometry) the Euclidean space $\mathbb{R}^n$, the real projective space $\mathbb{RP}^n$ and the real hyperbolic space $\mathbb{H}^n$. The results in this work extend some of the classical results in integral geometry from real space forms to complex space forms. Santaló [San52] and Griffiths [Gri78], among others, obtained some results of classical integral geometry in the standard Hermitian space and in the complex projective space taking complex submanifolds. In this work, we deal with non-empty domains and, thus, with real hypersurfaces.

Definition 1.1.16. Given two planes Π and Π' of real dimension 2 in a vector space with a scalar product, the *angle between the two planes* is defined as the infimum among the angles between a pair of vectors, one in Π and the other one in Π'.

Definition 1.1.17. Let Π be a plane with real dimension 2 in the tangent space of a point in a Kähler manifold with complex structure J. The *holomorphic angle* $\mu(\Pi)$ is the angle between Π and $J(\Pi)$.

Proposition 1.1.18 ([KN69] page 167)**.** *Let M be a Kähler manifold with complex structure J and Hermitian metric g. The holomorphic angle of a plane $\Pi \subset T_xM$, $x \in M$, is given by*

$$\cos\mu(\Pi) = |g(u, Jv)|$$

where u, v form an orthonormal basis of Π.

Remark 1.1.19. The holomorphic angle of a plane takes values between 0 and $\pi/2$. In the extreme cases we have holomorphic planes, when the holomorphic angle is 0; and *totally real planes* defined as the planes with holomorphic angle $\pi/2$.

In a complex space form, the sectional curvature of any plane can be computed from the holomorphic curvature and the holomorphic angle of the plane.

Proposition 1.1.20 ([KN69] page 167)**.** *Let M be a Kähler manifold with constant holomorphic curvature 4ϵ. Then, the sectional curvature of any plane $\Pi \subset T_xM$, $x \in M$ is given by*

$$K(\Pi) = \epsilon\left(1 + 3\cos^2\mu(\Pi)\right) \tag{1.1}$$

where $\mu(\Pi)$ is the holomorphic angle of the plane Π.

Corollary 1.1.21. *Sectional curvature of any plane in the tangent space of a point in a complex space form with constant holomorphic curvature 4ϵ lies in the interval $[\epsilon, 4\epsilon]$, if $\epsilon > 0$ and in the interval $[4\epsilon, \epsilon]$, if $\epsilon < 0$.*

1.2 Projective model

Along this work, we shall use the projective model of $\mathbb{CK}^n(\epsilon)$, which we describe here briefly (cf. [Gol99]).

If $\epsilon = 0$, we are considering the standard Hermitian space $\mathbb{C}^n$ with the standard Hermitian product. Along this section we suppose $\epsilon \neq 0$, unless otherwise stated.

1.2.1 Points

Endow $\mathbb{C}^{n+1}$ with the Hermitian product

$$(z, w) = \mathrm{sign}(\epsilon) z_0\overline{w_0} + \sum_{j=1}^{n} z_j\overline{w_j}. \tag{1.2}$$

Define

$$\mathbb{H} := \{z \in \mathbb{C}^{n+1} \mid (z, z) = \epsilon\}.$$

$\mathbb{H}$ is a real hypersurface of $\mathbb{C}^{n+1}$ (i.e. it has real dimension $2n+1$). We define the points of $\mathbb{CK}^n(\epsilon)$ as

$$\mathbb{CK}^n(\epsilon) := \pi(\mathbb{H})$$

where

$$\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{C}^{n+1}\backslash\{0\}/\mathbb{C}^* = \mathbb{CP}^n. \tag{1.3}$$

Remarks 1.2.1. (i) The fiber of Π for the points in $\pi(\mathbb{H}) =: \mathbb{CK}^n(\epsilon)$ is S^1. Indeed, let z, $w \in \mathbb{H}$ such that $\pi(z) = \pi(w)$. By definition of π, we have $w = \alpha z$. On the other hand, it holds $(z, z) = (\alpha z, \alpha z) = \epsilon$. Thus, $\alpha = e^{i\theta}$ with $\theta \in \mathbb{R}$ and $\pi^{-1}([z]) \cong S^1$.

(ii) If $\epsilon > 0$, then $\mathbb{CK}^n(\epsilon)$ coincides as a subset with $\mathbb{CP}^n$. But, if $\epsilon < 0$, then $\mathbb{CK}^n(\epsilon)$ is an open set of $\mathbb{CP}^n$.

The differentiable structure and the structure of a complex manifold we take in $\mathbb{CK}^n(\epsilon)$ is the same as the one of an open set of $\mathbb{CP}^n$.

1.2.2 Tangent space

The tangent space of a point $z \in \mathbb{H}$ is

$$T_z\mathbb{H} = \{w \in \mathbb{C}^{n+1} \mid \mathrm{Re}(z, w) = 0\}.$$

The elements in the tangent space of $\pi(z) \in \mathbb{CK}^n(\epsilon)$ are obtained from the image of the elements in the tangent space of the point z under $d\pi$. Moreover, the kernel of $d\pi$ has dimension 1.

The direction that its image under $d\pi$ is the null vector is Jz, since it is the tangent direction to the fiber. (Note that $Jz \in T_z\mathbb{H}$ since $\mathrm{Re}(z, Jz) = 0$.) Indeed, the fiber of a point $[z]$ is $\{e^{i\theta} z \mid \theta \in \mathbb{R}\}$, then

$$\left.\frac{\partial(e^{i\theta} z)}{\partial\theta}\right|_{\theta=0} = iz = Jz.$$

Thus, the tangent space at $z \in \mathbb{H}$ can be decomposed as

$$T_z\mathbb{H} = \langle Jz\rangle \oplus \langle Jz\rangle^{\perp}.$$

The tangent space at points in $\mathbb{CK}^n(\epsilon)$ coincides with the image by the differential map of the projection of vectors $\langle Jz\rangle^{\perp}$ at $T_z\mathbb{H}$.

Given a vector $v \in T_{\pi(z)}\mathbb{CK}^n(\epsilon)$ there are infinitely many vectors of $T_z\mathbb{H}$ such that under the differential map $d\pi$ give the same vector v, but we can distinguish the one lying in $\langle Jz\rangle^{\perp}$, which is called *horizontal lift* and we denote by v^L. All other vectors are obtained as linear combination of this vector and a multiple of Jz.

1.2.3 Metric

Let v, $w \in T_{\pi(z)}\mathbb{CH}^n$. The Hermitian product at $\mathbb{CK}^n(\epsilon)$ between v, w is defined by

$$(v, w)_\epsilon := (v^L, w^L), \tag{1.4}$$

that is, the Hermitian product defined at $\mathbb{C}^{n+1}$ applied to the horizontal lift of the vectors.

The real part of this product gives a Hermitian metric on $\mathbb{CK}^n(\epsilon)$

$$\langle v, w\rangle_\epsilon := \mathrm{Re}(v^L, w^L).$$

This metric coincides with the so-called *Fubini-Study metric*, if $\epsilon > 0$ and with the so-called *Bergmann metric*, if $\epsilon < 0$ (cf. [Gol99, page 74]).

Along this work we denote the Hermitian metric of $\mathbb{CK}^n(\epsilon)$ by $\langle\,,\rangle$ instead of $\langle\,,\rangle_\epsilon$.

Notation 1.2.2. In order to unify the study of the complex space forms, we define, in the same way as it is classically done in real space forms, the following trigonometric generalized functions

$$\sin_\epsilon(\alpha) = \begin{cases} \dfrac{\sin(\alpha\sqrt{\epsilon})}{\sqrt{\epsilon}}, & \text{if } \epsilon > 0 \\ \alpha, & \text{if } \epsilon = 0 \\ \dfrac{\sinh(\alpha\sqrt{-\epsilon})}{\sqrt{-\epsilon}}, & \text{if } \epsilon < 0 \end{cases}$$

$$\cos_\epsilon(\alpha) = \begin{cases} \cos(\alpha\sqrt{\epsilon}), & \text{if } \epsilon > 0 \\ 1, & \text{if } \epsilon = 0 \\ \cosh(\alpha\sqrt{|\epsilon|}), & \text{if } \epsilon < 0 \end{cases}$$

and

$$\cot_\epsilon(\alpha) = \frac{\cos_\epsilon(\alpha)}{\sin_\epsilon(\alpha)}.$$

1.2.4 Geodesics

Geodesics in the projective model of $\mathbb{CK}^n(\epsilon)$ are given by the projection of the intersection points between $\mathbb{H}$ and a plane in $\mathbb{C}^{n+1}$ such that it is spanned by a vector corresponding to a representative in $\mathbb{H} \subset \mathbb{C}^{n+1}$ of a point z in the geodesic at $\mathbb{CK}^n(\epsilon)$, and a vector u tangent to the geodesic at z.

Then, the expression of a geodesic at $\mathbb{CK}^n(\epsilon)$ is given by $[\gamma(t)] = [\cos_\epsilon(t)z + \sin_\epsilon(t)u]$ where $u \in \langle Jz\rangle^\perp \subset T_z\mathbb{H}$.

The distance between two points in the complex projective and hyperbolic space can be expressed in terms of the Hermitian product defined at $\mathbb{C}^{n+1}$.

Proposition 1.2.3 ([Gol99] page 76)**.** *Let $x, y \in \mathbb{CK}^n(\epsilon)$, $\epsilon \neq 0$, and let d be the distance between the two given points. If x' and y' are representatives of x and y, respectively, in the projective model, then the distance between the two points is given by*

$$(\cos_\epsilon d(x,y))^2 = \frac{(x',y')(y',x')}{(x',x')(y',y')}$$

where $(\,,)$ denotes the Hermitian product in $\mathbb{C}^{n+1}$ defined at (1.2).

1.2.5 Isometries

Let us recall the definition of the matrix Lie group $U(p,q)$.

Definition 1.2.4. Let $(x,y) = -\sum_{j=0}^{p-1} x_j\overline{y_j} + \sum_{j=p}^{n} x_j\overline{y_j}$ be a Hermitian product in $\mathbb{C}^n$ and $p, q \in \mathbb{N} \cup \{0\}$ such that $p+q = n+1$. Then it is defined

$$U(p,q) = \{A \in \mathcal{M}_{n\times n}(\mathbb{C}) \mid (Av, Aw) = (v,w) \text{ with } v, w \in \mathbb{C}^n\}.$$

The matrix group $U(n)$ coincides with $U(0,n)$, that is, we consider the standard Hermitian product on $\mathbb{C}^n$.

The matrices of $PU(n+1) = U(n+1)/(\text{multiplication by scalars})$, if $\epsilon > 0$ (resp. the matrices of $PU(1,n) = U(1,n)/(\text{multiplication by complex scalars})$, if $\epsilon < 0$) act naturally on $\mathbb{CK}^n(\epsilon)$. Moreover, they preserve the metric defined in the model since preserve the Hermitian product defined at $\mathbb{C}^{n+1}$. Then, the matrices in $PU(n+1)$ (resp. $PU(1,n)$) are isometries of $\mathbb{CP}^n$ (resp. $\mathbb{CH}^n$).

Proposition 1.2.5 ([Gol99] page 68)**.**
- *Every isometry of $\mathbb{CK}^n(\epsilon)$ comes from a linear map in $\mathbb{C}^{n+1}$.*
- *The isometry group of $\mathbb{CK}^n(\epsilon)$ is $PU_\epsilon(n)$ with*

$$U_\epsilon(n) = \begin{cases} \mathbb{C}^n \rtimes U(n), & \text{if } \epsilon = 0, \\ U(n+1) = U(0,n+1), & \text{if } \epsilon > 0, \\ U(1,n), & \text{if } \epsilon < 0. \end{cases} \tag{1.5}$$

In order to unify the study of the complex space forms, independently of ϵ, we represent the elements in the group $\mathbb{C}^n \rtimes U(n)$ as matrices

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline \mathbb{C}^n & U(n) \end{array}\right). \tag{1.6}$$

The transitivity of the isometry group at different levels is given in the following proposition.

Proposition 1.2.6 ([Gol99] page 70)**.** *The isometry group of $\mathbb{CK}^n(\epsilon)$ acts transitively*

- *on the points in $\mathbb{CK}^n(\epsilon)$,*
- *on the unit tangent bundle. That is, given $(p,v),(q,w)$ in the unit tangent bundle there exists an isometry σ such that $\sigma(p)=q$ and $d\sigma(v)=w$.*
- *on the holomorphic sections (see Definition 1.1.10).*

In the following lemma, we give a basis of left-invariant forms of $U(p,q)$. We will prove that these forms are also right-invariants.

Lemma 1.2.7. *Let $A=(a_0,\dots,a_m)\in U(p,q)$, with p, q, $m\in\mathbb{N}\cup\{0\}$ such that $p+q=m+1$. A basis of left-invariant forms in $U(p,q)$ is given by $\{\mathrm{Re}(\varphi_{jk}),\mathrm{Im}(\varphi_{jk}),\mathrm{Re}(\varphi_{jj})\}$, $0\le j\le k\le m$, $j\neq k$ where $\varphi_{ij}=(da_i,a_j)$ and $(x,y)=-\sum_{j=0}^{p-1}x_j\overline{y_j}+\sum_{j=p}^{m}x_j\overline{y_j}$ in $\mathbb{C}^{m+1}$.*

Proof. From Definition 1.2.4 of $U(p,q)$ it follows that $A\in U(p,q)$ if and only if $A^{-1}=\varepsilon\overline{A}^t\varepsilon$ where

$$\varepsilon=\begin{pmatrix} -\mathrm{Id}_p & 0 \\ 0 & \mathrm{Id}_q \end{pmatrix}. \tag{1.7}$$

In order to find a basis of left-invariant forms we compute $A^{-1}dA$ with $A\in U(p,q)$. If we denote $A=(a_0,\dots,a_m)$, then

$$A^{-1}dA=\varepsilon\overline{A}^T\varepsilon dA=\begin{pmatrix} (da_0,-a_0) & \dots & (da_m,-a_0) \\ \vdots & & \vdots \\ (da_0,-a_{p-1}) & \dots & (da_m,-a_{p-1}) \\ (da_0,a_p) & \dots & (da_m,a_p) \\ (da_0,a_m) & \dots & (da_m,a_m) \end{pmatrix}=(\varphi_{ij})_{ij}. \tag{1.8}$$

Each entry of this matrix is a 1-form given by $\varphi_{ij}=\pm(da_i,a_j)$.

Note that each a_j is a m-tuple of complex numbers, so that the 1-forms φ_{ij} are complex-valued.

In order to find a basis of left-invariant real-valued 1-forms from the entries of the former matrix we use the following

- $(a_j,a_j)=\pm1$ and differentiating

  $$0=(a_j,da_j)+(da_j,a_j)=\overline{(da_j,a_j)}+(da_j,a_j)=2\mathrm{Re}(da_j,a_j).$$

  Thus, $\varphi_{jj}=-\overline{\varphi_{jj}}$ and each φ_{jj} takes only imaginary values.

- $(a_j,a_k)=0$ if $k\neq j$ and differentiating

  $$0=(da_j,a_k)+(a_j,da_k).$$

  Thus,

  $$\begin{cases} \varphi_{jk}=\overline{\varphi_{kj}} & \text{if} \quad j\in\{0,\dots,p-1\} \quad \text{or} \quad k\in\{0,\dots,p-1\} \\ \varphi_{jk}=-\overline{\varphi_{kj}} & \text{otherwise.} \end{cases}$$

On the other hand, it follows directly from the definition of $U(p,q)$ that its dimension is $(p+q)^2$. Then, $\{\mathrm{Re}(\varphi_{jk}), \mathrm{Im}(\varphi_{jk}), \mathrm{Re}(\varphi_{jj})\}$, $0 \leq j \leq k \leq m$, $j \neq k$ constitutes a basis of left-invariant 1-forms since they generate all the space and there are $(p+q)^2$ forms. □

For $\epsilon = 0$, if we denote the elements $A \in \mathbb{C}^n \rtimes U(n)$ as the matrices of (1.8), we define the forms

$$\varphi_{ij} = (da_i, a_j)$$

with $(\,,)$ the standard product in $\mathbb{C}^{n+1}$.

Definition 1.2.8. A *group* G is said to be *unimodular* if there exists a volume element of G left and right-invariant.

Lemma 1.2.9. *$U(p+q)$ is a unimodular group.*

Proof. From Lemma 1.2.7 we have a basis of left-invariant forms of $U(p,q)$. We prove that each of these forms is also right-invariant, i.e. it satisfies

$$R_B^*\varphi_{ij}(A;v) = \varphi_{ij}(A;v) \quad \forall A, B \in U(p,q), v \in T_AU(p,q).$$

We use the expression $\varphi_{ij} = \pm(da_i, a_j)$ and we denote by $a_k(A)$ the map taking the k-th column of a matrix A. Then, if $A, B \in U(p,q)$ and $v \in T_AU(p,q)$

$$\begin{aligned}
R_B^*\varphi_{ij}(A;v) &= \varphi_{ij}(R_B(A); d(R_B)(v)) = \pm(da_i(d(R_B)(v)), a_j(R_B(A))) = \pm(da_i(vB), a_j(AB)) \\
&= \pm\Big(-\sum_{k,l=0}^{m}\sum_{r=0}^{p-1} v_i^k b_k^r \overline{a_j^l}\,\overline{b_l^r} + \sum_{k,l=0}^{m}\sum_{r=p}^{m} v_i^k b_k^r \overline{a_j^l}\,\overline{b_l^r}\Big) \\
&= \pm\Big(-\sum_{k,l=0}^{p-1} \delta_{kl} v_i^k \overline{a_j^l} + \sum_{k,l=p}^{m} \delta kl v_i^k \overline{a_j^l}\Big) \\
&= \pm\Big(-\sum_{k=0}^{p-1} v_i^k \overline{a_j^k} + \sum_{k=p}^{m} v_i^k \overline{a_j^k}\Big) = \pm(da_i(v), a_j(A)) = \varphi_{ij}(A;v).
\end{aligned}$$

Then, $U(p,q)$ is a unimodular group since the volume element obtained from the product of the forms φ_{ij} is left-invariant and right-invariant (it is a product of forms with this property). □

1.2.6 Structure of homogeneous space

Definition 1.2.10. Let (M,g) be a Riemannian manifold. If given any two points $x, y \in M$ there exists an isometry σ of M such that $\sigma(x) = y$, then M is a *homogeneous space.* That is, a Riemannian manifold is homogeneous if it is a homogeneous space of its isometry group.

By Proposition 1.2.6 we have that complex space forms are homogeneous spaces. It will be interesting to represent them as a quotient of Lie groups.

Proposition 1.2.11 ([War71] Theorem 3.62 page 123)**.** *Let $\eta : G \times M \to M$ be a transitive action of the Lie group G over the manifold M. Let $m_0 \in M$ and H the isotropy group of m_0. Then, the map*

$$\begin{array}{rccc}
\beta : & G/H & \longrightarrow & M \\
& gH & \mapsto & \eta(g, m_0)
\end{array}$$

is a diffeomorphism.

The Lie group $PU_\epsilon(n)$ acts transitively over $\mathbb{CK}^n(\epsilon)$ and the isotropy group of a point in $\mathbb{CK}^n(\epsilon)$, for each ϵ, is isomorphic to $P(U(1)\times U(n))$, a closed Lie subgroup of $PU_\epsilon(n)$. Thus, we can represent $\mathbb{CK}^n(\epsilon)$ as a quotient of Lie groups

$$\mathbb{CK}^n(\epsilon) \cong PU_\epsilon(n)/P(U(1)\times U(n)) \cong U_\epsilon(n)/(U(1)\times U(n)),$$

where the first diffeomorphism is given by $x \mapsto g\cdot P(U(1)\times U(n))$ with $g\in PU_\epsilon(n)$ such that, if $x_0\in\mathbb{CK}^n(\epsilon)$ is fixed then $g(x_0)=x$.

Definition 1.2.12. A Riemannian manifold is a *2-point homogeneous space* if the isometry group of the manifold acts transitively in the unit tangent bundle.

Thus, complex space forms are 2-point homogeneous spaces. Real space forms are also 2-point homogeneous spaces but, they are also 3-point homogeneous spaces, that is, the isometry group acts transitively for triplets of a point and two orthonormal vectors in the tangent space of the point. Complex space forms ($\epsilon\neq 0$) cannot be 3-point homogeneous spaces since the sectional curvature is preserved by isometries and the sectional curvature is not constant.

1.3 Moving frames

Definition 1.3.1. Let $U\subset M$ be an open set of a differentiable manifold. An *orthonormal moving frame of* $\mathbb{CK}^n(\epsilon)$ *defined at* U is a map $g_0: U\to\mathbb{CK}^n(\epsilon)$ together with a collection of $g_i: U\to T\mathbb{CK}^n(\epsilon)$ $(i\in\{1,\dots,2n\})$ such that $\langle g_i,g_j\rangle_\epsilon=\delta_{ij}$ where $\langle\,,\rangle_\epsilon$ denotes the Hermitian product of $\mathbb{CK}^n(\epsilon)$ (defined at (1.4)) and $\pi: T\mathbb{CK}^n(\epsilon)\to\mathbb{CK}^n(\epsilon)$ is the canonical projection.

Definition 1.3.2. Let V be a $2n$-dimensional real vector space endowed with a complex structure J. An orthonormal basis $\{v_1,v_2,\dots,v_{2n}\}$ is said to be a *J-basis* if $v_{2i}=Jv_{2i-1}$ for every $i\in\{1,\dots,n\}$.

We denote by $\{e_1,e_{\bar 1}=Je_1,\dots,e_n,e_{\bar n}=Je_n\}$ the J-bases of V, and by $\{\omega_1,\omega_{\bar 1},\dots,\omega_n,\omega_{\bar n}\}$ the dual basis of a J-basis.

Remark 1.3.3. A J-basis is a special type of an orthonormal basis in a real vector space with an almost complex structure J.

Definition 1.3.4. An orthonormal moving frame of $\mathbb{CK}^n(\epsilon)$ such that vectors $\{g_1(p),\dots,g_{2n}(p)\}$ constitute a J-basis for all $x\in U$, is called a *J-moving frame.*

J-moving frames in $\mathbb{CK}^n(\epsilon)$ play an important role since they are in correspondence with the elements of the isometry group of $\mathbb{CK}^n(\epsilon)$.

Consider $\mathcal{F}(\mathbb{CK}^n(\epsilon))$ the bundle of J-moving frames of $\mathbb{CK}^n(\epsilon)$, constituted by J-moving frames $(g_0;g_1,Jg_1,\dots,g_n,Jg_n)$ with $g_0\in\mathbb{CK}^n(\epsilon)$ and $\{g_1,Jg_1,\dots,g_n,Jg_n\}$ a J-basis of $T_{g_0}\mathbb{CK}^n(\epsilon)$.

Proposition 1.3.5. *The bundle of J-moving frames $\mathcal{F}(\mathbb{CK}^n(\epsilon))$ is identified with the isometry group of $\mathbb{CK}^n(\epsilon)$.*

Proof. We study the case $\epsilon\neq 0$. Let $A\in U_\epsilon(n)$. By definition (1.5) of $U_\epsilon(n)$ we have that A is an $(n+1)\times(n+1)$ matrix with complex entries and such that its columns $\{a_0,\dots,a_n\}$ satisfy

$$\begin{cases} (a_0,a_0)=\mathrm{sign}(\epsilon)1, & \\ (a_0,a_i)=0, & i\in\{1,\dots,n\}, \\ (a_i,a_j)=\delta_{ij} & i,j\in\{1,\dots,n\}, \end{cases} \tag{1.9}$$

where $(\,,)$ denotes the Hermitian product in $\mathbb{C}^{n+1}$ defined at (1.2).

From the first property in the former list, we can take a_0 as a representative of $g_0 = \pi(a_0) \in \mathbb{CK}^n(\epsilon)$.

From the second property of (1.9) we have that a_i satisfies the condition of being a vector in $T_{a_0}\mathbb{H}$ (cf. Section 1.2.2). Moreover, $\mathrm{Re}(Ja_0, a_i) = \mathrm{Im}(a_0, a_i) = 0$ and $\mathrm{Re}(Ja_0, Ja_i) = \mathrm{Re}(a_0, a_i) = 0$. Let us consider $g_i := d\pi(a_i)$ and $Jg_i := d\pi(Ja_i)$ where π denotes the projection defined at (1.3).

From the third condition in (1.9), vectors $\{g_1, Jg_1, \ldots, g_n, Jg_n\}$ constitute a J-basis of the tangent space at g_0.

Reciprocally, given a J-moving frame $\{g_0; g_1, Jg_1, \ldots, g_n, Jg_n\}$ defined on an open set, we can define a matrix of $U_\epsilon(n)$ (with the entries depending continuously on a parameter) just taking as the first column the representative a_0 of g_0 with norm $\mathrm{sign}(\epsilon)1$. For the other columns we consider the horizontal lift of g_j at a_0. As $\{g_1, Jg_1, \ldots, g_n, Jg_n\}$ is, in each point g, a J-basis and we choose the horizontal lift, the columns of the constructed matrix verify the conditions in (1.9) and are in $U_\epsilon(n)$. $\square$

Definition 1.3.6. The *unit tangent bundle of* $\mathbb{CK}^n(\epsilon)$, denoted by $S(\mathbb{CK}^n(\epsilon))$, is defined as

$$S(\mathbb{CK}^n(\epsilon)) = \bigcup_{p \in \mathbb{CK}^n(\epsilon)} T'_p\mathbb{CK}^n(\epsilon)$$

where $T'_p\mathbb{CK}^n(\epsilon)$ denotes the sphere of unit vectors in the tangent vector space of $\mathbb{CK}^n(\epsilon)$ at p.

In Lemma 1.2.7 we defined the invariant forms $\{\varphi_{ij}\}$ of $U_\epsilon(n)$ as

$$\varphi_{ij}(A; \cdot) = (da_i(\cdot), a_j)$$

where $A = (a_0, \ldots, a_n) \in U_\epsilon(n)$. As forms $\{\varphi_{ij}\}$ takes complex values we consider

$$\varphi_{jk} = \alpha_{jk} + i\beta_{jk} \tag{1.10}$$

Using the identification between $U_\epsilon(n)$ and $\mathcal{F}(\mathbb{CK}^n(\epsilon))$, we can consider forms $\{\varphi_{ij}\}$ as forms of $\mathcal{F}(\mathbb{CK}^n(\epsilon))$.

On the other hand, consider the canonical projections

$$\begin{array}{ccccc} \mathcal{F}(\mathbb{CK}^n(\epsilon)) & \xrightarrow{\pi_1} & S(\mathbb{CK}^n(\epsilon)) & \xrightarrow{\pi_2} & \mathbb{CK}^n(\epsilon) \\ (g; g_1, \ldots, Jg_n) & \mapsto & (g, g_1) & \mapsto & g \end{array}$$

and local sections

$$\mathbb{CK}^n(\epsilon) \supset U \xrightarrow{s_2} S(\mathbb{CK}^n(\epsilon)) \supset V \xrightarrow{s_1} \mathcal{F}(\mathbb{CK}^n(\epsilon)).$$

Using the forms φ_{ij} defined in $\mathcal{F}(\mathbb{CK}^n(\epsilon))$ and the previous local sections, we define the following local invariant forms in $S(\mathbb{CK}^n(\epsilon))$

$$s_1^*(\varphi_{ij}), \qquad s_1^*(\alpha_{ij}) \quad \text{and} \quad s_1^*(\beta_{ij})$$

and the local invariant forms in $\mathbb{CK}^n(\epsilon)$

$$s_2^*s_1^*(\varphi_{ij}), \qquad s_2^*s_1^*(\alpha_{ij}) \quad \text{and} \quad s_2^*s_1^*(\beta_{ij}).$$

Lemma 1.3.7. *Forms $s_1^*\alpha_{01}$, $s_1^*\beta_{01}$ and $s_1^*\beta_{11}$ are global forms in $S(\mathbb{CK}^n(\epsilon))$.*

Proof. If $V \in T_{(p,v)}(S(\mathbb{CK}^n(\epsilon)))$ then

$$\begin{aligned} s_1^*\alpha_{01}(V)_{(p,v)} &= \mathrm{Re}((d\pi_2(V), v)) = \langle d\pi_2(V), v\rangle_\epsilon \\ s_1^*\beta_{01}(V)_{(p,v)} &= \mathrm{Im}((d\pi_2(V), v)), \\ s_1^*\beta_{11}(V)_{(p,v)} &= \mathrm{Im}((\nabla V, v)), \end{aligned}$$

where ∇ denotes the Levi-Civita connection defined by $\nabla : T(S\mathbb{CK}^n(\epsilon)) \to T\mathbb{CK}^n(\epsilon)$, from the Levi-Civita connection of $\mathbb{CK}^n(\epsilon)$. Inded, a vector $V \in T(S\mathbb{CK}^n(\epsilon))$ is a tangent vector to a curve of unit vectors, and in each of them we can apply the Levi-Civita connection of $\mathbb{CK}^n(\epsilon)$. □

We denote by α, β, γ the forms $s_1^*\alpha_{01}$, $s_1^*\beta_{01}$, $s_1^*\beta_{11}$, respectively.

Remark 1.3.8. The 1-form α coincides with the standard contact form of the unit tangent bundle $S(\mathbb{CK}^n(\epsilon))$.

On the other hand, forms $s_2^*s_1^*(\varphi_{ij})$ coincide with forms ϕ_{ij} of $\mathbb{CK}^n(\epsilon)$ we define in the following.

Let $\{g; g_1, Jg_1, \ldots, g_n, Jg_n\}$ be a J-moving frame on $\mathbb{CK}^n(\epsilon)$. As in the tangent space of each point g, $\{g_1, Jg_1, \ldots, g_n, Jg_n\}$ defines a J-basis, we can consider the vectors $\{g_1, \ldots, g_n\}$ as complex vectors. Then, the following differential forms are well-defined

$$\phi_j(\cdot) = (dg(\cdot), g_j)_\epsilon \quad \text{and} \quad \phi_{jk}(\cdot) = (\nabla g_j(\cdot), g_k)_\epsilon \tag{1.11}$$

where $j, k \in \{1, ..., n\}$, and ∇ denotes the Levi-Civita of $\mathbb{CK}^n(\epsilon)$ (i.e. we consider g_j as a real vector, we apply the Levi-Civita connection and we consider the result again as a complex vector). Note that the differential forms ϕ_j and ϕ_{jk} are complex valued. We denote

$$\begin{aligned} \phi_j &= \alpha_j + i\beta_j \qquad (1.12) \\ \phi_{jk} &= \alpha_{jk} + i\beta_{jk}. \end{aligned}$$

At Chapter 3, we work with orthonormal moving frames not necessarily J-moving frames. Analogously, if $\{g; g_1, g_2, \ldots, g_{2n-1}, g_{2n}\}$ is a moving frame on $\mathbb{CK}^n(\epsilon)$, we define the dual and connection forms for this moving frame. We denote

$$\omega_j(\cdot) = \langle dg(\cdot), g_j\rangle_\epsilon \quad \text{and} \quad \omega_{jk}(\cdot) = \langle \nabla g_j(\cdot), g_k\rangle_\epsilon \tag{1.13}$$

with $j, k \in \{1, ..., 2n\}$, $\langle\, ,\rangle_\epsilon$ the Hermitian product defined on $\mathbb{CK}^n(\epsilon)$ (see (1.4)), and ∇ the Levi-Civita connection of $\mathbb{CK}^n(\epsilon)$.

Note that the differential forms $\{\alpha_j, \beta_j\}$ are a particular case of forms $\{\omega_j\}$: they are obtained if we consider a J-moving frame.

Notation 1.3.9. Along this work we use invariant forms defined at $\mathbb{CK}^n(\epsilon)$, $S(\mathbb{CK}^n(\epsilon))$ or $\mathcal{F}(\mathbb{CK}^n(\epsilon))$, but we denote all of them by φ_{ij}, α_{ij}, β_{ij}, without the pull-back of the sections, if it is clear by the context.

Definition 1.3.10. Given a domain $\Omega \subset \mathbb{CK}^n(\epsilon)$ we define the *unit normal bundle of* $\partial\Omega$ by

$$N(\Omega) = \{(p, v) : p \in \partial\Omega, v \text{ such that } \langle v, w\rangle_\epsilon \geq 0 \;\forall\, w \text{ tangent to a curve at } \Omega \text{ by } p \text{ and } ||v||_\epsilon = 1\}.$$

Remark 1.3.11. The main results of this work, given at Chapters 3 and 4, have as a hypothesis that the domain $\Omega \subset \mathbb{CK}^n(\epsilon)$ which we take is compact with $\mathcal{C}^2$ boundary. We denote by *regular domain* a domain satisfying these hypothesis. We suppose that domains are regular in order to simplify the arguments and to use techniques of differential geometry (for instance, to have a well-defined second fundamental form in the whole boundary of the domain). These hypothesis can be relaxed since most of the used results, mainly in valuations (see Chapter 2), are known for a more general class of domains (cf. [Ale07a]).

Lemma 1.3.12. *Let $\Omega \subset \mathbb{CK}^n(\epsilon)$ be a regular domain. Forms α and $d\alpha$ vanishes at $N(\Omega) \subset S(\mathbb{CK}^n(\epsilon))$.*

Proof. Let $V \in T_{(p,v)}\Omega$. Then, $\alpha(V)_{(p,N)} = \langle d\pi_2(V), N\rangle_\epsilon = 0$ since $d\pi_2(V)$ is a tangent vector tangent at $\partial\Omega$ at point p.

In order to prove that the 2-form $d\alpha$ vanishes at the unit normal bundle, we consider the inclusion of the unit normal bundle to the unit tangent bundle $i : N(\Omega) \to S(\Omega)$. Using that the differential map commutes with the inclusion we have the result

$$d\alpha|_{N(\Omega)} = (i^* \circ d)(\alpha) = (d \circ i^*)(\alpha) = d(i^*\alpha) = d(0) = 0.$$

□

1.4 Submanifolds

1.4.1 Totally geodesic submanifolds

Definition 1.4.1. Let M be a Riemannian manifold. A submanifold $N \subset M$ is *totally geodesic* if every geodesic in the submanifold N is also a geodesic in M.

As $\mathbb{C}^n$ is metrically equivalent to $\mathbb{R}^{2n}$, totally geodesic submanifolds in $\mathbb{C}^n$ coincide with the ones in $\mathbb{R}^{2n}$. For the other complex space forms, the totally geodesic submanifolds are classified.

Definition 1.4.2. Let V be a real vector space of dimension $2n$ endowed with an almost complex structure J compatible with a scalar product $\langle\, ,\rangle$. It is said that vectors $\{e_1,\ldots,e_m\}$ expand a *complex subspace* if the space generated by these vectors is J-invariant, i.e. $J(\mathrm{span}\{e_1,\ldots,e_m\}) = \mathrm{span}\{e_1,\ldots,e_m\}$.

It is said that a *submanifold* of a complex manifold is *complex* if at each point, the tangent space of the submanifold is complex subspace of the tangent space of the manifold.

Definition 1.4.3. Let V be a real vector space of dimension $2n$ endowed with an almost complex structure J compatible with a scalar product $\langle\, ,\rangle$. It is said that vectors $\{e_1,\ldots,e_m\}$ expand a *totally real subspace* if

$$\langle e_i, Je_j\rangle = 0, \quad \forall i,j \in \{1,\ldots,m\}.$$

It is said that a *submanifold* of a complex manifold is *totally real* if at each point, the tangent space of the submanifold is a totally real subspace of the tangent space of the manifold.

Theorem 1.4.4 ([Gol99] pages 75 and 80)**.** *Let $z \in \mathbb{CK}^n(\epsilon)$.*

1. *If $L \subset T_z\mathbb{CK}^n(\epsilon)$ is a complex vector subspace with complex dimension r, then there exists a unique complete complex totally geodesic submanifold through z and tangent to L at z.*

2. *If $L \subset T_z\mathbb{CK}^n(\epsilon)$ is a totally real vector space of real dimension k, then there exists a unique complete totally geodesic totally real submanifold through z and tangent to L at z.*

Definition 1.4.5. The complex submanifold defined at 1. in the previous theorem is called *complex r-plane*, and denoted by L_r.

The totally real submanifold defined at 2. in the previous theorem is called *totally real k-plane*, and denoted by $L_k^{\mathbb{R}}$.

In the projective model, complex r-planes are obtained from the projection of a subspace $F \subset \mathbb{C}^{n+1}$ intersection $\mathbb{CK}^n(\epsilon)$. The subspace F is the $(r+1)$-dimensional complex vector subspace spanned by a representative z' of $z = \pi(z')$ and by the horizontal lift of vectors in $L \subset T_z\mathbb{CK}^n(\epsilon)$ (cf. [Gol99, Section 3.1.4]).

Analogously, totally real k-planes are obtained from the projection of $F^{\mathbb{R}} \subset \mathbb{C}^{n+1}$ intersection $\mathbb{CK}^n(\epsilon)$, where $F^{\mathbb{R}}$ is the $(k+1)$-dimensional real vector subspace spanned by a representative z' of $z = \pi(z')$ and by the horizontal lift of vectors in $L \subset T_z\mathbb{CK}^n(\epsilon)$.

Theorem 1.4.6 ([Gol99] p. 82)**.** *The unique complete totally geodesic submanifolds in $\mathbb{CK}^n(\epsilon)$ are the complex r-planes, $r \in \{1, \dots, n-1\}$ and the totally real k-planes, $k \in \{1, \dots, n\}$.*

Corollary 1.4.7. *In $\mathbb{CK}^n(\epsilon)$, $\epsilon \neq 0$, there are not totally geodesic (real) hypersurfaces.*

That is, it does not exist the equivalent hypersurface to a hyperplane in a real space form. The more reasonable substitutes of hyperplanes are the so-called *bisectors*, which we study on page 82.

Theorem 1.4.6 will be important along this work since it will be interesting to know which totally geodesic submanifolds can be taken in a complex space form as a substitutes of (totally geodesic) planes in real space forms.

1.4.2 Geodesic balls

A *geodesic ball* in a Riemannian manifold is the set of points equidistant from a fixed point called center.

In real space forms, geodesic balls are *totally umbilical* real hypersurfaces, i.e. the second fundamental form is, at every point, a multiple of the identity and the same multiple for every point.

This fact does not hold in complex projective and hyperbolic spaces. Moreover, in these spaces, there are no totally umbilical real hypersurface.

Proposition 1.4.8 ([Mon85])**.** *The principal curvatures of a sphere of radius r in $\mathbb{CK}^n(\epsilon)$, $\epsilon \neq 0$ are*

i) $2\cot_\epsilon(2r)$ *with multiplicity 1 and principal direction $-JN$ (where N denotes the inward normal vector to the sphere),*

ii) $\cot_\epsilon(r)$ *with multiplicity $2n-2$.*

Recall that $\cos_\epsilon, \sin_\epsilon$ denote the generalized trigonometric functions defined at Notation 1.2.2.

Along this work, we use the value of the mean curvature integrals for a geodesic ball of radius R in $\mathbb{CK}^n(\epsilon)$ (cf. Definition 2.1.4).

From the previous proposition we have that the symmetric elementary functions are

$$\begin{cases} \sigma_0 = 1 \\ \sigma_i = \binom{2n-1}{i}^{-1}\left(\binom{2n-2}{i}\cot_\epsilon^i(R) + \binom{2n-2}{i-1} 2\cot_\epsilon^{i-1}(R)\cot_\epsilon(2R)\right) \\ \sigma_{2n-1} = 2\cot_\epsilon^{2n-2}(R)\cot_\epsilon(2R). \end{cases}$$

By a straightforward computation, we obtain that the expression of the mean curvature integrals is

$$\begin{cases} M_0 = \mathrm{vol}(\partial B_R) = \frac{2\pi^n}{(n-1)!}\sin_\epsilon^{2n-1}(R)\cos_\epsilon(R) \\ M_i = \frac{2\pi^n}{(2n-1)(n-1)!}((2n+i-1)\cos_\epsilon^{i+1}(R)\sin_\epsilon^{2n-i-1}(R) - i\cos_\epsilon^{i-1}(R)\sin_\epsilon^{2n-i-1}(R)) \\ M_{2n-1} = \frac{2\pi^n}{(n-1)!}(\cos_\epsilon^{2n}(R) + \cos_\epsilon^{2n-2}(R)\sin_\epsilon^2(R)) \end{cases}$$

and

$$\mathrm{vol}(B_R) = \frac{\pi^n}{n!}(\sin_\epsilon(R))^{2n}.$$

1.5 Space of complex r-planes

We denote the space of all complex r-planes in $\mathbb{CK}^n(\epsilon)$ by $\mathcal{L}_r^{\mathbb{C}}$ and the space of all totally real r-planes in $\mathbb{CK}^n(\epsilon)$ by $\mathcal{L}_r^{\mathbb{R}}$ (cf. Definition 1.4.5).

We shall use that $\mathcal{L}_r^{\mathbb{C}}$ is a homogeneous space with respect to the isometry group of $\mathbb{CK}^n(\epsilon)$. In order to prove this fact, we need the following result.

Lemma 1.5.1. *The group of isometries of $\mathbb{CK}^n(\epsilon)$ acts transitively on the J-bases.*

Proof. We study the case $\epsilon \neq 0$. Fix the canonical J-basis $\{e_1, Je_1, \dots, e_n, Je_n\}$ at $e_0 \in \mathbb{CK}^n(\epsilon)$. It is enough to prove that given another J-basis $\{g_1, Jg_1, \dots, g_n, Jg_n\}$ of $T_{g_0}\mathbb{CK}^n(\epsilon)$, there exists an isometry ρ which takes this J-basis to the fixed one.

Take as isometry $\rho \in U_\epsilon(n)$ the matrix with columns $(\tilde{g}_0, \tilde{g}_1, \dots, \tilde{g}_n)$ where $\tilde{g}_0$ is a representative of g_0 with norm ϵ and $\tilde{g}_i$ is the horizontal lift of g_i at $\tilde{g}_0$. In the same way as in the proof of Proposition 1.3.5 we have that ρ is a matrix in $U_\epsilon(n)$. Moreover, it carries the fixed J-basis to the given one. $\square$

From the previous lemma we get

Lemma 1.5.2. *The space of complex r-planes is a homogeneous space with respect to the isometry group of $\mathbb{CK}^n(\epsilon)$.*

Proof. We define a J-basis $\{e_1, Je_1, \dots, e_r, Je_r\}$ of the tangent space in any point of a complex r-plane. Completing this J-basis to a J-basis of the whole space $\mathbb{CK}^n(\epsilon)$, and applying the previous lemma we get the result. $\square$

In order to study integral geometry in $\mathbb{CK}^n(\epsilon)$, it is necessary that the space $\mathcal{L}_r^{\mathbb{C}}$ admits an invariant density under the isometries of $\mathbb{CK}^n(\epsilon)$. In general, the absolute value of a form of maximum degree is called a *density*. By the following lemma, it is enough to prove that $\mathcal{L}_r^{\mathbb{C}}$ is the quotient of unimodular groups.

Lemma 1.5.3 ([San04])**.** *If G, H are unimodular groups, then G/H admits an invariant density.*

The isotropy group of a complex r-plane is isomorphic to

$$U_\epsilon(r) \times U(n-r) = \left\{ X \in \mathcal{M}_{(n+1)\times(n+1)}(\mathbb{C}) : X = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right), A \in U_\epsilon(r), B \in U(n-r) \right\} \tag{1.14}$$

since these matrices (and only these) leave invariant a complex r-plane and its orthogonal.

Then,

$$\mathcal{L}_r^{\mathbb{C}} \cong U_\epsilon(n)/(U_\epsilon(r) \times U(n-r)).$$

The group $U_\epsilon(n)$, by Lemma 1.2.9, is unimodular, and $U_\epsilon(r) \times U(n-r)$ is also a unimodular group. Thus, there exists an invariant density in the quotient space, that is, in the space of complex r-planes.

The following result give a method to obtain explicitly the density in the quotient space.

Theorem 1.5.4 ([San04] page 147)**.** *Let G be a Lie group with dimension n and H a closed subgroup of G with dimension $n-m$. Then, G/H is a homogeneous space with dimension m. Let $\tilde{\omega}$ be the m-form obtained from the product of all invariant 1-forms of G such that they vanish on H. Then, there exists an invariant density ω in G/H if and only if $d\tilde{\omega} = 0$. In this case, $\tilde{\omega}$ is the pull-back of ω for the canonical projection of G at G/H (up to constants factors).*

Proposition 1.5.5. *Let $\pi : U_\epsilon(n) \longrightarrow U_\epsilon(n)/(U_\epsilon(r) \times U(n-r)) \cong \mathcal{L}_r^{\mathbb{C}}$. If dL_r is an invariant density of $\mathcal{L}_r^{\mathbb{C}}$ then*

$$\pi^* dL_r = \bigwedge_{\substack{i=0,\dots,r\\ j=r+1,\dots,n}} \varphi_{ij} \wedge \overline{\varphi_{ij}}$$

or, equivalently,

$$\pi^* dL_r = \bigwedge_{\substack{i=0,\dots,r\\ j=r+1,\dots,n}} \alpha_{ij} \wedge \beta_{ij},$$

where $\{\varphi_{ij}, \alpha_{ij}, \beta_{ij}\}$ are the forms defined at Lemma 1.2.7 and at (1.11).

Proof. The result is known for $\mathbb{C}^n$, see [San04].

For the other complex space forms, we note that the (real) dimension of the space of complex r-plans coincides with the degree of $\pi^* dL_r$ (and it is equal to $2(r+1)(n-r)$). Thus, it suffices to prove that each form φ_{ij} with $i \in \{0, \dots, r\}$, $j \in \{r+1, \dots, n\}$ vanishes over $U_\epsilon(n) \times U(n-r)$. But, from the form of matrices in $U_\epsilon(n) \times U(n-r)$ given at (1.14), the result follows immediately. □

Let us give an example of moving frames in the space of complex r-planes using Definition 1.3.1, which will be used in the next section.

Let us take as the open set $U \subset M$ (see Definition 1.3.1) an open set in $\mathcal{L}_r^{\mathbb{C}}$. An orthonormal frame is given by

$$\begin{array}{rccc} g: & U \subset \mathcal{L}_r^{\mathbb{C}} & \longrightarrow & \mathbb{CK}^n(\epsilon) \\ & L_r & \mapsto & p \in L_r \end{array} \quad \text{and} \quad \begin{array}{rccc} g_i: & U \subset \mathcal{L}_r^{\mathbb{C}} & \longrightarrow & T\mathbb{CK}^n(\epsilon) \\ & L_r & \mapsto & v_i \in T_{g(L_r)} L_r \end{array}, \tag{1.15}$$

$i \in \{1, \dots, 2r\}$, such that $\langle v_i, v_j \rangle_\epsilon = \delta_{ij}$. It will be interesting to consider that $J g_{2k-1}(L_r) = g_{2k}(L_r)$, $k \in \{1, \dots, r\}$, that is, vectors $\{g_1(L_r), \dots, g_{2r}(L_r)\}$ constitute a J-basis at $g(L_r)$. By abuse of notation, we denote $g(L_r)$ by g and $g_i(L_r)$ by g_i.

Remark 1.5.6. From the correspondence between $U_\epsilon(n)$ and the bundle of J-moving frames $\mathcal{F}(\mathbb{CK}^n(\epsilon))$, and between $U_\epsilon(n)/(U_\epsilon(r) \times U(n-r))$ and $\mathcal{L}_r^{\mathbb{C}}$ we have that $\{p; g_1(L_r), \dots, g_{2r}(L_r)\}$ are sections of $U_\epsilon(n) \to U_\epsilon(n)/(U_\epsilon(r) \times U(n-r))$.

1.5.1 Expression for the invariant density in terms of a parametrization

Sometimes it is interesting to have a more geometric expression for the invariant density of complex r-planes. For example, Santaló proved

Proposition 1.5.7 ([San04]). *The invariant density of the space of totally geodesic planes in a space form of constant sectional curvature k is*

$$dL_r = \cos_k^r(\rho) dx_{n-r} \wedge dL_{(n-r)[O]} \tag{1.16}$$

where dx_{n-r} is the volume element of the $(n-r)$-plane orthogonal to L_r containing the origin O and $dL_{(n-r)[O]}$ is the volume element of the Grassmannian of $(n-r)$-planes containing the origin.

Now, we give a similar expression in complex space forms, for the density of complex r-planes in $\mathbb{CK}^n(\epsilon)$.

Proposition 1.5.8. *The invariant density of the space of complex r-planes in a space form of constant holomorphic curvature 4ϵ is*

$$dL_r = \cos_\epsilon^{2r}(\rho) dx_{n-r} \wedge dL_{(n-r)[O]}$$

where dx_{n-r} is the volume element of the complex $(n-r)$-plane orthogonal to L_r containing the origin O and $dL_{(n-r)[O]}$ is the volume element of the Grassmannian of complex $(n-r)$-planes containing the origin.

Proof. In order to obtain the expression of the invariant density of $\mathcal{L}_r^{\mathbb{C}}$ in terms of $dL_{(n-r)[O]}$ and dx_{n-r}, we follow the same idea as in [San04] (where it is proved for the Euclidean and the real hyperbolic space). That is, we fix an adapted J-moving frame to the complex r-plane, defined in a neighborhood of the point at minimum distance from the origin O, then, by parallel translation, we translate this moving frame to the origin O. Finally, we relate both moving frames from the pull-back of a section.

Denote by $G = U_\epsilon(n)$ and by $H = U_\epsilon(r) \times U(n-r)$ the isotropy group of a complex r-plane.

The projection $\pi : G \longrightarrow G/H$ gives, with respect to a J-moving frame adapted to the complex r-plane and the forms defined at Lemma 1.2.7,

$$\pi^* dL_r = \bigwedge_{j=r+1}^{n} \alpha_{j0} \wedge \beta_{j0} \bigwedge_{\substack{i=1,\ldots,r \\ j=r+1,\ldots,n}} \alpha_{ji} \wedge \beta_{ji}.$$

Let $O \in \mathbb{CK}^n(\epsilon)$ be the fixed origin and let $L_r \in \mathcal{L}_r^{\mathbb{C}}$. We denote by $p(L_r)$ the point in L_r at a minimum distance from O.

Let i be a local section of π. Then, $\pi \circ i = \mathrm{id}$ and $i^*\pi^* dL_r = dL_r$, so that, we can obtain the expression of dL_r. From the identifications explained in Remark 1.5.6, we take as a section i the defined by $\{p(L_r); g_1, Jg_1, \ldots, g_n, Jg_n\}$, a J-moving frame defined in a neighborhood V of L_r, adapted to L_r and such that g_{r+1} is the tangent vector to the geodesic joining $p(L_r)$ and O. Denote by $\{g^1, Jg^1, \ldots, g^n, Jg^n\}$ the dual basis of $\{g_1, Jg_1, \ldots, g_n, Jg_n\}$ at g_0. From Proposition 1.3.5, we consider the matrix in G corresponding to this J-moving frame. Denote the columns of G also by $(g_0, g_1, \ldots, g_n)$, so that $(g_i \circ i)$ denotes the i-th column of the matrix. Then, using the same notation as in (1.12), we have

$$\begin{aligned} i^*(\bigwedge_{j=r+1}^{n} \alpha_{j0} \wedge \beta_{j0}) &= \prod_{j=r+1}^{n} \alpha_{j0}(di) \wedge \beta_{j0}(di) = \prod_{j=r+1}^{n} \langle dg_0(di), g_j \rangle \langle dg_0(di), Jg_j \rangle \\ &= \prod_{j=r+1}^{n} \langle d(g_0 \circ i), g_j \rangle \langle d(g_0 \circ i), Jg_j \rangle = (g_0 \circ i)^*(\bigwedge_{j=r+1}^{n} g^j \wedge Jg^j) \end{aligned}$$

but $(g_0 \circ i) = p(L_r)$ and the previous form coincides with the volume element of p in the subspace generated by $\{g_{r+1}, Jg_{r+1}, \ldots, g_n, Jg_n\}$, which is a complex $(n-r)$-plane. Thus,

$$i^*(\bigwedge_{j=r+1}^{n} \alpha_{j0} \wedge \beta_{j0}) = dx_{n-r}.$$

Let $G' \subset G$ be the subgroup of all J-moving frames of G such that g_{r+1} is the tangent vector to the geodesic containing O and g_0, and let $G_0 \subset G$ be the subgroup of all J-moving frames with base point O. Let ρ be the distance from L_r to O. Denote by s_ρ the parallel translation from O to $p(L_r)$ along the geodesic. Consider the following maps

$$\begin{array}{cccc} \pi_1 : & G' & \longrightarrow & G_0 \\ & (g_0; g_1, \ldots, g_n) & \mapsto & (O; s_\rho^{-1}(g_1), \ldots, s_\rho^{-1}(g_n)) \end{array},$$

$$\begin{array}{rccl} \pi_2: & G_0 & \longrightarrow & \mathcal{L}^{\mathbb{C}}_{r[O]} \\ & G_0 & \mapsto & \text{complex } r\text{-plane containing } O \text{ generated by } g_1,\dots,g_r \end{array},$$

$$\begin{array}{rccl} \pi_3: & \mathcal{L}^{\mathbb{C}}_r & \longrightarrow & \mathcal{L}^{\mathbb{C}}_{r[0]} \\ & L_r & \mapsto & ((L_r)^{\perp}_O)^{\perp}_O \end{array},$$

where $(L_r)^{\perp}_O$ denotes the orthogonal space to L_r by O. Then, the following diagram is commutative

$$\begin{tikzcd} G' \arrow[r, "\pi_1"] \arrow[d, "\pi"'] & G_0 \arrow[d, "\pi_2"] \\ \mathcal{L}^{\mathbb{C}}_r \arrow[r, "\pi_3"] & \mathcal{L}^{\mathbb{C}}_{r[0]}. \end{tikzcd} \tag{1.17}$$

We define curves x_{ij} in $G_0 \subset G$ as follows

$$x_{ij}(t) = (O; g_1,\dots,g_i(t), Jg_i(t),\dots,g_j(t), Jg_j(t)\dots,g_n, Jg_n)$$

where $g_i(t) = \cos(t)g_i + \sin(t)g_j$ and $g_j(t) = -\sin(t)g_i + \cos(t)g_j$ and curves

$$x_i^j(t) = (O; g_1,\dots,g_i(t), Jg_i(t),\dots,g_j(t), Jg_j(t),\dots,g_n, Jg_n)$$

where $g_i(t) = \cos(t)g_i + \sin(t)Jg_j$ and $g_j(t) = -\sin(t)Jg_i - \cos(t)g_j$.

From the local section i, we have

$$i^*\alpha_{ji} = i^*((\pi_1^* \circ s^*)(\alpha_{ji})) = (\pi_1 \circ i)^*(s_\rho^*\alpha_{ji}),$$

$$i^*\beta_{ji} = i^*((\pi_1^* \circ s^*)(\beta_{ji})) = (\pi_1 \circ i)^*(s_\rho^*\beta_{ji}).$$

Thus, we have to study $(s_\rho^*\alpha_{ij})(\dot{x}_{kl})$ and $(s_\rho^*\alpha_{ij})(\dot{x}_k^l)$ (and the same for β_{ij}). Then, we need $(g_l \circ s_\rho)(x_{ij})$ since

$$(s_\rho^*\alpha_{ij})(\dot{x}_{kl}) = \alpha_{ij}(ds_\rho(\dot{x}_{kl})) = \langle g_j\big|_{s_\rho(x_{kl}(t))}, d(g_i \circ s_\rho)\big|_{s_\rho(x_{kl}(t))}(\dot{x}_{kl})\rangle. \tag{1.18}$$

Note that

$$(g_i \circ s_\rho)(x_{kl}) = \ i\text{-th column of the matrix } s_\rho(x_{kl})$$

and that $s_\rho(x_{kl}) \in G'$ is obtained from the parallel translation along the geodesic for O and with tangent vector $g_{r+1}(t)$ in O. Hence, for curves x_{ij}, x_i^j with $i,j \neq r+1$ we always take parallel translation along the same geodesic, when we apply s_ρ.

When we move $g_{r+1}(t)$ we consider the parallel translation along different geodesics for each t. But, as curves $x_{r+1,j}$, x_{r+1}^j, $x_{1,r+1}$ just move the vector $g_{r+1}(t)$ in a real plane generated by $\{g_{r+1}(0), g_j(0)\}$ or $\{g_{r+1}(0), Jg_j(0)\}$, we have that $g_0(s_\rho(x_{r+1,j})(t))$ or $g_0(s_\rho(x_{r+1}^j(t)))$ describes a circle in $\mathbb{CK}^n(\epsilon)$ contained in the plane generated by $\{g_{r+1}(0), g_j(0)\}$ (or $\{g_{r+1}(0), Jg_j(0)\}$).

From these remarks we have

- $(g_0 \circ s_\rho)(x_{kl})$: point at a distance ρ from O in which we arrive along the geodesic with tangent vector g_{r+1} at O.

  ◇ $x_{r+1,l}$, $l > r+1$.

$$(g_0 \circ s_\rho)(x_{r+1,l}) = \cos_\epsilon(\rho)O + \sin_\epsilon(\rho)(\cos(t)g_{r+1} + \sin(t)g_l).$$

  ◇ x_{r+1}^l, $l \geq r+1$.

$$(g_0 \circ s_\rho)(x_{r+1}^l) = \cos_\epsilon(\rho)O + \sin_\epsilon(\rho)(\cos(t)g_{r+1} + \sin(t)(Jg_l)).$$

◇ $x_{1,r+1}$.

$$(g_0 \circ s_\rho)(x_{1,r+1}) = \cos_\epsilon(\rho)O + \sin_\epsilon(\rho)(-\sin(t)g_1 + \cos(t)g_{r+1})$$

since $g_1(t) = g_1 \cos(t) + g_{r+1} \sin(t)$ and $g_{r+1}(t)$ is perpendicular to $g_1(t)$ for all t. Then, $g_{r+1}(t) = -g_1 \sin(t) + g_{r+1} \cos(t)$.

◇ x_1^{r+1}.

$$(g_0 \circ s_\rho)(x_1^{r+1}) = \cos_\epsilon(\rho)O + \sin_\epsilon(\rho)(\sin(t)(Jg_1) + \cos(t)g_{r+1}).$$

◇ x_{kl}, $k < l$, $k, l \neq r+1$.

$$(g_0 \circ s_\rho)(x_{kl}) = \cos_\epsilon(\rho)O + \sin_\epsilon(\rho)g_{r+1}.$$

◇ x_k^l, $k \leq l$, $k, l \neq r+1$.

$$(g_0 \circ s_\rho)(x_k^l) = \cos_\epsilon(\rho)O + \sin_\epsilon(\rho)g_{r+1}.$$

- $(g_{r+1} \circ s_\rho)(x_{kl})$:

◇ $x_{r+1,l}$, $l > r+1$.

$$(g_{r+1} \circ s_\rho)(x_{r+1,l}) = s_\rho^{-1}(\cos(t)g_{r+1} + \sin(t)g_l)$$

but this parallel translation coincides with the tangent vector to the geodesic at time ρ, that is,

$$\begin{aligned} s_\rho^{-1}(\cos(t)g_{r+1}+\sin(t)g_l) &= \text{tangent vector to } (\cos_\epsilon(\rho)O+\sin_\epsilon(\rho)(\cos(t)g_{r+1}+\sin(t)g_l)) \\ &= \sin_\epsilon(\rho)O + \cos_\epsilon(\rho)(\cos(t)g_{r+1} + \sin(t)g_l). \end{aligned}$$

◇ x_{r+1}^l, $l \geq r+1$.

$$(g_{r+1} \circ s_\rho)(x_{r+1}^l) = \sin_\epsilon(\rho)O + \cos_\epsilon(\rho)(\cos(t)g_{r+1} + \sin(t)(Jg_l)).$$

◇ $x_{1,r+1}$.

$$(g_{r+1} \circ s_\rho)(x_{1,r+1}) = \sin_\epsilon(\rho)O + \cos_\epsilon(\rho)(-\sin(t)g_1 + \cos(t)g_{r+1}).$$

◇ x_1^{r+1}.

$$(g_{r+1} \circ s_\rho)(x_1^{r+1}) = \sin_\epsilon(\rho)O + \cos_\epsilon(\rho)(\sin(t)(Jg_1) + \cos(t)g_{r+1}).$$

◇ x_{kl}, $k < l$, $k, l \neq j$.

$$(g_{r+1} \circ s_\rho)(x_{kl}) = \sin_\epsilon(\rho)O + \cos_\epsilon(\rho)g_{r+1}.$$

◇ x_k^l, $k \leq l$, $k, l \neq j$.

$$(g_{r+1} \circ s_\rho)(x_k^l) = \sin_\epsilon(\rho)O + \cos_\epsilon(\rho)g_{r+1}.$$

- $(g_j \circ s_\rho)(x_{kl})$, $j > r+1$:

◇ x_{jl}, $l > j$.

$$(g_j \circ s_\rho)(x_{jl}) = s_\rho^{-1}(\cos(t)g_j + \sin(t)g_l).$$

◇ x_j^l, $l \geq j$.
$$(g_j \circ s_\rho)(x_j^l) = s_\rho^{-1}(\cos(t)g_j + \sin(t)(Jg_l)).$$

◇ x_{lj}, $l < j$.
$$(g_j \circ s_\rho)(x_{lj}) = s_\rho^{-1}(-\sin(t)g_l + \cos(t)g_j).$$

◇ x_l^j, $l \leq j$.
$$(g_j \circ s_\rho)(x_l^j) = s_\rho^{-1}(\sin(t)(Jg_l) + \cos(t)g_j).$$

◇ x_{kl}, $k < l$, $k, l \neq j$.
$$(g_j \circ s_\rho)(x_{kl}) = s_\rho^{-1}(g_j).$$

◇ x_k^l, $k \leq l$, $k, l \neq j$.
$$(g_j \circ s_\rho)(x_k^l) = s_\rho^{-1}(g_j).$$

Now, we compute $s_\rho^*\alpha_{ij}$ (and $s_\rho^*\beta_{ij}$) in terms of α_{ij}, β_{ij} using (1.18) and evaluating $s_\rho^*\alpha_{ij}$ (and $s_\rho^*\beta_{ij}$) to each curve x_{kl}. Doing so, we obtain

$(s_\rho^*\alpha_{j1})_g = -(\alpha_{j1})_{g'}$, $j > r+1$.

$(s_\rho^*\alpha_{r+1,1})_g = -\cos_\epsilon(\rho)(\alpha_{r+1,1})_{g'}$.

$(s_\rho^*\beta_{j1})_g = (\beta_{j1})_{g'}$, $j > r+1$.

$(s_\rho^*\alpha_{r+1,1})_g = -\cos_\epsilon(\rho)(\beta_{r+1,1})_{g'}$.

Finally, we have

$$s_\rho^*(\bigwedge_{\substack{j=r+1,\dots,n\\ i=1,\dots,r}} \alpha_{ji} \wedge \beta_{j1})_g = \cos_\epsilon^{2r}(\rho)(\bigwedge_{\substack{j=r+1,\dots,n\\ i=1,\dots,r}} \alpha_{j1} \wedge \beta_{j1})_{g'}. \tag{1.19}$$

To get an expression for

$$i^*(\bigwedge_{\substack{j=r+1,\dots,n\\ i=1,\dots,r}} \alpha_{ji} \wedge \beta_{j1})$$

we use the diagram (1.17) and the computation in (1.19), so that

$$\begin{aligned} i^*(\bigwedge_{\substack{j=r+1,\dots,n\\ i=1,\dots,r}} \alpha_{ji} \wedge \beta_{ji})_g &= i^* \circ \pi_1^* \circ s_\rho^*(\bigwedge_{\substack{j=r+1,\dots,n\\ i=1,\dots,r}} \alpha_{ji} \wedge \beta_{ji})_g \\ &= i^* \circ \pi_1^*(\cos_\epsilon^{2r}(\rho)(\bigwedge_{\substack{j=r+1,\dots,n\\ i=1,\dots,r}} \alpha_{ji} \wedge \beta_{ji})'_g) \\ &= \cos_\epsilon^{2r}(\rho)\pi_3^*(dL_{r[0]}) = \cos_\epsilon^{2r}(\rho)dL_{(n-r)[0]} \end{aligned}$$

where we used the expression for the invariant density of complex r-planes through a point, cf. (1.20), and the duality between complex r-planes through a point and the complex $(n-r)$-planes through the same point.

Hence, we get
$$dL_r = \cos_\epsilon^{2r}(\rho)dx_{n-r}dL_{(n-r)[O]}.$$

□

1.5.2 Density of complex r-planes containing a fixed complex q-plane

We denote by $\mathcal{L}^{\mathbb{C}}_{r[q]}$ the space of complex r-planes containing a fixed complex q-plane.

We denote by $H_{[q]} := U_\epsilon(q)$ the isotropy group of a complex q-plane in $\mathbb{CK}^n(\epsilon)$ and by $H_{r[q]}$ the isotropy group of a complex r-plane containing the fixed complex q-plane. Note that $H_{[q]}$ acts transitively on $\mathcal{L}^{\mathbb{C}}_{r[q]}$.

On the other hand, if we suppose that L^0_q is the fixed complex q-plane, then we can define the projection

$$\begin{array}{rccc} \pi: & H_{[q]} & \longrightarrow & H_{[q]}/H_{r[q]} \\ & g & \mapsto & L_{r[q]} = gL^0_q. \end{array}$$

As the elements in $H_{r[q]}$ do not mix tangent vectors to the fixed complex q-plane with orthogonal vectors to this complex q-plane, we have

$$H_{r[q]} \cong \left\{ \left(\begin{array}{c||c|c} A & 0 & 0 \\ \hline\hline 0 & B & 0 \\ \hline 0 & 0 & C \end{array} \right),\ A \in U_\epsilon(q), B \in U(r-q), C \in U(n-r) \right\},$$

$$\pi^* dL_{r[q]} = \bigwedge_{\substack{q+1 \leq i \leq r \\ r+1 \leq j \leq n}} \alpha_{ij} \wedge \beta_{ij}. \tag{1.20}$$

(The forms vanishing, with respect to the isometry group, in this case $H_{[q]}$, are the ones inside the big box.)

1.5.3 Density of complex q-planes contained in a fixed complex r-plane

Denote by $\mathcal{L}^{(r)}_q$ the space of complex q-planes contained in a fixed complex r-plane. Let us fix a complex r-plane L_r and a complex q-plane $L^{(r)}_q$ contained in L_r. Consider the projection $\pi : U_\epsilon(r) \to U_\epsilon(r)/U_\epsilon(q) \times U(r-q)$ where $U_\epsilon(r)$ denotes the isometry group of the fixed complex r-plane and $U_\epsilon(q) \times U(r-q)$ the isotropy group of a complex q-plane contained in the complex r-plane. Then, as in the previous case we get

$$\pi^* dL^{(r)}_q = \bigwedge_{\substack{1 \leq i \leq q \\ q+1 \leq j \leq r}} \alpha_{ij} \wedge \beta_{ij} \bigwedge_{q+1 \leq j \leq r} \alpha_{j0} \wedge \beta_{j0} = \bigwedge_{\substack{0 \leq i \leq q \\ q+1 \leq j \leq r}} \alpha_{ij} \wedge \beta_{ij}.$$

1.5.4 Measure of complex r-planes intersecting a geodesic ball

In order to obtain the value of this measure we use the expression for the invariant density of complex r-planes in (1.16) and the expression of the Jacobian of the map of changing to spherical coordinates. This is given by (cf. [Gra73])

$$\frac{\cos_\epsilon(R) \sin^{2n-1}_\epsilon(R)}{|\epsilon|^{(n-1)/2}}.$$

Recall that $\cos_\epsilon$ and $\sin_\epsilon$ denote the generalized trigonometric functions defined at Notation 1.2.2.

We fix as a origin of the spherical coordinates the center of the geodesic ball. Then, using the expression in spherical coordinates for the element volume of $\mathbb{CK}^{n-r}(\epsilon)$ (the orthogonal space to a complex r-plane intersecting the sphere), we obtain

$$
\begin{aligned}
m(L_r \in \mathcal{L}_r^{\mathbb{C}} \ : \ B_R \cap L_r \neq \emptyset) &= \int_{B_R \cap L_r \neq \emptyset} dL_r = \int_{G^{\mathbb{C}}_{n,n-r}} \int_{B_R \cap L_r} \cos_\epsilon^{2r}(\rho) dx_{n-r} \wedge dG^{\mathbb{C}}_{n,n-r[O]} \\
&= \frac{\mathrm{vol}(G^{\mathbb{C}}_{n,n-r})}{|\epsilon|^{(n-1)/2}} \int_{S^{2(n-r)-1}} \int_0^R \cos_\epsilon^{2r+1}(\rho) \sin_\epsilon^{2(n-r)-1}(\rho) d\rho dS^{2(n-r)-1} \\
&= \frac{\mathrm{vol}(G^{\mathbb{C}}_{n,n-r})}{|\epsilon|^{(n-1)/2}} \mathrm{vol}(S^{2(n-r)-1}) \int_0^R \cos_\epsilon^{2r+1}(\rho) \sin_\epsilon^{2(n-r)-1}(\rho) d\rho \\
&= \frac{\mathrm{vol}(G^{\mathbb{C}}_{n,n-r})}{|\epsilon|^{(n-1)/2}} \mathrm{vol}(S^{2(n-r)-1}) \int_0^R \cos_\epsilon(\rho)(1+\sin_\epsilon^2(\rho))^r \sin_\epsilon^{2(n-r)-1}(\rho) \\
&= \frac{\mathrm{vol}(G^{\mathbb{C}}_{n,n-r})}{|\epsilon|^{(n-1)/2}} \mathrm{vol}(S^{2(n-r)-1}) \sum_{i=0}^r \binom{r}{i} \int_0^R \cos_\epsilon(\rho) \sin_\epsilon^{2(n-r+i)-1}(\rho) \\
&= \frac{\mathrm{vol}(G^{\mathbb{C}}_{n,n-r})}{|\epsilon|^{(n-1)/2}} \mathrm{vol}(S^{2(n-r)-1}) \sum_{i=0}^r \binom{r}{i} \frac{\sin_\epsilon^{2(n-r+i)}(R)}{2(n-r+i)}.
\end{aligned}
$$

At Chapter 4 we give a general expression for the measure of complex r-planes intersecting a regular domain, in a way such that the previous expression can be interpreted in terms of mean curvature integrals and other valuations defined at Chapter 2.

1.5.5 Reproductive property of Quermassintegrale

At Chapter 3 we prove that the mean curvature integral does not satisfy a reproductive property (see Definition 3.4.1). In this section we prove that Quermassintegrale do satisfy a reproductive property.

Definition 1.5.9. Let Ω be a domain in $\mathbb{CK}^n(\epsilon)$. For $r \in \{1, \ldots, n-1\}$ we define

$$W_r(\Omega) = \frac{(n-r) \cdot O_{r-1} \cdots O_0}{n \cdot O_{n-2} \cdots O_{n-r-1}} \int_{\mathcal{L}_r^{\mathbb{C}}} \chi(\Omega \cap L_r) dL_r$$

where O_i denotes the area of the sphere of radius in the standard Euclidean space. Moreover, we define

$$W_0(\Omega) = \mathrm{vol}(\Omega) \qquad \text{and} \qquad W_n(\Omega) = \frac{O_{n-1}}{n} \chi(\Omega).$$

Constants are chosen for analogy to the case of real space forms.

Proposition 1.5.10. *Let Ω be a domain in $\mathbb{CK}^n(\epsilon)$. Then,*

$$W_r(\Omega) = c \int_{\mathcal{L}_q^{\mathbb{C}}} W_r(\Omega \cap L_q) dL_q$$

for $1 \leq r \leq q \leq n-1$ and c is a constant depending only on n, r and q.

Proof. This proof is analogous to the one given by Santaló (cf. [San04]) to obtain the result in the Euclidean space.

By definition, it is satisfied

$$\int_{\mathcal{L}_q^{\mathbb{C}}} W_r(\Omega \cap L_q) dL_q = \frac{(q-r) O_{r-1} \ldots O_0}{q O_{q-1} \ldots O_{q-r-1}} \int_{\mathcal{L}_q^{\mathbb{C}}} \int_{\Omega \cap L_r^{(q)} \neq \emptyset} dL_r^{(q)} \wedge dL_q.$$

We express the densities $dL_r^{(q)}$, dL_q, $dL_{q[r]}$, dL_r in terms of the forms φ_{ij} defined at Lemma 1.2.7 (we omit the absolute value for the densities),

$$dL_q = \bigwedge_{\substack{q<i\leq n \\ 0\leq j\leq q}} \varphi_{ij}\overline{\varphi_{ij}},$$

$$dL_r^{(q)} = \bigwedge_{\substack{1\leq i\leq r \\ r+1\leq j\leq q}} \varphi_{ij}\overline{\varphi_{ij}},$$

$$dL_{q[r]} = \bigwedge_{\substack{r+1\leq i\leq q \\ q+1\leq j\leq n}} \varphi_{ij}\overline{\varphi_{ij}},$$

$$dL_r = \bigwedge_{\substack{r<i\leq n \\ 0\leq j\leq r}} \varphi_{ij}\overline{\varphi_{ij}},$$

and

$$dL_r^{(q)} \wedge dL_q = \bigwedge_{r+1\leq j\leq n} \varphi_{j0}\overline{\varphi_{j0}} \bigwedge_{\substack{r+1\leq i\leq n \\ 1\leq j\leq r}} \varphi_{ij}\overline{\varphi_{ij}} \bigwedge_{\substack{q+1\leq i\leq n \\ r+1\leq j\leq q}} \varphi_{ij}\overline{\varphi_{ij}},$$

$$dL_{q[r]} \wedge dL_r = \bigwedge_{\substack{r+1\leq j\leq q \\ q+1\leq j\leq n}} \varphi_{ij}\overline{\varphi_{ij}} \bigwedge_{\substack{r+1\leq i\leq n \\ 1\leq j\leq r}} \varphi_{ij}\overline{\varphi_{ij}} \bigwedge_{r+1\leq i\leq n} \varphi_{ij}\overline{\varphi_{i0}}.$$

Thus, the equality

$$dL_r^{(q)} \wedge dL_q = dL_{q[r]} \wedge dL_r$$

holds.

Applying it we get the result

$$\begin{aligned}\int_{\mathcal{L}_q^{\mathbb{C}}} \int_{\Omega\cap L_r^{(q)}\neq\emptyset} dL_r^{(q)} \wedge dL_q &= \int_{\Omega\cap L_r\neq\emptyset} \int_{L_{q[r]}} dL_{q[r]} \wedge dL_r \\ &= \int_{\mathcal{L}_{q[r]}^{\mathbb{C}}} dL_{q[r]} \int_{\Omega\cap L_r\neq\emptyset} dL_r = cW_r(\Omega).\end{aligned}$$

$\square$

Chapter 2

Introduction to valuations

The notion of valuation in $\mathbb{R}^n$ was introduced by Blaschke in 1955 at [Bla55]. Recently, Alesker, among others, has extended this notion to differentiable manifolds. A survey about the development of valuations is given at [MS83] where some references are given.

In this chapter we briefly introduce the theory of valuations, focusing on the concepts and results we shall use in the following chapters.

In the last section, we define some valuations in the spaces of constant holomorphic curvature, generalizing the definition of some valuations in $\mathbb{C}^n$. We also give relations among the defined valuations.

2.1 Definition and basic properties

Let V be a vector space of real dimension n. We denote by $\mathcal{K}(V)$ the set of non-empty compact convex domains in V.

Definition 2.1.1. A functional $\phi : \mathcal{K}(V) \to \mathbb{R}$ is called a *valuation* if

$$\phi(A \cup B) = \phi(A) + \phi(B) - \phi(A \cap B)$$

whenever $A, B, A \cup B \in \mathcal{K}(V)$.

Remark 2.1.2. The extension theorem of Groemer states that every valuation extends uniquely to the set of finite union of convex set.

First examples of valuations in $\mathbb{R}^n$ are the volume of a convex domain, the area of its boundary and its Euler characteristic. Intrinsic volumes, defined by the coefficients of the polynomial obtained in the Steiner's formula, are also valuations.

Consider in $\mathbb{R}^n$ a convex domain Ω and denote by Ω_r the parallel domain at a distance r from Ω. Recall that the parallel domain is constituted by all points at a distance less or equal than r.

The Steiner formula relates the volume of the parallel domain with the volume and some other functionals of the initial domain.

Proposition 2.1.3 (Steiner's formula)**.** *Let $\Omega \subset \mathbb{R}^n$ be a compact domain and Ω_r the parallel domain at distance r. Then, the volume of Ω_r can be expressed as a polynomial in r and its coefficients are multiples of the valuations $V_i : \mathcal{K}(V) \to \mathbb{R}$, $i \in \{0, \ldots, n\}$, called* intrinsic volumes*. The explicit expression is*

$$\operatorname{vol}(\Omega_r) = \sum_{i=0}^{n} r^{n-i} \omega_{n-i} V_i(\Omega) \tag{2.1}$$

where ω_{n-i} denotes the volume of the $(n-i)$-dimensional ball of radius 1 in the Euclidean space.

Proof. (of the fact that V_i are valuations.)

Let A, B, $A \cup B \in \mathcal{K}(V)$. Then, it is satisfied $(A \cap B)_r = A_r \cap B_r$ and $(A \cup B)_r = A_r \cup B_r$. Thus,

$$\mathrm{vol}((A \cup B)_r) = \mathrm{vol}(A_r) + \mathrm{vol}(B_r) - \mathrm{vol}(A_r \cap B_r) = \mathrm{vol}(A_r) + \mathrm{vol}(B_r) - \mathrm{vol}((A \cap B)_r) \quad \forall r.$$

Applying (2.1) we deduce that the intrinsic volumes are valuations. □

Some particular cases of intrinsic volumes are

- $V_n(\Omega) = \mathrm{vol}(\Omega)$,
- $V_{n-1}(\Omega) = \mathrm{vol}(\partial\Omega)/2$,
- $V_0(\Omega) = \chi(\Omega)$.

Note that the Steiner formula has sense for any convex domain, without any assumption on the regularity of the boundary. In some cases it is interesting to consider convex domains such that its boundary is an oriented hypersurface of class $\mathcal{C}^2$. Applying the previous formula in this case we obtain the so-called *mean curvature integrals*.

Definition 2.1.4. Let S be a hypersurface of class $\mathcal{C}^2$ in a Riemannian manifold M of dimension n. If $x \in S$, we denote the second fundamental form of S at x by II_x. We define the *i-th mean curvature integral of* S as

$$M_i(S) = \binom{n-1}{i}^{-1} \int_S \sigma_i(\mathrm{II}_x) dx$$

where $\sigma_i(\mathrm{II}_x)$ denotes the r-th symmetric elementary function of the second fundamental form II_x.

Then, the Steiner formula in $\mathbb{R}^n$ is

$$\mathrm{vol}(\Omega_r) = \mathrm{vol}(\Omega) + \sum_{i=0}^{n-1} r^{n-i} \frac{\binom{n}{i}}{n} M_{n-i-1}(\partial\Omega).$$

Sometimes, it is defined $M_{-1}(\partial\Omega) := \mathrm{vol}(\Omega)$.

The relation among the intrinsic volumes and the mean curvature integrals, for convex domains with boundary of class $\mathcal{C}^2$, is

$$V_i(\Omega) = \frac{\binom{n}{i}}{n\omega_{n-i}} M_{n-i-1}(\partial\Omega).$$

Definition 2.1.5. A valuation ϕ is *continuous* if it is continuous with respect to the Hausdorff metric.

Recall that the *Hausdorff distance* between two compact sets A, B is given by

$$d_{\mathrm{Haus}}(A,B) = \max\{\sup_{a\in A} \inf_{b \in B} \{d(a,b)\}, \sup_{b\in B} \inf_{a\in A} \{d(a,b)\}\}$$

where $d(a,b)$ is the distance defined in the ambient space for A and B.

Example 2.1.6. The intrinsic volumes are continuous valuations. Anyway, there are some interesting examples of non-continuous valuations in $\mathbb{R}^n$. For example, the *affine surface area* is a valuation in the Euclidean space, but it is not continuous (cf. [KR97]). The affine surface area of a convex domain $\Omega \subset \mathbb{R}^n$ is defined as the integral of the $(n+1)$-th root of the generalized Gauss curvature (cf. [Sch93] notes of Sections 1.5 and 2.5) of the boundary of the domain with respect to the $(n-1)$-dimensional Hausdorff measure of the boundary

$$AS(\Omega) = \int_{\partial\Omega} \sqrt[n+1]{K_x}dx.$$

One of the most important property of this valuation is that it is invariant under translations and linear transformations with determinant 1.

Definition 2.1.7. Given a $(2n-1)$-form ω defined on $S(V)$, and a smooth measure, η we consider, for each regular domain Ω,

$$\int_\Omega \eta + \int_{N(\Omega)} \omega$$

where $N(\Omega)$ denotes the normal bundle of the boundary of the domain. The obtained functional is called *smooth valuation.*

Definition 2.1.8. Let $\Omega \in \mathcal{K}(V)$. A valuation $\phi : \mathcal{K}(V) \to \mathbb{R}$ is called

- *translation invariant* if
$$\phi(\psi\Omega) = \phi(\Omega)$$
for every ψ translation of the vector space V;
- *invariant with respect to a group G acting on V* if
$$\phi(g\Omega) = \phi(\Omega)$$
for every $g \in G$;
- *homogeneous of degree k* if
$$\phi(\lambda\Omega) = \lambda^k\phi(\Omega) \text{ for every } \lambda > 0,\ k \in \mathbb{R};$$
- *even* (resp. *odd*) if
$$\phi(-1 \cdot \Omega) = (-1)^\epsilon\phi(\Omega)$$
with ϵ even (resp. odd);
- *monotone* if
$$\phi(\Omega_1) \geq \phi(\Omega_2) \text{ for every } \Omega_1, \Omega_2 \in \mathcal{K}(V) \text{ and } \Omega_1 \supset \Omega_2.$$

The space of continuous invariant translation valuations is denoted by $\mathrm{Val}(V)$, the subspace of $\mathrm{Val}(V)$ of the homogeneous valuations of degree k by $\mathrm{Val}_k(V)$ and the subspace of $\mathrm{Val}(V)$ of even valuations (resp. odd valuations) by $\mathrm{Val}^+(V)$ (resp. $\mathrm{Val}^-(V)$).

Example 2.1.9. The intrinsic volume V_i is a continuous invariant translation valuation homogeneous of degree i.

Remark 2.1.10. The space $\mathrm{Val}(V)$ has structure of infinite dimensional vector space.

The following result of P. McMullen [McM77] gives a decomposition of the space of valuations depending on the degree, and another depending on the parity

Theorem 2.1.11 ([McM77])**.** *Let $n = \dim V$. Then,*

$$\mathrm{Val}(V) = \bigoplus_{i=0}^n \mathrm{Val}_i(V) \quad \textit{and} \quad \mathrm{Val}(V) = \mathrm{Val}^+(V) \oplus \mathrm{Val}^-(V).$$

The linear group $GL(V)$ of invertible linear transformations of V acts transitively on $\mathrm{Val}(V)$

$$(g\phi)(\Omega) = \phi(g^{-1}(\Omega)) \text{ for } g \in GL(V), \phi \in \mathrm{Val}(V), \Omega \in \mathcal{K}(V).$$

This action is continuous and preserves the homogeneous degree of the valuation.

Theorem 2.1.12 (Irreducibility Theorem). *Let V be an n-dimensional vector space. The natural representation of $GL(V)$ at $\mathrm{Val}_i^+(V)$ and $\mathrm{Val}_i^-(V)$ is irreducible for any $i \in \{0, \dots, n\}$. That is, there is no proper closed $GL(V)$-invariant subspace.*

From this theorem, it can be proved the following result which relates continuous valuations with smooth valuations.

Theorem 2.1.13 ([Ale01]). *In a vector space V, the smooth translation invariant valuations are dense in the space of continuous translation invariant valuations.*

If V has an Euclidean metric, then every group G subgroup of the orthogonal group, acts on $\mathrm{Val}(V)$ and it has sense to consider the space $\mathrm{Val}^G(V) \subset \mathrm{Val}(V)$, i.e. the space of G-invariant valuations under the action of the group $G \ltimes V$. The following result by Alesker gives the necessary and sufficient condition to be this space of finite dimension.

Corollary 2.1.14 ([Ale07a] Proposition 2.6). *The space $\mathrm{Val}^G(V)$ is finite dimensional if and only of G acts transitively on the unit sphere of V.*

2.2 Hadwiger Theorem

In 1957 Hadwiger proved the following result concerning valuations.

Theorem 2.2.1 ([Had57]). *The dimension of the space of continuous translation and $O(n)$-invariant valuations is*

$$\dim \mathrm{Val}^{O(n)}(\mathbb{R}^n) = n + 1$$

and a basis of this space is given by

$$V_0, V_1, \dots, V_{n-1}, V_n$$

where V_i denotes the i-th intrinsic volume.

Remark 2.2.2. From this theorem it follows that in $\mathbb{R}^n$ the subspace of valuations of homogeneous degree $k \in \{0, \dots, n\}$ is of dimension 1.

Last remark allows us to prove, in an easy way, some of the classical results of integral geometry (in $\mathbb{R}^n$), such as reproductive or the kinematic formulas.

Example 2.2.3.

- *Crofton formula.* Let $\Omega \subset \mathbb{R}^n$ be a compact convex domain, and let $\mathcal{L}_r$ be the space of all planes of dimension r with dL_r the (unique up to a constant factor) invariant density. The measure of the set of planes a convex body in $\mathbb{R}^n$ can be expressed in terms of the intrinsic volumes as

$$\int_{\mathcal{L}_r} \chi(\Omega \cap L_r) dL_r = cV_{n-r}(\Omega).$$

  This expression is obtained from the fact that the integral in the left hand side is a valuation of homogeneous degree $(n - r)$.

  At Chapter 4, we study the expression of the measure of complex planes meeting a domain in $\mathbb{CK}^n(\epsilon)$, and at Chapter 5, the measure of totally real planes of dimension n meeting a domain.

- *Reproductive property of the intrinsic volumes.* If Ω is a compact convex domain, the reproductive formula for the intrinsic volumes in $\mathbb{R}^n$ is given by

$$\int_{\mathcal{L}_r} V_i^{(r)}(\partial\Omega \cap L_r)dL_r = cV_i(\partial\Omega)$$

where $\mathcal{L}_r$ denotes the space of r-planes in $\mathbb{R}^n$ and c is a constant depending only on n, r and i.

Clearly, the integral in the left hand side is a continuous valuation of homogeneous degree i. By the Hadwiger Theorem the equality follows directly, since the i-th intrinsic volume is a homogeneous valuation of this degree and the dimension of the space of homogeneous valuations of degree i is 1. In order to compute the value of c it is used the so-called *template method*, which consists on evaluating each side of the equation in an easy domain (for instance, a sphere) and then compute the value of c. In [San04] it is given a way to prove this reproductive formula, but using the mean curvature integrals. The value of c it is also computed.

- *Kinematic formula.* Although in this work we do not study kinematic formulas, we would like to give, for completion, the classical kinematic formula of Blaschke-Santaló.

One of the problems of study of the integral geometry consists on measuring the movements of $\mathbb{R}^n$ which takes one convex domain to another fixed one. In $\mathbb{R}^n$, if we denote by $\overline{O(n)}$ the movements group of the space, we obtain the classical kinematic formula

$$\int_{\overline{O(n)}} \chi(\Omega_1 \cap g\Omega_2)dg = \sum_{i=0}^{n} c_{n,i}V_i(\Omega_1)V_{n-i}(\Omega_2). \tag{2.2}$$

This formula can be proved from applying twice the Hadwiger Theorem, and then, applying the obtained formula to spheres of different radius.

Note that the integral on the left hand side is a functional on the first convex domain Ω_1, but also on the second convex domain Ω_2. As a functional on the second convex domain, from the Hadwiger Theorem, we have

$$\int_{\overline{O(n)}} \chi(\Omega_1 \cap g\Omega_2)dg = \sum_{i=0}^{n} c_i(\Omega_1)V_i(\Omega_2)$$

where the coefficients $c_i(\Omega_1)$ depend on Ω_1. But, the integral under consideration is also a valuation with respect to Ω_1, thus, the coefficients $c_i(\Omega_1)$ are valuations and, again by Hadwiger Theorem, we obtain that it is satisfied

$$\int_{\overline{O(n)}} \chi(\Omega_1 \cap g\Omega_2)dg = \sum_{i=0}^{n}\sum_{j=0}^{n} c_{ij}V_j(\Omega_1)V_i(\Omega_2).$$

To obtain the expression (2.2), first, note that the desired expression have to be symmetric with respect to Ω_1 and Ω_2, hence, $c_{ij} = c_{ji}$. To prove that most of the constants c_{ij} vanishes we use the *template method*, i.e. we apply the equality for a sphere of radius r and for a sphere of radius R.

Using the invariance with respect to the rotations of a sphere we have

$$\begin{aligned}\int_{\overline{O(n)}} \chi(B_r \cap gB_R)dg &= \int_{O(n)}\int_{\mathbb{R}^n} \chi(B_r \cap (\phi B_R + v))dvd\phi \\ &= \mathrm{vol}(O(n))\int_{\mathbb{R}^n} \chi(B_r \cap (B_R + v))dv = \mathrm{vol}(O(n))(r+R)^n\omega_n.\end{aligned} \tag{2.3}$$

On the other hand, as the intrinsic volume V_i is a homogeneous valuation of degree i we get

$$\sum_{i=0}^{n}\sum_{j=0}^{n} c_{ij} V_j(B_r)V_i(B_R) = \sum_{i,j=0}^{n} c_{ij} r^j R^i V_j(B_1)V_i(B_1). \tag{2.4}$$

Thus, in both cases (2.3) and (2.4), we obtained a polynomial on r and R. Comparing the coefficients we get $i + j = n$ in the last summation in (2.4).

To get the explicit value for the constant, it remains only to use the expression of the intrinsic volume of sphere of any radius. As the parallel domain at distance r of a sphere of radius R is a sphere of radius $r + R$, we can easily compute its intrinsic volume using the Steiner formula, and we get

$$V_i(B_r) = r^i \binom{n}{i} \frac{\omega_n}{\omega_{n-i}}.$$

2.3 Alesker Theorem

Recently, Alesker gave an analogous theorem of Hadwiger Theorem for the Hermitian standard space $V = \mathbb{C}^n$, with isometry group $IU(n) = \mathbb{C}^n \rtimes U(n)$. As the isometry group of $\mathbb{C}^n$ is smaller than the isometry group of $\mathbb{R}^{2n}$ it may happen that some non-invariant valuations under the isometry group of $\mathbb{R}^{2n}$ is invariant under the isometry group of $\mathbb{C}^n$, and this occurs.

Theorem 2.3.1 ([Ale03] Theorem 2.1.1). *Let* $\mathrm{Val}^{U(n)}(\mathbb{C}^n)$ *be the space of continuous translation and* $U(n)$*-invariant valuations in* $\mathbb{C}^n$*. Then,*

$$\dim \mathrm{Val}^{U(n)}(\mathbb{C}^n) = \binom{n+2}{2},$$

and the dimension of the subspace of degree k homogeneous valuations is $\dfrac{\min\{k, 2n-k\}}{2} + 1.$

Alesker [Ale03] also gave two bases of continuous isometry invariant valuations on $\mathbb{C}^n$. One of these bases is defined as the integral of the projection volume. That is, let $\Omega \subset \mathbb{C}^n$ be a convex domain and k, l integers such that $0 \leq k \leq 2l \leq 2n$, then

$$C_{k,l}(\Omega) := \int_{G^{\mathbb{C}}_{n,l}} V_k(\mathrm{Pr}_{L_l}(\Omega))dL_l$$

where $G^{\mathbb{C}}_{n,l}$ denotes the space of complex l-planes in $\mathbb{C}^n$ through the origin (see Section 1.5), $\mathrm{Pr}_{L_l}(\Omega)$ denotes the orthogonal projection of Ω at L_l and V_k the k-th intrinsic volume. Valuations $\{C_{k,l}\}$ with $0 \leq k \leq 2l \leq 2n$ define a basis of $\mathrm{Val}^{U(n)}(\mathbb{C}^n)$. The index k coincides with the homogeneous degree of the valuation.

The other basis of $\mathrm{Val}^{U(n)}(\mathbb{C}^n)$ is $\{U_{k,p}\}$ with k, p integers such that $0 \leq 2p \leq k \leq 2n$ with

$$U_{k,p}(\Omega) := \int_{\mathcal{L}^{\mathbb{C}}_{n-p}} V_{k-2p}(\Omega \cap L_{n-p})dL_{n-p} \tag{2.5}$$

where $\mathcal{L}^{\mathbb{C}}_{n-p}$ denotes the space of complex affine $(n-p)$-planes of $\mathbb{C}^n$ (see Section 1.5), and V_{k-2p} the $(k-2p)$-th intrinsic volume. The index k coincides with the homogeneous degree of the valuation.

Example 2.3.2. We describe explicitily the elements of $\{U_{k,p}\}$ in $\mathbb{C}^3$. In $\mathbb{C}^3$ there are $\binom{5}{2} = 10$ linearly independent valuations. For each degree k we have as much valuations as linearly independent integers p such that

$$0 \leq p \leq \frac{\min\{k, 2n-k\}}{2}.$$

Suppose that Ω is a convex domain in $\mathbb{C}^3$. Then, a basis of $\mathrm{Val}^{U(3)}(\mathbb{C}^3)$ is given by the following valuations.

$k = 0$: There is only one value of p, $p = 0$ and

$$U_{0,0}(\Omega) = V_0(\Omega) = \chi(\Omega).$$

$k = 1$: There is only one value of p, $p = 0$ and

$$U_{1,0}(\Omega) = V_1(\Omega).$$

$k = 2$: There are two values of p, $p = 0, 1$.

If $p = 0$ then

$$U_{2,0}(\Omega) = V_2(\Omega).$$

If $p = 1$ then

$$U_{2,1}(\Omega) = \int_{\mathcal{L}_2} V_0(\Omega \cap L_2) dL_2.$$

$k = 3$: There are two values of p, $p = 0, 1$.

If $p = 0$ then

$$U_{3,0}(\Omega) = V_3(\Omega).$$

If $p = 1$ then

$$U_{3,1}(\Omega) = \int_{\mathcal{L}_2} V_1(\Omega \cap L_2) dL_2.$$

$k = 4$: There are two values of p, $p = 0, 1$.

If $p = 0$ then

$$U_{4,0}(\Omega) = V_4(\Omega).$$

If $p = 1$ then

$$U_{4,1}(\Omega) = \int_{\mathcal{L}_2} V_2(\Omega \cap L_2) dL_2.$$

$k = 5$: There is only one value of p, $p = 0$ and

$$U_{5,0}(\Omega) = V_5(\Omega) = \frac{1}{2}\mathrm{vol}(\partial\Omega).$$

$k = 6$: There is only one value of p, $p = 0$ and

$$U_{6,0}(\Omega) = V_6(\Omega) = \mathrm{vol}(\Omega).$$

Note that we obtained all intrinsic volumes $V_j(K)$, $j \in \{0, \ldots, 6\}$, in the same way as in $\mathbb{R}^6$, but at $\mathbb{C}^3$ appear three new linear independent valuations.

In the same way as in $\mathbb{R}^n$, having a basis for $\mathrm{Val}^{U(n)}(\mathbb{C}^n)$ allows us to establish a kinematic formula. The difference is that, now, does not seem possible to find the constants using the *template method*. Alesker, in the same paper [Ale03] establish the following result.

Theorem 2.3.3 ([Ale03] Theorem 3.1.1)**.** *Let Ω_1, $\Omega_2 \subset \mathbb{C}^n$ be domains with piecewise smooth boundaries such that for any $U \in IU(n)$ the intersection $\Omega_1 \cap U(\Omega_2)$ has a finite number of components. Then,*

$$\int_{U\in IU(n)} \chi(\Omega_1 \cap U(\Omega_2))dU = \sum_{k_1+k_2=2n} \sum_{p_1,p_2} \kappa(k_1,k_2,p_1,p_2)U_{k_1,p_1}(\Omega_1)U_{k_2,p_2}(\Omega_2),$$

where the index of the inner summation runs over $0 \le p_i \le k_i/2$, $i = 1, 2$, and $\kappa(k_1,k_2,p_1,p_2)$ are constants depending only on n, k_1, k_2, p_1, p_2.

Theorem 2.3.4 ([Ale03] Theorem 3.1.2)**.** *Let $\Omega \subset \mathbb{C}^n$ be a domain with piecewise smooth boundary and $0 < q < n$, $0 < 2p < k < 2q$. Then,*

$$\int_{\mathcal{L}_r^{\mathbb{C}}} U_{k,p}(\Omega \cap L_r)dL_r = \sum_{p=0}^{[k/2]+n-q} \gamma_p \cdot U_{k+2(n-q),p}(\Omega),$$

where γ_p are constants depending only on n, q, and p.

Theorem 2.3.5 ([Ale03] Theorem 3.1.3)**.** *Let $\Omega \subset \mathbb{C}^n$ be a domain with piecewise smooth boundary. Then,*

$$\int_{\mathcal{L}_n^{\mathbb{R}}} \chi(\Omega \cap L_n)dL_n = \sum_{p=0}^{[n/2]} \beta_p \cdot U_{n,p}(\Omega),$$

where $\mathcal{L}_n^{\mathbb{R}}$ denotes the space of Lagrangian planes (i.e. totally real planes of dimension n) in $\mathbb{C}^n$ and β_p are constants depending only on n and p.

The constants for Theorem 2.3.3 are given by Bernig-Fu at [BF08]. These constants were computed using indirect methods and others bases of valuations in $\mathbb{C}^n$. In this work, we give the constant, in some cases, for Theorems 2.3.4 and 2.3.5.

In [BF08] are given some other bases for $\mathrm{Val}^{U(n)}(\mathbb{C}^n)$. In Chapters 4 and 5, we use a basis defined in [BF08] and we extend it to all complex space forms $\mathbb{CK}^n(\epsilon)$. The definition of theses valuations and its extension in $\mathbb{CK}^n(\epsilon)$ is given at Section 2.4.2.

Finally, we note that Proposition 2.1.14 gives another way to generalize the theory of valuations in vector spaces. From this proposition, we have that for any group acting transitively on the sphere can be stated a Hadwiger type theorem, i.e. the space of continuous translation invariant valuations has finite dimension, thus, it has sense to compute its dimension and give a basis. An expression for the kinematic formula can also be given.

In this section, we recalled the case in which the acting group is $U(n)$ and befors we studied $SO(n)$, but there are some known result for other groups.

The groups acting over a sphere are classified and they are (cf. [Bor49], [Bor50], [MS43]) six infinite series

$$SO(n), U(n), SU(n), Sp(n), Sp(n)\cdot U(1), Sp(n)\cdot Sp(1)$$

and three exceptional groups

$$G_2, \mathrm{Spin}(7), \mathrm{Spin}(9).$$

Alesker and Bernig obtained a Hadwiger type theorem, a kinematic formula (and the algebraic structure) for some of these groups (cf. [Ale04] for $G = SU(2)$, [Ber08a] for $G = SU(n)$ and [Ber08b] for $G = G_2$ and $G = \mathrm{Spin}(7)$).

2.4 Valuations on complex space forms

In this section we define some valuations on $\mathbb{CK}^n(\epsilon)$ and we give relations among them.

2.4.1 Smooth valuations on manifolds

The notion of smooth valuation in a differentiable manifold was recently studied (cf. [Ale06a], [Ale06b], [AF08], [Ale07b], [AB08]). First, a definition of smooth valuation on a manifold was given, and then it was proved that some important properties, which we do not study in this work, of smooth valuations, such as the duality property, still hold.

Definition 2.1.7 can be extended for differentiable manifolds.

Definition 2.4.1. Let M be a differentiable manifold and Ω a compact submanifold. Given a $(2n-1)$-form ω in $S(M)$, and a smooth measure η, we consider for any Ω

$$\int_\Omega \eta + \int_{N(\Omega)} \omega,$$

where $N(\Omega)$ denotes the normal cycle (cf. [Ale07a]). The obtained functional is called *smooth valuation.*

Remark 2.4.2. A more general definition analogue to Definiton 2.1.1 appears in [Ale06b], and it is called *finitely additive measure.*

In spaces of constant sectional curvatures, invariant smooth valuations are well-known. These spaces have the same isotropy group of a point as a point in $\mathbb{R}^n$, and from the point of view of a homogeneous spaces they can be studied in an analogous way. Despite this fact, a Hadwiger type theorem for continuous valuations (and not only for smooth valuations) it is not known, i.e. it is not known a basis of continuous translation invariant valuations invariant also for the isometry group of the space. The dimension of this space of valuations it is not known an analogous result to Theorem 2.1.13.

Anyway, a big amount of the results in integral geometry are known in these spaces. For instance, Santaló [San04, page 309] proved that a reproductive formula holds for any real space form and also obtained an expression for the measure of totally geodesic planes meeting a convex domain.

In view of these results of Santaló and the knowledge of a basis of continuous invariant valuations on $\mathbb{C}^n$, the aim of this work is to study the classical formulas in integral geometry described in the last paragraph in complex space forms, i.e. in the standard Hermitian space, and in the complex projective and hyperbolic space.

2.4.2 Hermitian intrinsic volumes

Bernig and Fu [BF08] defined the Hermitian intrinsic volumes in $\mathbb{C}^n$. In this section we recall this definition and its extension to $\mathbb{CK}^n(\epsilon)$ following [Par02].

Bernig and Fu at [BF08, page 14] defined in $T\mathbb{C}^n$, the following invariant 1-forms α, β and γ and the invariant 2-forms θ_0, θ_1, θ_2 and θ_s. Let $(z_1,\dots,z_n,\zeta_1,\dots,\zeta_n)$ be the canonical coordinates of $T\mathbb{C}^n \simeq \mathbb{C}^n \times \mathbb{C}^n$ with $z_i = x_i + \sqrt{-1}y_i$ and $\zeta_i = \xi_i + \sqrt{-1}\eta_i$. Then,

$$\theta_0 := \textstyle\sum_{i=1}^n d\xi_i \wedge d\eta_i, \qquad \theta_1 := \sum_{i=1}^n \left(dx_i \wedge d\eta_i - dy_i \wedge d\xi_i\right),$$

$$\theta_2 := \textstyle\sum_{i=1}^n dx_i \wedge dy_i, \qquad \theta_s := \sum_{i=1}^n \left(dx_i \wedge d\xi_i + dy_i \wedge d\eta_i\right),$$

$$\alpha := \textstyle\sum_{i=1}^n \xi_i dx_i + \eta_i dy_i, \qquad \beta := \sum_{i=1}^n \xi_i dy_i - \eta_i dx_i,$$

$$\gamma := \textstyle\sum_{i=1}^n \xi_i d\eta_i - \eta_i d\xi_i.$$

α is the contact form (see Remark 1.3.8) and θ_s is the symplectic form of $T\mathbb{C}^n$. Recall that a *2-form* ω defined in a manifold of dimension $2m$ is said *symplectic* if it is closed, non-degenerated and $\omega^m \neq 0$.

The previous forms are only defined for $\epsilon = 0$ since canonical coordinates exist only at $\mathbb{C}^n$. But, we can express them in terms of a moving frame on $S(\mathbb{C}^n)$ (and, then extend them for any $\epsilon \in \mathbb{R}$). The expression in terms of a moving frame allows us to prove that these forms are well-defined, i.e. they do not depend on the chosen coordinates. Let $(x, v) \in S(\mathbb{CK}^n(\epsilon))$ and let $\{x; e_1 := v, Je_1, \ldots, e_n, Je_n\}$ be a J-moving frame defined in a neighborhood of x. Then

$$\begin{aligned}
\theta_0 &= \sum_{i=2}^{n} \alpha_{1i} \wedge \beta_{1i}, \\
\theta_1 &= \sum_{i=2}^{n} (\alpha_i \wedge \beta_{1i} - \beta_i \wedge \alpha_{1i}), \\
\theta_2 &= \sum_{i=2}^{n} \alpha_i \wedge \beta_i,
\end{aligned} \tag{2.6}$$

where α_i, β_i, α_{ij}, β_{ij} are the forms given at (1.12) but interpreted as forms in $S(\mathbb{CK}^n(\epsilon))$.

Remark 2.4.3. From the expression of θ_0, θ_1 and θ_2 in terms of a moving frame we can define these 2-forms in $S(\mathbb{CK}^n(\epsilon))$ for any ϵ.

Remark 2.4.4. In [Par02] invariant 2-forms at $\mathbb{CK}^n(\epsilon)$ are defined in the same way as before (see Section 2.4.3).

Proposition 2.4.5 ([Par02] Proposition 2.2.1)**.** *The algebra $\Omega^*(S(\mathbb{CK}^n(\epsilon)))$ of $\mathbb{R}$-valued invariant differential forms on the unit tangent bundle $S(\mathbb{CK}^n(\epsilon))$ is generated by*

$$\alpha,\ \beta,\ \gamma,\ \theta_0,\ \theta_1,\ \theta_2,\ \theta_s.$$

From this proposition, Bernig and Fu define the families of $(2n-1)$-forms $\{\beta_{k,q}\}$ and $\{\gamma_{k,q}\}$ at $S(\mathbb{C}^n)$, but from the definition of θ_0, θ_1 and θ_2 at $S(\mathbb{CK}^n(\epsilon))$ these families of forms can be defined in the same way at $S(\mathbb{CK}^n(\epsilon))$. By the previous proposition we have, as it is note in [Par02], that all $(2n-1)$-form invariant on $S(\mathbb{CK}^n(\epsilon))$ such that they do not vanish over the normal bundle of a domain Ω (cf. Lemma 1.3.12) are the ones given in the next definition.

Definition 2.4.6. Let $k, q \in \mathbb{N}$ be such that $\max\{0, k-n\} \leq q \leq \frac{k}{2} < n$. Then, the following differential $(2n-1)$-forms at $S(\mathbb{CK}^n(\epsilon))$ are defined

$$\begin{aligned}
\beta_{k,q} &:= c_{n,k,q} \beta \wedge \theta_0^{n-k+q} \wedge \theta_1^{k-2q-1} \wedge \theta_2^q, \quad k \neq 2q \\
\gamma_{k,q} &:= \frac{c_{n,k,q}}{2} \gamma \wedge \theta_0^{n-k+q-1} \wedge \theta_1^{k-2q} \wedge \theta_2^q, \quad n \neq k-q
\end{aligned}$$

where

$$c_{n,k,q} := \frac{1}{q!(n-k+q)!(k-2q)!\omega_{2n-k}}$$

and ω_{2n-k} denotes the volume of the Euclidean ball of radius 1 and dimension $2n-k$.

Definition 2.4.7. Given a regular domain $\Omega \subset \mathbb{CK}^n(\epsilon)$, forms $\beta_{k,q}$ and $\gamma_{k,q}$ define the following invariant valuations (see Section 2.4.3) in $\mathbb{CK}^n(\epsilon)$ (for $\max\{0, k-n\} \leq q \leq \frac{k}{2} < n$)

$$B_{k,q}(\Omega) := \int_{N(\Omega)} \beta_{k,q} \quad (k \neq 2q) \qquad \text{and} \qquad \Gamma_{k,q}(\Omega) = \int_{N(\Omega)} \gamma_{k,q} \quad (n \neq k-q)$$

where $N(\Omega)$ denotes the normal fiber bundle of Ω (see Definition 1.3.10).

In $\mathbb{C}^n$ the previous valuations satisfy $B_{k,q}(\Omega) = \Gamma_{k,q}(\Omega)$ since $d\beta_{k,q} = d\gamma_{k,q}$. For $\epsilon \neq 0$ no form $\beta_{k,q}$ has the same differential as $\gamma_{k,q}$.

The differential of the forms θ_0, θ_1 and θ_2 is given in [BF08] for $\epsilon = 0$. From the structure equations in $\mathbb{CK}^n(\epsilon)$ (cf. [KN69]), in the same way as in [Par02], we compute the differential for any ϵ.

Lemma 2.4.8. *In $S(\mathbb{CK}^n(\epsilon))$ it is satisfied*

$$\begin{aligned} d\alpha &= -\theta_s, & d\theta_0 &= -\epsilon(\alpha \wedge \theta_1 + \beta\theta_s),\\ d\beta &= \theta_1, & d\theta_1 &= d\theta_2 = d\theta_s = 0\\ d\gamma &= 2\theta_0 - 2\epsilon(\alpha \wedge \beta + \theta_2). \end{aligned}$$

In the following proposition we give the relation between valuations $\{B_{k,q}(\Omega)\}$ and $\{\Gamma_{k,q}(\Omega)\}$ in $\mathbb{CK}^n(\epsilon)$.

Proposition 2.4.9. *In $\mathbb{CK}^n(\epsilon)$, for any pair of integers (k, q) such that* $\max\{0, k-n\} < q < k/2 < n$ *it is satisfied*

$$\begin{aligned} \Gamma_{k,q}(\Omega) &= B_{k,q}(\Omega) - \epsilon \frac{c_{n,k,q}}{c_{n,k+2,q+1}} B_{k+2,q+1}(\Omega)\\ &= B_{k,q}(\Omega) - \epsilon \frac{(q+1)(2n-k)}{2\pi(n-k+q)} B_{k+2,q+1}(\Omega). \end{aligned}$$

Proof. Denote by I the ideal generated by α, $d\alpha$ and all the exact forms in $N(\Omega)$ (see Definition 1.3.10). If two forms λ and ρ of degree $2n-1$ coincide modulo I, then by Lemma 1.3.12

$$\int_{N(\Omega)} \lambda = \int_{N(\Omega)} \rho.$$

Thus, it is enough to prove

$$\gamma_{k,q} \equiv \beta_{k,q} - \epsilon \frac{c_{n,k,q}}{c_{n,k+2,q+1}} \beta_{k+2,q+1} \qquad \text{mod } I. \tag{2.7}$$

Consider the form $\eta = (\theta_s - \beta \wedge \gamma) \wedge \theta_0^{n-k+q-1} \theta_1^{k-2q-1} \theta_2^q$. As $d\eta$ is exact we have $d\eta \equiv 0$ mod I. On the other hand, from the differentials given in Lemma 2.4.8 we obtain

$$d\eta \equiv -\gamma\theta_0^{n-k+q-1}\theta_1^{k-2q}\theta_2^q + 2\beta\theta_0^{n-k+q}\theta_1^{k-2q-1}\theta_2^q - 2\epsilon\beta\theta_0^{n-k+q-1}\theta_1^{k-2q-1}\theta_2^{q+1} \quad \text{mod } I.$$

Using the definition of $\gamma_{k,q}$ and $\beta_{k,q}$ we get the relation (2.7). □

Remark 2.4.10. For $n = 2, 3$, the previous relations are given in [Par02].

By the relation in Proposition 2.4.9, we define the Hermitian intrinsic volumes in $\mathbb{CK}^n(\epsilon)$.

Definition 2.4.11. For $\max\{0, k-n\} \leq q \leq \frac{k}{2} < n$, we define the *Hermitian intrinsic volumes* $\mu_{k,q}$ *in* $\mathbb{CK}^n(\epsilon)$

$$\mu_{k,q}(\Omega) := \begin{cases} B_{k,q}(\Omega) & \text{si } k \neq 2q \\ \Gamma_{2q,q}(\Omega) & \text{si } k = 2q. \end{cases} \tag{2.8}$$

Remark 2.4.12. In $\mathbb{C}^n$, Hermitian intrinsic volumes form a basis of continuous valuations invariant under the isometry group of $\mathbb{C}^n$ (cf. [BF08]).

In the previous definition of $\mu_{k,q}$ we take an arbitrary choice, despite of the relations in Proposition 2.4.9. It would be interesting to know if there is a better choice, such that it satisfies some more natural geometric or algebraic properties.

2.4.3 Relation between Hermitian intrinsic volumes and the valuations given by Park

We give, by completeness, the valuations defined by Park in $\mathbb{CK}^n(\epsilon)$.

Definition 2.4.13. Denote $\theta_{00} = -\alpha\wedge\beta+\theta_2$, $\theta_{01} = -\beta\wedge\gamma+\theta_s$, $\theta_{10} = -\alpha\wedge\gamma+\theta_1$, $\theta_{11} = \theta_0$. Let $\kappa = \sigma\wedge\{a,b,c\}$ where $\sigma\in\{\alpha,\beta\}$ and $\{a,b,c\} = \frac{1}{a!b!c!}\theta_{11}^a\wedge\theta_{00}^b\wedge\theta_{10}^c$. Then

$$\Phi^{\kappa}(\Omega) = \int_{N(\Omega)}\kappa$$

are smooth valuation invariant under the action of the isometry group of $\mathbb{CK}^n(\epsilon)$.

The relation between these valuations and the Hermitian intrinsic volumes is given straightforward from the definition of each valuation.

Proposition 2.4.14. *Let $\Omega\subset\mathbb{CK}^n(\epsilon)$ be a regular domain. Then,*

$$\begin{aligned} B_{k,q}(\Omega) &= \frac{1}{(k-2q)\omega_{2n-k}}\Phi^{\beta\{n-k+q,q,k-2q-1\}}(\Omega),\\ \Gamma_{k,q}(\Omega) &= \frac{1}{2(n-k+q)\omega_{2n-k}}\Phi^{\gamma\{n-k+q-1,q,k-2q\}}(\Omega). \end{aligned}$$

2.4.4 Other curvature integrals

Let M be a Kähler manifold of complex dimension n and suppose that $S\subset M$ is a real hypersurface. Then, we can canonically define a distribution of complex dimension $n-1$ in the tangent fiber bundle of S in the following way.

Let N_x be the normal fiber bundle of S at x. Let J be the complex structure in M. The vector JN_x is a tangent vector to S at x. Consider the orthogonal vectors to JN_x inside the tangent space of S at x. These form a complex subspace of dimension $n-1$. Denote by $\mathcal{D}$ the distribution defined by these subspaces. Then, we define the mean curvature integrals restricted to the distribution $\mathcal{D}$ as

Definition 2.4.15. Let S be a hypersurface of a Kähler manifold M of complex dimension n. If $x\in S$, we denote the second fundamental form of S at x by II_x and the second fundamental form restricted to $\mathcal{D}$ by $\mathrm{II}_x|_{\mathcal{D}}$. The *$r$-th mean curvature integral of S restricted at $\mathcal{D}$*, $1\le r\le 2n-2$, is defined as

$$M_r^{\mathcal{D}}(S) = \binom{2n-2}{r}^{-1}\int_S \sigma_r(\mathrm{II}_x|_{\mathcal{D}})dx$$

where $\sigma_r(\mathrm{II}_x|_{\mathcal{D}})$ denotes the r-th symmetric elementary function of $\mathrm{II}_x|_{\mathcal{D}}$.

Along this work, we use the idea of restricting mean curvature integrals to the distribution $\mathcal{D}$. The first mean curvature integral restricted to $\mathcal{D}$ will play an important role, i.e. the integral of the trace of the second fundamental form restricted to the distribution. Also the integral over the normal curvature JN will have an important role. If Ω is a regular domain, we have the following relations

$$(2n-1)M_1(\partial\Omega)-(2n-2)M_1^{\mathcal{D}}(\partial\Omega) = \int_{\partial\Omega}k_n(JN)dp = 2\omega_2\Gamma_{2n-2,n-1}(\Omega). \tag{2.9}$$

2.4.5 Relation between the Hermitian intrinsic volumes and the second fundamental form

In this section we give another expression for the Hermitian intrinsic volumes in terms of the second fundamental form.

In the proof of Theorem 4.3.1 we shall use some properties, interesting by their own, of the expression of the invariant $(2n-1)$-forms expressed in terms of the second fundamental form of $\partial\Omega$, and not only in terms of the connection forms of a moving frame. In order to give this expression in terms of the second fundamental form (with respect to a fixed basis) it is necessary to consider the pull-back of the following canonical map

$$\begin{array}{cccc} \varphi: & \partial\Omega & \longrightarrow & N(\Omega) \\ & x & \mapsto & (x, N_x) \end{array}. \tag{2.10}$$

Let us study some properties of $\varphi^*(\beta_{k,q})$ and $\varphi^*(\gamma_{k,q})$.

Let $x \in \partial\Omega \subset \mathbb{CK}^n(\epsilon)$ and let $\{e_1 = \varphi(x), e_{\overline{1}} = Je_1, \ldots, e_n, e_{\overline{n}} = Je_n\}$ be a J-moving frame defined in a neighborhood of x. Consider the 1-forms $\{\alpha_i, \beta_i, \alpha_{1j}, \beta_{1j}\}$, and the 2-forms $\{\theta_0, \theta_1, \theta_2, \theta_s\}$ given at (2.6).

Notation 2.4.16. In order to simplify the notation in the following expressions we denote β_i by $\alpha_{\overline{i}}$ and β_{1i} by $\alpha_{1\overline{i}}$ and we define $I := \{\overline{1}, 2, \overline{2}, \ldots, n, \overline{n}\}$.

Now, using the relation between the connection forms α_{1i}, $i \in I$, and the second fundamental form

$$\alpha_{1i} = \sum_{j \in I} h_{ij}\alpha_i, \tag{2.11}$$

we obtain

Lemma 2.4.17. *In the previous notation,*

$$\begin{aligned}
\varphi^*(\beta) &= \alpha_{\overline{1}}, \\
\varphi^*(\gamma) &= \sum_{j\in I} h_{\overline{1}j}\alpha_j, \\
\varphi^*(\theta_0) &= \sum_{i=2}^{n} \sum_{j,l\in I} h_{ij}h_{\overline{i}l}\alpha_j \wedge \alpha_l, \\
\varphi^*(\theta_1) &= \sum_{i=2}^{n} \left(\sum_{j\in I} h_{\overline{i}j}\alpha_i \wedge \alpha_j - \sum_{l\in I} h_{il}\alpha_{\overline{i}} \wedge \alpha_l \right), \\
\varphi^*(\theta_2) &= \sum_{i=2}^{n} \alpha_i \wedge \alpha_{\overline{i}}.
\end{aligned}$$

On the other hand, each form $\varphi^*(\beta_{k,q})$ is a form of maximum degree, and, thus, a multiple of the volume element $dx = \alpha_{\overline{1}} \wedge \alpha_2 \wedge \alpha_{\overline{2}} \wedge \cdots \wedge \alpha_{\overline{n}}$ of $\partial\Omega$. Thus, $\varphi^*(\beta_{k,q})$ is determined by this multiple, which can be interpreted as a polynomial with variables the entries of the second fundamental form h_{ij} (with respect to the fixed J-moving frame).

In [Par02], it is computed explicitly the pull-back of the forms $\beta_{k,q}$ and $\gamma_{k,q}$ for dimensions $n = 2, 3$. In the following lemma we give some general properties for the pull-back of these forms for any dimension n.

Lemma 2.4.18. *Let $\Omega \subset \mathbb{CK}^n(\epsilon)$ be a regular domain and let $\varphi : \partial\Omega \to N(\Omega)$ be the canonical map defined at (2.10). Let us fix a point $x \in \partial\Omega$ and a J-moving frame $\{e_1 = \varphi(x), e_{\overline{1}} = Je_1, \ldots, e_n, e_{\overline{n}} = Je_n\}$ at x. If*

$$\varphi^*(\beta_{k,q}) = Q_{k,q}dx, \quad \varphi^*(\gamma_{k,q}) = P_{k,q}dx$$

where dx is the volume element of $\partial\Omega$, then

1. *$Q_{k,q}$ is a polynomial of degree $2n-k-1$ with variables the entries of the second fundamental form $h_{ij} = \mathrm{II}(e_i, e_j)$, $i, j \in \{2, \overline{2}, \dots, \overline{n}\}$;*

2. *$P_{k,q}$ is a polynomial of degree $2n-k-1$ with variables the entries of the second fundamental form $h_{ij} = \mathrm{II}(e_i, e_j)$, $i, j \in \{\overline{1}, 2, \overline{2}, \dots, \overline{n}\}$;*

3. *each monomial of $P_{k,q}$ containing only entries of the form h_{ii} also contains $h_{\overline{1}\overline{1}}$ and exactly $n+q-k-1$ factors of the form $h_{jj}h_{\overline{j}\overline{j}}$, $i \in \{\overline{1}, 2, \overline{2}, \dots, \overline{n}\}, j \in \{2, \dots, n\}$;*

4. *the polynomials $P_{k,q}$ and $Q_{k,q}$ can be written as a sum of minors of the second fundamental form with rank $r = 2n-k-1$;*

5. *among the minors described at 4. appear all minors centered at the diagonal with degree r containing $h_{\overline{1}\overline{1}}$ for $P_{k,q}$, and not containing $h_{\overline{1}\overline{1}}$ for $Q_{k,q}$. It also appears all non-centered minors such that the indices of the rows and the indices of the columns determining a minor satisfy*

   (a) *contain $n-k+q$ indices, in the case of $Q_{k,q}$, and $n-k+q-1$, in the case of $P_{k,q}$, such that, if the index j appears as an index in the rows (resp. columns), then the index $\overline{j}$ also appears as an index in the rows (resp. columns) of the minor. We say that the index j is paired in the rows (or in the columns);*

   (b) *contain $k-2q-1$ indices non-paired neither in the rows nor in the columns for $Q_{k,q}$, and $k-2q$ for $P_{k,q}$;*

   (c) *if the index j is in the non-paired indices of the rows, then the index $\overline{j}$ is not in the index of the columns.*

Proof. From Lemma 2.4.17 we have

$$\begin{aligned}\varphi^*(\beta_{k,q}) = c_{n,k,q}\alpha_{\overline{1}} \wedge \left(\sum_{i=2}^{n}\sum_{j,l\in I} h_{ij}h_{\overline{i}l}\alpha_j \wedge \alpha_l\right)^{n+q-k} &\wedge \\ \wedge \left(\sum_{i=2}^{n}\left(\sum_{j\in I} h_{\overline{i}j}\alpha_i \wedge \alpha_j - \sum_{l\in I} h_{il}\alpha_{\overline{i}} \wedge \alpha_l\right)\right)^{k-2q-1} &\wedge \left(\sum_{i=2}^{n}\alpha_i \wedge \alpha_{\overline{i}}\right)^{q},\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}\varphi^*(\gamma_{k,q}) = \frac{c_{n,k,q}}{2}\left(\sum_{j\in I} h_{\overline{1}j}\alpha_j\right) \wedge \left(\sum_{i=2}^{n}\sum_{j,l\in I} h_{ij}h_{\overline{i}l}\alpha_j \wedge \alpha_l\right)^{n+q-k-1} &\wedge \\ \wedge \left(\sum_{i=2}^{n}\left(\sum_{j\in I} h_{\overline{i}j}\alpha_i \wedge \alpha_j - \sum_{l\in I} h_{il}\alpha_{\overline{i}} \wedge \alpha_l\right)\right)^{k-2q} &\wedge \left(\sum_{i=2}^{n}\alpha_i \wedge \alpha_{\overline{i}}\right)^{q}.\end{aligned}$$

Thus, as $\varphi^*(\beta_{k,q})$ and $\varphi^*(\gamma_{k,q})$ are differential forms defined on $\partial\Omega$ of maximum degree, we have that $\varphi^*(\beta_{k,q}) = Q_{k,q}dx$ and $\varphi^*(\gamma_{k,q}) = P_{k,q}dx$ satisfy that $Q_{k,p}$ and $P_{k,q}$ are polynomials of degree $2n-k-1$. Moreover, polynomials $P_{k,q}$ cannot contain $h_{\overline{1}\overline{1}}$ since this variable is multiplied by $\alpha_{\overline{1}}$ (see formula (2.11)), but this differential form is a common factor in the expression $\varphi^*(\beta_{k,q})$.

In order to prove 3. we observe that terms in the expression $\varphi^*(\gamma_{k,q})$ containing only entries of the type h_{ii} are

$$\frac{c_{n,k,q}}{2} h_{\overline{1}\overline{1}} \alpha_{\overline{1}} \wedge \left(\sum_{i=2}^{n} h_{ii} h_{\overline{i}\overline{i}} \alpha_i \wedge \alpha_{\overline{i}} \right)^{n+q-k-1} \wedge \left(\sum_{i=2}^{n} h_{\overline{i}\overline{i}} \alpha_i \wedge \alpha_{\overline{i}} - h_{ii} \alpha_{\overline{i}} \wedge \alpha_i \right)^{k-2q} \wedge \left(\sum_{i=2}^{n} \alpha_i \wedge \alpha_{\overline{i}} \right)^{q}.$$

So, the variable $h_{\overline{1}\overline{1}}$ always appears and there are exactly $n+q-k-1$ factors of the form $h_{jj}h_{\overline{j}\overline{j}}$, which come from $(\sum_{i=2}^{n} h_{ii}h_{\overline{i}\overline{i}}\alpha_i \wedge \alpha_{\overline{i}})^{n+q-k-1}$.

A minor of rank r of a matrix is defined choosing r rows and r columns of the matrix and then taking the determinant of the square submatrix. To prove 4. and 5. we study in more detail the expression (2.12). (An analogous study can be done in the case $\varphi^*(\gamma_{k,q})$.) First, note that factors $\alpha_{\overline{1}}$ and $\varphi^*(\theta_2)$ do not give any term of the second fundamental form and, thus, they do not influence in the minor (but they do influence in which minors can be constructed). Thus, we have to prove that the expression

$$\varphi^*(\theta_0^{n+q-k}) \wedge \varphi^*(\theta_1^{k-2q-1}), \tag{2.13}$$

is a form of degree $2n-2q-2$ where each term $\alpha_{i_1} \wedge \alpha_{i_2} \wedge \cdots \wedge \alpha_{i_{2n-2q-2}}$ goes with a summation of minors.

Developing (2.13) we have that it is equivalent to

$$\varphi^* \left((\sum_{i=2}^{n} \alpha_{\overline{1}i} \wedge \alpha_{\overline{1}\overline{i}})^{n+q-k} \wedge (\sum_{j=2}^{n} (\alpha_j \wedge \alpha_{\overline{1}\overline{j}} - \alpha_j \wedge \alpha_{\overline{1}j}))^{k-2q-1} \right).$$

To develop this expression, first, we chose $a := n+q-k$ values $\{i_1, \ldots, i_a\}$ for the index i of the first summation and $b := k-2q-1$ values $\{j_1, \ldots, j_b\}$ (with $i_k, j_l \in \{2, \ldots, n\}$) for the index j of the second summation. Note that some indexes can be repeated. Thus, we get $\binom{n-1}{n+q-k} \cdot \binom{n-1}{k-2q-1}$ summands

$$\varphi^*(\alpha_{\overline{1}i_1} \wedge \alpha_{\overline{1}\overline{i_1}} \wedge \cdots \wedge \alpha_{\overline{1}i_a} \wedge \alpha_{\overline{1}\overline{i_a}} \wedge (\alpha_{j_1} \wedge \alpha_{\overline{1}\overline{j_1}} - \alpha_{\overline{j_1}} \wedge \alpha_{\overline{1}j_1}) \wedge \cdots \wedge (\alpha_{j_b} \wedge \alpha_{\overline{1}\overline{j_b}} - \alpha_{\overline{j_b}} \wedge \alpha_{\overline{1}j_b})),$$

which can be decomposed, for example, in the form

$$\begin{aligned} &\varphi^*(\alpha_{\overline{1}i_1} \wedge \alpha_{\overline{1}\overline{i_1}} \wedge \cdots \wedge \alpha_{\overline{1}i_a} \wedge \alpha_{\overline{1}\overline{i_a}} \wedge \alpha_{j_1} \wedge \alpha_{\overline{1}\overline{j_1}} \wedge \cdots \wedge \alpha_{j_b} \wedge \alpha_{\overline{1}\overline{j_b}}) \\ &\qquad = \alpha_{j_1} \wedge \cdots \wedge \alpha_{j_b} \varphi^*(\alpha_{\overline{1}i_1} \wedge \alpha_{\overline{1}\overline{i_1}} \wedge \cdots \wedge \alpha_{\overline{1}i_a} \wedge \alpha_{\overline{1}\overline{i_a}} \wedge \alpha_{\overline{1}\overline{j_1}} \wedge \cdots \wedge \alpha_{\overline{1}\overline{j_b}}). \end{aligned}$$

From (2.11) the form in which we take pull-back can be expressed as the summation of the minors with rows given by the indices $\mathcal{I} := \{i_1, \overline{i_1}, \ldots, i_a, \overline{i_a}, \overline{j_1}, \ldots, \overline{j_b}\}$, and by columns each of the possible permutations of the elements without repetition among the indexes in $\mathcal{J} := I \setminus \{\overline{1}, j_1, \ldots, j_b\}$. Note that index $\alpha_{\overline{1}}$ cannot be taken since we are considering the form in (2.12), which is multiplied by $\alpha_{\overline{1}}$. (In the case of $\varphi^*(\gamma_{k,q})$ we do not have this restriction, and, so, can also appear minors with the term $h_{\overline{1}\overline{1}}$.)

If we chose for $\mathcal{J}$ the same indexes as in $\mathcal{I}$, then we get all the minors centered at the diagonal. Condition 5.(c) is obtained directly from the fact that the indices which determines the columns have to be in $\mathcal{J}$, and if $\overline{j_k}$ is an index in the rows, the index j_k is not in $\mathcal{J}$.

Conditions 5.(a) and 5.(b) are obtained when we recall that we are not studying the differential form in (2.13) but in (2.12), i.e. we have to take the product with $\alpha_{\overline{1}} \wedge (\sum_{i=2}^{n} \alpha_i \wedge \alpha_{\overline{i}})^q$. But, if this product contains the form α_k, then it also contains the form $\alpha_{\overline{k}}$ (except for $k=1$). Thus, to complete the form in (2.13) to a $(2n-1)$-differential form we have to take the products $\alpha_k \wedge \alpha_{\overline{k}}$, so that, in order to not obtain a vanishing term, the quantity of paired indices in the rows and in the columns must be the same, and, thus, it also coincides the quantity of no-paired indices. □

Remarks 2.4.19.
1. Each of the minor goes with a constant depending on the number of permutations that allows us to obtain it. Thus, not all the minors have the same constant, but, for instance all the minors (fixed $\beta_{k,q}$ or $\gamma_{k,q}$) centered at the diagonal have the same one.

2. The degree of the polynomial given by $\beta_{k,q}$ or $\gamma_{k,q}$ does not depend on q, just on k, but two polynomials coming from $\beta_{k,q}$ and $\beta_{k,q'}$ (or $\gamma_{k,q}$ and $\gamma_{k,q'}$) are distinguished by the number of paired indexes.

Example 2.4.20. We give the explicit relation between some Hermitian intrinsic volumes and the second fundamental form.

1. $\varphi^*(\gamma_{00}) = \dfrac{c_{n,0,0}}{2}\varphi^*(\gamma \wedge \theta_0^{n-1}) = (n-1)!\dfrac{c_{n,0,0}}{2}\det(\mathrm{II})dx = \dfrac{1}{2n\omega_{2n}}\det(\mathrm{II})dx.$

2. $\varphi^*(\beta_{10}) = c_{n,1,0}\varphi^*(\beta \wedge \theta_0^{n-1}) = (n-1)!c_{n,1,0}\det(\mathrm{II}|_{\mathcal{D}})dx = \dfrac{1}{\omega_{2n-1}}\det(\mathrm{II}|_{\mathcal{D}})dx.$

3. $\varphi^*(\gamma_{2n-2,n-1}) = \dfrac{c_{n,2n-2,n-1}}{2}\varphi^*(\gamma\wedge\theta_2^{n-1}) = \dfrac{c_{n,2n-1,n-1}(n-1)!}{2}k_n(JN)dx = k_n(JN)\dfrac{dx}{2\omega_2}.$

4. $\varphi^*(\beta_{2n-2,n-2}) = c_{n,2n-2,n-2}\varphi^*(\beta \wedge \theta_1 \wedge \theta_2^{n-2}) = \dfrac{(n-2)!}{(n-2)!2\omega_2}\mathrm{tr}(\mathrm{II}|_{\mathcal{D}})dx = \mathrm{tr}(\mathrm{II}|_{\mathcal{D}})\dfrac{dx}{2\omega_2}.$

5. $\varphi^*(\beta_{2n-1,n-1}) = c_{n,2n-1,n-1}\varphi^*(\beta \wedge \theta_2^{n-1}) = c_{n,2n-1,n-1}(n-1)!\alpha_1 \wedge \alpha_{\overline{1}} \wedge \cdots \wedge \alpha_{\overline{n}} = \dfrac{dx}{2}.$

Chapter 3

Average of the mean curvature integral

For the real space forms ($\mathbb{R}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$), it is known that the *reproductive property* holds for mean curvature integrals. That is, given a regular domain Ω, it is satisfied (cf. Example 2.2.3)

$$\int_{\mathcal{L}_s} M_r^{(s)}(\partial\Omega \cap L_s)dL_s = cM_r(\partial\Omega).$$

On the other hand, by Section 2.3, this property may not hold in $\mathbb{C}^n$, when we integrate over the space of complex planes. Thus, it is natural to study, in $\mathbb{C}^n$, the value of

$$\int_{\mathcal{L}_s^{\mathbb{C}}} M_r^{(s)}(\partial\Omega \cap L_s)dL_s.$$

In the same way, we will study the value of this integral but in the other complex space forms, $\mathbb{CP}^n$ and $\mathbb{CH}^n$. Recall that we denote by $\mathbb{CK}^n(\epsilon)$ the space of constant holomorphic curvature 4ϵ.

In this chapter we deduce the expression of the integral of $M_1^{(s)}(\partial\Omega \cap L_s)$ in terms of the mean curvature integral of the convex domain $M_1(\partial\Omega)$ and the integral of the normal curvature in the direction JN, $\int_{\partial\Omega} k_n(JN)$ (see Theorem 3.3.2). We also find a partial result for the integral over any other mean curvature integral $M_r^{(s)}(\partial\Omega \cap L_s)$, $0 \leq r \leq 2s-1$ (see Proposition 3.2.2).

3.1 Previous lemmas

First of all, we state some lemmas that will be necessary in order to prove Theorem 3.3.2 and some other results.

Lemma 3.1.1. *Let V be a complex vector space of complex dimension 2 endowed with an inner product $\langle\, ,\rangle$ compatible with the complex structure J and let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis of V. Then $\langle e_a, Je_b\rangle^2 = \langle e_c, Je_d\rangle^2$ with $\{a, b, c, d\} = \{1, 2, 3, 4\}$.*

Proof. We express Je_b and Je_d in terms of the orthonormal, and we obtain

$$\begin{aligned} Je_b &= \langle Je_b, e_a\rangle e_a + \langle Je_b, e_c\rangle e_c + \langle Je_b, e_d\rangle e_d = Ae_a + Be_c + Ce_d, \\ Je_d &= \langle Je_d, e_a\rangle e_a + \langle Je_d, e_b\rangle e_b + \langle Je_d, e_c\rangle e_c = De_a + Ee_b + Fe_c. \end{aligned}$$

Now, using that $\langle Je_b, Je_b\rangle = \langle Je_d, Je_d\rangle = 1$, $\langle Je_b, Je_d\rangle = 0$ and $\langle Je_b, e_d\rangle = -\langle Je_d, e_b\rangle$, we get

$$\begin{aligned} A^2+B^2+C^2 &= 1,\\ D^2+E^2+F^2 &= 1,\\ AD+BF &= 0,\\ C &= -E, \end{aligned}$$

and

$$A^2+B^2 = D^2+F^2 \quad \text{and} \quad A^2D^2 = B^2F^2.$$

Finally, we substitute $D^2 = A^2+B^2-F^2$ in the second equation and we obtain $A^2 = F^2$. □

Lemma 3.1.2. *If $u \in S^{2n-3}$, then*

$$\int_{S^{2n-3}} \langle u, v\rangle dv = 0 \quad \textit{and} \quad \int_{S^{2n-3}} \langle u, v\rangle^2 dv = \omega_{2n-2}.$$

Proof. The first equality follows since the integral is over an odd function. For the second one, we decompose $v = \cos\theta u + \sin\theta w$ with $w \in \langle u\rangle^\perp$, then using polar coordinates with respect to u, we have

$$\int_{S^{2n-3}} \langle u, v\rangle^2 dv = O_{2n-4}\int_0^\pi \cos^2\theta \sin^{2n-4}\theta d\theta = O_{2n-4}\frac{O_{2n-4+2+1}}{O_2 O_{2n-4}} = \omega_{2n-2}.$$

where O_n denotes the volume of the n-dimensional Euclidean sphere and ω_n the volume of the the n-dimensional Euclidean ball. □

The following lemma gives a generalized version of the Meusiner Theorem.

Lemma 3.1.3. *Let $S \subset M$ be a hypersurface of class $\mathcal{C}^2$ of a Riemannian manifold M, $p \in S$ and $L \subset T_pM$ a vector subspace. We denote by II_S the second fundamental form of S and by II_C^L the second fundamental form of $C = S \cap \exp_p L$ as a hypersurface of $\exp_p L$. We also denote $u = T_pS \cap L$. Then,*

$$\sigma_i(II_S|_u) = \cos^i\theta \sigma_i(II|_C^L)$$

where θ denotes the angle at p between a normal vector of S and a normal vector of C in $\exp_p L$, and $\sigma_i(Q)$ denotes the i-th symmetric elementary function of the bilineal form Q.

Proof. If $A \subset B \subset M$ are submanifolds, then we denote the second fundamental form of A as a submanifold of B by $h_A^B : T_pA \times T_pA \to (T_pA)^\perp$. If $B = M$, we just put h_A instead of h_A^M.

Then, for all $X, Y \in T_pC$

$$h_C(X,Y) = h_C^L(X,Y) + h_L(X,Y) = h_C^L(X,Y)$$

since the second fundamental form of L vanishes at p. Moreover,

$$h_C(X,Y) = h_C^S(X,Y) + h_S(X,Y).$$

Let N be a normal vector to S. Note that $h_C^S(X,Y)$ is a multiple of a normal vector to C in S, so $\langle h_C^S(X,Y), N\rangle = 0$ (for $X, Y \in T_pC$).

If $X, Y \in T_pC$, then

$$\begin{aligned} \mathrm{II}_S(X,Y) &:= \langle h_S(X,Y), N\rangle = \langle h_C(X,Y) - h_C^S(X,Y), N\rangle\\ &= \langle h_C(X,Y), N\rangle = \langle h_C^L(X,Y), N\rangle = \langle \mathrm{II}_C^L(X,Y)n, N\rangle \end{aligned}$$

where n denotes a normal vector of C in L. So,

$$\mathrm{II}_C^L(X,Y) = \frac{1}{\langle N, n\rangle}\mathrm{II}_S(X,Y). \tag{3.1}$$

Since $\sigma_i(\mathrm{II}_C^L)$ is the sum of the minors of order i of II_C^L, by replacing by (3.1) each entry of the second fundamental form, we obtain the result. □

The following lemma generalizes in $\mathbb{C}^n$ a result given by Langevin and Shifrin [LS82] in $\mathbb{R}^n$.

Lemma 3.1.4. *Let E be a complex vector space of complex dimension n and let II be a real bilinear form defined on E. We denote by $G_{n,s}^{\mathbb{C}}$ the Grassmanian of s-dimensional complex planes on E. Then,*

$$\int_{G_{n,s}^{\mathbb{C}}} \mathrm{tr}(II|_V)dV = \frac{s\,\mathrm{vol}(G_{n,s}^{\mathbb{C}})}{n}\mathrm{tr}(II|_E).$$

Proof. First, recall that

$$U(n-s)\times U(s) \longrightarrow U(n) \longrightarrow G_{n,s}^{\mathbb{C}} \tag{3.2}$$

is a fibration for each $s\in\{1,\dots,n-1\}$.

We prove the case $\dim_{\mathbb{C}} V \leq \frac{n}{2}$ by induction on the complex dimension of V. The case $\dim_{\mathbb{C}} V > \frac{n}{2}$ can be proved using similar arguments.

Suppose $\dim_{\mathbb{C}} V = 1$, that is, $s=1$. Then,

$$\int_{G_{n,1}^{\mathbb{C}}} \mathrm{tr}(\mathrm{II}|_V)dV = \frac{1}{\mathrm{vol}(U(n-1))\mathrm{vol}(U(1))}\int_{U(n)} \mathrm{tr}(\mathrm{II}|_{V_1^1})dU$$

since $\mathrm{tr}(\mathrm{II}|_{V_1^1})$ is constant along the fiber. We denote by V_1^1 the complex vector subspace generated by the first column of the matrix $U\in U(n)$. In general, for $U\in U(n)$, we will denote by V_a^b the complex vector subspace generated by the columns b to $b+a-1$. The subscript a denotes the dimension of V_a^b, or equivalently, the number of columns we consider and the upperscript b denotes from which column we start to consider them. Then

$$\begin{aligned}\int_{U(n)} \mathrm{tr}(\mathrm{II}|_{V_1^1})dU &= \frac{1}{n}\int_{U(n)} (\mathrm{tr}(\mathrm{II}|_{V_1^1}) + \mathrm{tr}(\mathrm{II}|_{V_1^2}) + \cdots + \mathrm{tr}(\mathrm{II}|_{V_1^n}))dU\\ &= \frac{1}{n}\int_{U(n)} \mathrm{tr}(\mathrm{II}|_E)dU = \frac{\mathrm{vol}(U(n))}{n}\mathrm{tr}(\mathrm{II}|_E).\end{aligned}$$

Thus,

$$\int_{G_{n,1}^{\mathbb{C}}} \mathrm{tr}(\mathrm{II}|_V)dV = \frac{\mathrm{vol}(U(n))}{n\mathrm{vol}(U(n-1))\mathrm{vol}(U(1))}\mathrm{tr}(\mathrm{II}|_E) = \frac{\mathrm{vol}(G_{n,1}^{\mathbb{C}})}{n}\mathrm{tr}(\mathrm{II}|_E).$$

Suppose now that the result is true till $\dim_{\mathbb{C}} V = r-1$. We shall prove it for $\dim_{\mathbb{C}} V = r \leq \frac{n}{2}$. If R denotes the remainder of $\frac{n}{r}$, then $R<r$ and we can apply the induction hypothesis in

R at equality (*). Thus, using similar arguments as before, we obtain

$$\begin{aligned}
\int_{G^{\mathbb{C}}_{n,r}} \operatorname{tr}(\mathrm{II}|_V) &= \frac{1}{\operatorname{vol}(U(n-r))\operatorname{vol}(U(r))} \int_{U(n)} \operatorname{tr}(\mathrm{II}|_{V_r^1}) \\
&= \frac{1}{\operatorname{vol}(U(n-r))\operatorname{vol}(U(r))\lfloor \frac{n}{r} \rfloor} \int_{U(n)} (\operatorname{tr}(\mathrm{II}|_{V_r^1}) + \operatorname{tr}(\mathrm{II}|_{V_r^2}) + \cdots + \operatorname{tr}(\mathrm{II}|_{V_r^{n-R-r+1}})) \\
&= \frac{1}{\operatorname{vol}(U(n-r))\operatorname{vol}(U(r))\lfloor \frac{n}{r} \rfloor} \left(\int_{U(n)} \operatorname{tr}(\mathrm{II}|_E) - \int_{U(n)} \operatorname{tr}(\mathrm{II}|_{V_R^{n-R+1}}) \right) \\
&= \frac{\operatorname{vol}(U(n))\operatorname{tr}(\mathrm{II}|_E) - \operatorname{vol}(U(n-R))\operatorname{vol}(U(R)) \int_{G^{\mathbb{C}}_{n,R}} \operatorname{tr}(\mathrm{II}|_{V_R})}{\operatorname{vol}(U(n-r))\operatorname{vol}(U(r))\lfloor \frac{n}{r} \rfloor} \\
&\overset{(*)}{=} \frac{\left(\operatorname{vol}(U(n)) - \operatorname{vol}(U(n-R))\operatorname{vol}(U(R))\operatorname{vol}(G^{\mathbb{C}}_{n,R})\frac{R}{n}\right) \operatorname{tr}(\mathrm{II}|_E)}{\operatorname{vol}(U(n-r))\operatorname{vol}(U(r))\lfloor \frac{n}{r} \rfloor} \\
&= \operatorname{vol}(G^{\mathbb{C}}_{n,r}) \frac{r}{n-R} \left(1 - \frac{R}{n}\right) \operatorname{tr}(\mathrm{II}|_E) \\
&= \operatorname{vol}(G^{\mathbb{C}}_{n,r}) \frac{r}{n} \operatorname{tr}(\mathrm{II}|_E)
\end{aligned}$$

and the result follows when $2s \leq n$. □

3.2 Integral of the r-th mean curvature integral over the space of complex s-planes

Along this chapter we follow some conventions which we state in the following paragraphs.

We denote by $S \subset \mathbb{CK}^n(\epsilon)$ a hypersurface of class $\mathcal{C}^2$, compact and oriented (possibly with boundary). Given a complex s-plane L_s intersecting S, it is said that L_s is in *generic position* if $S \cap L_s$ is a submanifold of dimension $2s-1$ in $\mathbb{CK}^n(\epsilon)$. For hypersurfaces of class $\mathcal{C}^2$, the subset of generic planes (intersecting S) has full measure. Thus, we suppose that each complex s-plane is in generic position. Note that $S \cap L_s$ (if L_s is a complex s-plane in generic position) is a hypersurface in $L_s \cong \mathbb{CK}^s(\epsilon)$.

Suppose that N denotes a unit normal vector field on S. In this chapter we take, in $S \cap L_s$ as a submanifold in S, the normal vector field $\tilde{N}$ such that the angle between N and $\tilde{N}$ is acute. Along the proofs in this chapter, we denote $e_s := \pm J\tilde{N}$.

Note that if $p \in S \cap L_s$ and $\tilde{N}_p$ is the chosen normal vector field in $S \cap L_s$ inside L_s then $J\tilde{N} \in T_p(S \cap L_s)$. Indeed, $L_s \cong \mathbb{CK}^s(\epsilon)$, thus, the same structure for hypersurfaces hols inside L_s.

We denote by $E \subset T_p\mathbb{CK}^n(\epsilon)$, $p \in S \cap L_s$, the orthogonal subspace to the space generated by $\{N, JN, \tilde{N}, J\tilde{N}\}$. Note that $\exp_p(E) \cong \mathbb{CK}^{n-2}(\epsilon)$ and it is univocally determined for each L_s.

The fact stated in the following remark is used implicitly along this chapter, specially to define the moving frames g and g' in the proof of the next proposition.

Remark 3.2.1. Let V^n be an n-dimensional Hermitian space with complex structure J, $H \subset V^n$ a real hyperplane and $W_s \subset V^n$ a complex subspace of dimension s.

Consider the subspace $H \cap W_s$ and denote by N' an orthonormal vector to $H \cap W_s$ in W_s, and $\mathcal{D}' = \langle N', JN' \rangle^\perp \cap W_s$.

Consider also the subspaces $H \cap W_s^\perp$ and denote by N'' an orthonormal vector to $H \cap W_s^\perp$ in $W_s^\perp$, and $\mathcal{D}'' = \langle N'', JN'' \rangle^\perp \cap W_s''$.

Denote by N an orthonormal vector to H in V, and $\mathcal{D} = \langle N, JN\rangle^{\perp}$. Then,

$$\begin{aligned} V^n &= \mathcal{D}\perp\langle JN\rangle_{\mathbb{R}}\perp\langle N\rangle_{\mathbb{R}} \\ &= \mathcal{D}'\perp\langle JN'\rangle_{\mathbb{R}}\perp\langle N'\rangle_{\mathbb{R}}\perp\mathcal{D}''\perp\langle JN''\rangle_{\mathbb{R}}\perp\langle N''\rangle_{\mathbb{R}}. \end{aligned}$$

Indeed, from section 2.4.4 given a real hyperplane W in a Hermitian space V there exists a canonical decomposition of V as

$$V = \mathcal{D}\perp\langle JN\rangle_{\mathbb{R}}\perp\langle N\rangle_{\mathbb{R}},$$

where N is an orthogonal vector to W in V. Applying this fact to $V = W_s$ and to $V = W_s^{\perp}$ we get the result.

The following proposition shall be essential to prove Theorem 3.3.2 and other results, since it gives a first expression of the integral over the space of complex s-planes of the mean curvature integral in terms of an integral on the boundary of the domain.

Proposition 3.2.2. *Let $S \subset \mathbb{CK}^n(\epsilon)$ be a compact (possibly with boundary) hypersurface of class $\mathcal{C}^2$ oriented by a normal vector N, and let $r, s \in \mathbb{N}$ such that $1 \leq s \leq n$ and $0 \leq r \leq 2s-1$. Then*

$$\int_{\mathcal{L}_s^{\mathbb{C}}} M_r^{(s)}(S\cap L_s)dL_s = \binom{2s-1}{r}^{-1}\int_S\left(\int_{\mathbb{RP}^{2n-2}}\left(\int_{G^{\mathbb{C}}_{n-2,s-1}}\frac{|\langle JN, e_s\rangle|^{2s-r}}{(1-\langle JN, e_s\rangle^2)^{s-1}}\sigma_r(p; e_s\oplus V)dV\right)de_s\right)dp,$$

where $e_s \in T_pS$ unit vector, V denotes a complex $(s-1)$-plane by p contained in $\{N, JN, e_s, Je_s\}^{\perp}$, $\sigma_r(p; e_s \oplus V)$ denotes the r-th symmetric elementary function of the second fundamental form of S restricted to the real subspace $e_s\oplus V$ and the integration over $\mathbb{RP}^{2n-2}$ denotes the projective space of the unit tangent space of the hypersurface.

Remark 3.2.3. Using the previous remarks, it follows that the product $|\langle JN, e_s\rangle|$ in the last proposition gives the cosine of the acute angle between the normal vector to the hypersurface S in $\mathbb{CK}^n(\epsilon)$ and a normal vector to $S \cap L_s$ in L_s, that is,

$$|\langle JN, e_s\rangle| = |\langle N, \tilde{N}\rangle|.$$

Proof. Let L_s be a complex s-plane such that $S \cap L_s \neq \emptyset$ and let $p \in S \cap L_s$. We denote by $\tilde{\sigma}_r$ the r-th symmetric elementary function of the second fundamental form of $S \cap L_s$ as a hypersurface of L_s. Then, by definition

$$\int_{\mathcal{L}_s^{\mathbb{C}}} M_r^{(s)}(S\cap L_s)dL_s = \binom{2s-1}{r}^{-1}\int_{S\cap L_s\neq\emptyset}\int_{S\cap L_s}\tilde{\sigma}_r(s)dx\, dL_s.$$

We shall prove the result using moving frames adapted to $S \cap L_s$, L_s or S.

Let $g = \{e_1, e_{\overline{1}} = Je_1, e_2, e_{\overline{2}} = Je_2, \ldots, e_s, w_s, e_{s+1}, e_{\overline{s+1}} = Je_{s+1}\ldots, e_n, N\}$ be a moving frame adapted to $S \cap L_s$ and S (cf. Remark 3.2.1). That is, $\{e_1, e_{\overline{1}}, \ldots, e_s\}$ is an orthonormal basis of $T_p(S\cap L_s)$, $\{e_{s+1}, e_{\overline{s+1}}, \ldots, e_n\}$ is an orthonormal basis of $T_pS\cap(T_pL_s)^{\perp}$, N is a normal vector field to TS and w_s completes to an orthonormal basis of $T_p\mathbb{CK}^n(\epsilon)$. We denote by

$$\{\omega_1, \omega_{\overline{1}}, \ldots, \omega_{s-1}, \omega_{\overline{s-1}}, \omega_s, \omega_{\tilde{s}}, \omega_{s+1}, \omega_{\overline{s+1}}, \ldots, \omega_n, \omega_{\tilde{n}}\}$$

the dual basis of the vectors in g and by $\{\omega_{ij}\}$, $i, j \in \{1, \overline{1}, \ldots, s, \tilde{s}, s+1, \overline{s+1}, \ldots, n, \tilde{n}\}$, the connection forms (cf. (1.15)).

Let $g' = \{e'_1 = e_1, e'_{\overline{1}} = Je_1, e'_2 = e_2, e'_{\overline{2}} = Je_2, ..., e'_s = e_s, e'_{\overline{s}} = Je_s, e'_{s+1} = e_{s+1}, e'_{\overline{s+1}} = Je_{s+1}, ..., e'_n = e_n, e'_{\overline{n}} = Je_n\}$ be a moving frame adapted to $S \cap L_s$ and L_s. That is,

$\{e'_1, e'_{\overline{1}}, ..., e'_s\}$ is an orthonormal basis of $T_p(L_s \cap S)$ and $\{e'_1, e'_{\overline{1}}, \ldots, e'_s, e'_{\overline{s}}\}$ is an orthonormal basis of T_pL_s. Denote by

$$\{\omega'_1, \omega'_{\overline{1}}, \ldots, \omega'_n, \omega'_{\overline{n}}\}$$

the dual basis of vectors in g' and by $\{\omega'_{ij}\}$ the connection forms, $i, j \in \{1, \overline{1}, \ldots, n, \overline{n}\}$.

As the base g and g' are constituted by orthonormal vectors, we can easily give the relation among the elements in the frame g' and the ones in g just expressing the vectors in g' in coordinates with respect to the vectors in g:

$$\begin{aligned} e'_{\overline{s}} &= Je_s = \langle Je_s, w_s\rangle w_s + \langle Je_s, N\rangle N, \\ e'_{\overline{n}} &= Je_n = \langle Je_n, w_s\rangle w_s + \langle Je_n, N\rangle N \end{aligned}$$

and $e'_j = e_j$ when $j \in \{1, \overline{1}, \ldots, s-1, \overline{s-1}, s, s+1, \overline{s+1}, \ldots, n-1, \overline{n-1}, n\}$.

Then

$$\begin{cases} \omega'_j &= \omega_j, \text{ if } j \neq \overline{s}, \overline{n}, \\ \omega'_{\overline{n}} &= \langle Je_n, w_s\rangle \omega_{\tilde{s}} + \langle Je_n, N\rangle \omega_{\tilde{n}} \end{cases} \tag{3.3}$$

and

$$\begin{cases} \omega'_{i\overline{s}} &= \langle Je_s, w_s\rangle \omega_{i\tilde{s}} + \langle Je_s, N\rangle \omega_{i\tilde{n}}, \quad \text{if } i \neq \overline{s}, \overline{n} \\ \omega'_{i\overline{n}} &= \langle Je_n, w_s\rangle \omega_{i\tilde{s}} + \langle Je_n, N\rangle \omega_{i\tilde{n}}, \quad \text{if } i \neq \overline{s}, \overline{n} \\ \omega'_{ij} &= \omega_{ij}, \quad \text{if } i, j \neq \overline{s}, \overline{n}. \end{cases} \tag{3.4}$$

From now on, in order to simplify the notation, we omit the absolute value in densities. The expression of dx (the density of $S \cap L_s$), dL_s and $dL_{s[p]}$ in terms of ω' is

$$\begin{aligned} dx &= \omega'_1 \wedge \omega'_{\overline{1}} \wedge \cdots \wedge \omega'_s, \\ dL_s &= \omega'_{s+1} \wedge \omega'_{\overline{s+1}} \wedge \cdots \wedge \omega'_n \wedge \omega'_{\overline{n}} \wedge \bigwedge_{\substack{i=1,2,\ldots,s \\ j=s+1,\overline{s+1},\ldots,n,\overline{n}}} \omega'_{ij}, \\ dL_{s[p]} &= \bigwedge_{\substack{i=1,2,\ldots,s \\ j=s+1,\overline{s+1},\ldots,n,\overline{n}}} \omega'_{ij} \end{aligned}$$

and the expression of dp (the density of S) in terms of ω is $dp = \omega_1 \wedge \omega_{\overline{1}} \wedge \cdots \wedge \omega_n$.

On the other hand, by Lemma 3.1.1 it is satisfied

$$|\langle Je_n, w_s\rangle| = |\langle Je_s, N\rangle|. \tag{3.5}$$

Indeed, vectors $\{e_s, w_s, e_n, N\}$ are an orthonormal basis of a 2-dimensional complex plane, the orthogonal complement of the space generated by $\{e_1, Je_1, \ldots, e_{s-1}, Je_{s-1}, e_{s+1}, Je_{s+1}, \ldots, e_{n-1}, Je_{n-1}\}$.

By relations (3.3) and (3.5) we get

$$dx \wedge dL_s = |\langle Je_n, w_s\rangle| dL_{s[p]} \wedge dp = |\langle JN, e_s\rangle| dL_{s[p]} \wedge dp \tag{3.6}$$

since $\omega_{\tilde{n}}$ vanishes on TS.

Then, by Lemma 3.1.3,

$$\begin{aligned} \int_{S\cap L_s \neq \emptyset} M_r^{(s)}(S \cap L_s) dL_s &= \binom{2s-1}{r}^{-1} \int_S \int_{L_{s[p]}} |\langle JN, e_s\rangle| \tilde{\sigma}_r(p) dL_{s[p]} dp \\ &= \binom{2s-1}{r}^{-1} \int_S \int_{L_{s[p]}} \frac{|\langle JN, e_s\rangle|}{|\langle N, Je_s\rangle|^r} \sigma_r(p) dL_{s[p]} dp. \end{aligned}$$

Note that in the last integrand we consider the absolut value in the denominator to be sure that we consider the acute angle between the two intersecting subspace. This is not a priori true since e_s is any vector and, thus, not always the vector Je_s (a normal vector to $S \cap L_s \subset L_s$) has the desired orientation.

Now, we shall express $dL_{s[p]}$ in terms of $dV \wedge de_s$ where dV denotes the volume element of the Grassmannian $G^{\mathbb{C}}_{n-2,s-1}$ and de_s the volume element of S^{2n-2}. Fixed p, define the manifold

$$\mathbb{M} = \{(e_s, V) \,|\, e_s \in T_pS \text{ unit }, V \in \mathcal{L}_s \text{ containing } p \text{ and orthogonal to } \{N, JN, e_s, Je_s\}\},$$

which locally coincides with $S^{2n-2} \times G^{\mathbb{C}}_{n,s-1}$. Consider the fiber bundle

$$\begin{array}{rccl} \phi: & \mathbb{M} & \longrightarrow & \mathcal{L}^{\mathbb{C}}_{s[p]} \\ & (e_s, V) & \mapsto & \exp_p\{e_s, Je_s, V\} \end{array}$$

with fiber $G^{\mathbb{C}}_{n,s}$. This is a double covering of $\mathcal{L}^{\mathbb{C}}_{s[p]}$ since vectors v and $-v$ give the same complex s-plane.

The pull-back of $dL_{s[p]}$ by the last mapping give the desired relation among the densities.

The expression of de_s in terms of ω and the expression of dV in terms of ω' are

$$\begin{aligned} de_s &= \bigwedge_{j=1,\bar{1},\dots,s-1,\overline{s-1},\tilde{s},s+1,\overline{s+1},\dots,n} \omega_{sj}, \\ dV &= \bigwedge_{\substack{i=1,2,\dots,s-1 \\ j=s+1,\overline{s+1},\dots,n-1,\overline{n-1}}} \omega'_{ij}. \end{aligned} \tag{3.7}$$

By (3.4) and (3.7) we have

$$\begin{aligned} dL_{s[p]} &= \bigwedge_{\substack{i=1,2,\dots,s \\ j=s+1,\overline{s+1},\dots,n,\overline{n}}} \omega'_{ij} \\ &= dV \wedge \bigwedge_{i=1,2,\dots,s-1} \omega_{in} \wedge \bigwedge_{i=1,2,\dots,s} \omega'_{i\overline{n}} \wedge \bigwedge_{j=s+1,\overline{s+1},\dots,\overline{n-1},n} \omega_{sj}. \end{aligned}$$

Next, we relate $\bigwedge \omega'_{in} \bigwedge \omega'_{i\overline{n}}$ with $\bigwedge \omega_{sj}$. From (3.4) follows

$$\bigwedge_{i=1,2,\dots,s} \omega'_{i\overline{n}} = |\langle Je_n, w_s\rangle|^s \bigwedge_{i=1,2,\dots,s} \omega_{i\tilde{s}}$$

and also using that $\omega'_{in} = \omega'_{\bar{i}\overline{n}}$ since $\omega'_{in} = \langle de'_i, e'_n\rangle = \langle dJe'_i, Je'_n\rangle = \omega'_{\bar{i},\overline{n}}$ we obtain

$$\begin{aligned} \bigwedge_{i=1,2,\dots,s-1} \omega'_{in} = \bigwedge_{i=1,2,\dots,s-1} \omega'_{\bar{i}\overline{n}} &= \bigwedge_{i=1,2,\dots,s-1} \langle Je_n, w_s\rangle \omega_{\bar{i}\tilde{s}} \\ &= |\langle Je_n, w_s\rangle|^{s-1} \bigwedge_{i=1,2,\dots,s-1} \omega_{\bar{i}\tilde{s}}. \end{aligned}$$

In order to study

$$\bigwedge_{i=1,\bar{1},\dots,s-1,\overline{s-1}} \omega_{i\tilde{s}},$$

we use $e_{\overline{s}} = Je_s = \langle Je_s, w_s\rangle w_s + \langle Je_s, N\rangle N$ and we obtain

$$\begin{aligned} \bigwedge_{i=1,\bar{1},\dots,s-1,\overline{s-1}} \omega_{is} = \bigwedge_{i=\bar{1},1,\dots,\overline{s-1},s-1} \omega_{\bar{i}\overline{s}} &= \bigwedge_{i=1,\bar{1},\dots,s-1,\overline{s-1}} \omega_{i\overline{s}} \\ &= \langle Je_s, w_s\rangle^{2(s-1)} \bigwedge_{i=1,\bar{1},\dots,s-1,\overline{s-1}} \omega_{i\tilde{s}}. \end{aligned}$$

Thus,

$$\bigwedge_{i=1,\bar{1},\ldots,s-1,\overline{s-1}} \omega_{i\tilde{s}} = \langle Je_s, w_s\rangle^{-2(s-1)} \bigwedge_{i=1,\bar{1},\ldots,s-1,\overline{s-1}} \omega_{is}$$

and

$$dL_{s[p]} = \frac{|\langle Je_n, w_s\rangle|^{2s-1}}{\langle Je_s, w_s\rangle^{2(s-1)}} dV \wedge de_s. \tag{3.8}$$

Using $|\langle Je_n, w_s\rangle| = |\langle JN, e_s\rangle|$ and $\langle Je_s, w_s\rangle^2 = 1 - \langle JN, e_s\rangle^2$ we get the result. □

3.3 Mean curvature integral

First we give an expression for the integral over the space of complex r-planes of the integral of the normal curvature in the direction JN (see (2.9)). By Example 2.4.20 we have that this integral is a smooth valuation in $\mathbb{CK}^n(\epsilon)$. Moreover, it is not a multiple of the mean curvature integral.

Theorem 3.3.1. *Let $S \subset \mathbb{CK}^n(\epsilon)$ be compact oriented (possibly with boundary) hypersurface of class $\mathcal{C}^2$ oriented by a normal vector N, and $s \in \{1, \ldots, n-1\}$. Then*

$$\int_{\mathcal{L}_s^{\mathbb{C}}} \left(\int_{S\cap L_s} \tilde{k}_n(J\tilde{N})dx \right) dL_s$$
$$= \mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1}) \frac{\omega_{2n-2}}{2s} \binom{n}{s}^{-1} \left(\frac{2sn-s-n}{n-s} \int_S k_n(JN) + (2n-1)M_1(S) \right),$$

where $\tilde{k}_n(J\tilde{N})$ the normal curvature of $S \cap L_s$ in the direction $J\tilde{N}$, $k_n(JN)$ the normal curvature of JN in $\mathbb{CK}^n(\epsilon)$ and ω_{2n-2} denotes the volume of the unit ball in the standard Euclidean space of dimension $2n-2$.

Proof. Denote $J\tilde{N}$ by e_s.

By Lemma 3.1.3 we have $\tilde{k}_n(J\tilde{N}) = \dfrac{k_n(e_s)}{\langle JN, e_s\rangle}$. Using equalities (3.6) and (3.8) we obtain

$$I = \int_{\mathcal{L}_s} \int_{S\cap L_s} \tilde{k}_n(J\tilde{N}) dx dL_s$$
$$= \int_S \int_{\mathbb{RP}^{2n-2}} \int_{G^{\mathbb{C}}_{n-2,s-1}} \frac{\langle JN, e_s\rangle^{2s}}{(1-\langle JN, e_s\rangle^2)^{s-1}} \frac{k_n(e_s)}{\langle JN, e_s\rangle} dV de_s dp$$

As the integral over $G^{\mathbb{C}}_{n-2,s-1}$ is independent of V, it follows

$$I = \mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1}) \int_S \int_{\mathbb{RP}^{2n-2}} \frac{\langle JN, e_s\rangle^{2s-1}}{(1-\langle JN, e_s\rangle^2)^{s-1}} k_n(e_s) de_s dp. \tag{3.9}$$

In order to compute the integral over $\mathbb{RP}^{2n-2}$, we use polar coordinates and express the normal curvature of e_s in terms of the principal curvatures of T_pS.

That is, if $\{f_1, \ldots, f_{2n-1}\}$ is an orthonormal basis of principal directions of T_pS then $e_s = \sum_{j=1}^{2n-1} \langle e_s, f_j\rangle f_j$, and

$$k_n(e_s) = \sum_{j=1}^{2n-1} \langle e_s, f_j\rangle^2 k_n(f_j) = \sum_{j=1}^{2n-1} \langle e_s, f_j\rangle^2 k_j.$$

On the other hand, we consider a polar coordinates system θ_1, θ_2 with respect to JN defined by

$$|\langle JN, e_s\rangle| = \cos\theta_1, \tag{3.10}$$

and using spherical trigonometry,

$$\begin{aligned}\langle e_s, f_j\rangle &= \cos\theta_1\cos(JN, f_j) + \sin\theta_1\sin(JN, f_j)\cos(e_s, JN, f_j)\\ &= \cos\theta_1\cos\alpha_j + \sin\theta_1\sin\alpha_j\cos\theta_2 \end{aligned} \tag{3.11}$$

where $\cos(e_s, JN, f_j)$ denotes the cosine of the spherical angle with vertex JN. Note that α_j are constants when the point is fixed.

Then, from the relations

$$\frac{\Gamma(s)\Gamma(n-s)}{\Gamma(n+1)} = \frac{1}{s(n-s)}\binom{n}{s}^{-1} \quad \text{and} \quad \frac{\Gamma(s)\Gamma(n-s+1)}{\Gamma(n+1)} = \frac{1}{s}\binom{n}{s}^{-1},$$

we get

$$\begin{aligned}&\int_{\mathbb{RP}^{2n-2}} \frac{|\langle JN, e_s\rangle|^{2s-1}}{(1-\langle JN, e_s\rangle^2)^{s-1}} k_n(e_s) de_s\\ &= \sum_{j=1}^{2n-1} k_j\Big(\int_{S^{2n-3}}\int_0^{\pi/2} \frac{\cos^{2s-1}\theta_1}{\sin^{2s-2}\theta_1}\cos^2\theta_1\cos^2\alpha_j\sin^{2n-3}\theta_1 d\theta_1 dS_{2n-3}+\\ &+\int_{S^{2n-4}}\int_0^{\pi}\cos^2\theta_2\sin^{2n-4}\theta_2\int_0^{\pi/2}\frac{\cos^{2s-1}\theta_1}{\sin^{2s-2}\theta_1}\sin^2\theta_1\sin^2\alpha_j\sin^{2n-3}\theta_1 d\theta_1 d\theta_2 dS_{2n-4} + 0\Big)\\ &= \sum_{j=1}^{2n-1} k_j O_{2n-3}\left(\cos^2\alpha_j\frac{(2sn-s-n)\Gamma(s)\Gamma(n-s)}{4(n-1)\Gamma(n+1)} + \frac{\Gamma(s)\Gamma(n-s+1)}{4(n-1)\Gamma(n+1)}\right)\\ &= \frac{\omega_{2n-2}}{2s}\binom{n}{s}^{-1}\sum_{j=1}^{2n-1} k_j\left(\frac{2sn-n-s}{n-s}\cos^2\alpha_j + 1\right).\end{aligned}$$

Integrating over S and using

$$k_n(JN) = \sum_{j=1}^{2n-1} k_j\langle JN, f_j\rangle^2 = \sum_{j=1}^{2n-1} k_j\cos^2\alpha_j$$

we obtain the stated result. $\square$

Theorem 3.3.2. *Let $S \subset \mathbb{CK}^n(\epsilon)$ be a compact (possibly with boundary) hypersurface of class $\mathcal{C}^2$ oriented by a normal vector N, and let $s \in \{1, ..., n-1\}$. Then*

$$\int_{\mathcal{L}_s^{\mathbb{C}}} M_1^{(s)}(S\cap L_s)dL_s = \frac{\omega_{2n-2}\mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1})}{2s(2s-1)}\binom{n}{s}^{-1}\left((2n-1)\frac{2ns-n-s}{n-s}M_1(S) + \int_S k_n(JN)\right)$$

where $k_n(JN)$ denotes the normal curvature in the direction $JN \in TS$.

Proof. By Proposition 3.2.2 and Lemma 3.1.4 we have

$$\begin{aligned}&\int_{\mathcal{L}_s^{\mathbb{C}}} M_1^{(s)}(S\cap L_s)dL_s = \frac{1}{2s-1}\int_S\int_{\mathbb{RP}^{2n-2}}\int_{G^{\mathbb{C}}_{n-2,s-1}} \frac{|\langle JN, e_s\rangle|^{2s-1}}{(1-\langle JN, e_s\rangle^2)^{s-1}}\sigma_1(p; e_s, V)dV de_s dp\\ &= \frac{1}{2s-1}\int_S\int_{\mathbb{RP}^{2n-2}} \frac{|\langle JN, e_s\rangle|^{2s-1}}{(1-\langle JN, e_s\rangle^2)^{s-1}}\int_{G^{\mathbb{C}}_{n-2,s-1}}(\mathrm{tr}(\mathrm{II}|_V) + \mathrm{II}(e_s, e_s))dV de_s dp\\ &= \frac{\mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1})}{2s-1}\int_S\int_{\mathbb{RP}^{2n-2}} \frac{|\langle JN, e_s\rangle|^{2s-1}}{(1-\langle JN, e_s\rangle^2)^{s-1}}\left(\frac{s-1}{n-2}\mathrm{tr}(\mathrm{II}|_E) + k_n(e_s)\right)de_s dp,\end{aligned}$$

where $E = \langle N, JN, e_s, Je_s\rangle^{\perp}$.

Note that if $s=1$ then $\dim V=0$. Although the integral $\int_{G^{\mathbb{C}}_{n-2,s-1}} \mathrm{tr}(\mathrm{II}|V)dV$ has no sense, last equality above remains true since $\frac{s-1}{n-2}\mathrm{tr}\mathrm{II}|_E=0$.

We shall study the following integrals

$$J_E=\frac{\mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1})}{2s-1}\frac{s-1}{n-2}\int_S\int_{\mathbb{RP}^{2n-2}}\frac{|\langle JN,e_s\rangle|^{2s-1}}{(1-\langle JN,e_s\rangle^2)^{s-1}}\mathrm{tr}(\mathrm{II}|_E)de_sdp$$

$$J_s=\frac{\mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1})}{2s-1}\int_S\int_{\mathbb{RP}^{2n-2}}\frac{|\langle JN,e_s\rangle|^{2s-1}}{(1-\langle JN,e_s\rangle^2)^{s-1}}k_n(e_s)de_sdp.$$

The second integral is the same (except for a constant factor) as the integral (3.9) in Proposition 3.3.1. Thus, we know

$$J_s=\frac{O_{2n-3}\mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1})}{4s(2s-1)(n-1)}\binom{n}{s}^{-1}\left(\frac{2sn-s-n}{n-s}\int_S k_n(JN)+(2n-1)M_1(S)\right).$$

In order to study the integral J_E, we shall use polar coordinates in the same way and with the same notation as in the proof of Theorem 3.3.1. Let $\{e_1,Je_1,\dots,e_{s-1},Je_{s-1}\}$ be a J-basis of $E\cap T_pL_s$ and let $\{e_{s+1},Je_{s+1},\dots,e_{n-1},Je_{n-1}\}$ be a J-basis of $E\cap(T_pL_s)^{\perp}$. With respect to this orthonormal basis of E

$$\mathrm{tr}(\mathrm{II}|_E)=\sum_{i=1,i\neq s}^{n-1}(k_n(e_i)+k_n(Je_i)).$$

If we denote by $\{f_1,\dots,f_{2n-1}\}$ a basis of principal directions of S at p, we obtain

$$k_n(e_i)=\sum_{j=1}^{2n-1}k_n(f_j)\langle e_i,f_j\rangle^2=\sum_{j=1}^{2n-1}k_j\langle e_i,f_j\rangle^2,$$

and using polar coordinates with respect to JN, we get

$$\begin{aligned}\mathrm{tr}(\mathrm{II}|_E)&=\sum_{j=1}^{2n-1}k_j\left(\sum_{i=1,i\neq s}^{n-1}(\langle e_i,f_j\rangle^2+\langle Je_i,f_j\rangle^2)\right)\\&=\sum_{j=1}^{2n-1}k_j\left(\sum_{i=1,i\neq s}^{n-1}(\cos^2(e_i,JN,f_j)+\cos^2(Je_i,JN,f_j)\right).\end{aligned}$$

We denote by (u,v,w) the spherical angle with vertex v and sides on u and w, $\cos\phi_{ij}=\cos(e_i,JN,f_j)$ and $\cos\phi_{\bar{i}j}=\cos(Je_i,JN,f_j)$. Then, to study the integral J_E we have to deal with the following integral

$$\begin{aligned}&\int_{S^{2n-3}}\int_0^{\pi/2}\frac{\cos^{2s-1}\theta_1}{\sin^{2s-2}\theta_1}\sin^2\alpha_j\sin^{2n-3}\theta_1\cdot\\&\quad\cdot(\cos^2\phi_{1j}+\cdots+\cos^2\phi_{\overline{s-1},j}+\cos^2\phi_{s+1,j}+\cdots+\cos^2\phi_{\overline{n-1},j})d\theta_1dS_{2n-3}\\&=\sin^2\alpha_j\frac{\Gamma(s)\Gamma(n-s)}{2\Gamma(n)}\cdot\\&\quad\cdot\int_{S^{2n-3}}(\cos^2\phi_{1j}+\cdots+\cos^2\phi_{\overline{s-1},j}+\cos^2\phi_{s+1,j}+\cdots+\cos^2\phi_{\overline{n-1},j})dS_{2n-3}\end{aligned}$$

where θ_1 and α_i are defined in (3.10) and (3.11).

Denote by $\tilde{S}^{2n-1}$ the subset of S^{2n-1} defined by all points not in span$\{N, JN\}$. Consider the well-defined map

$$\begin{array}{rccc} \Pi: & \tilde{S}^{2n-1} & \longrightarrow & \{N,JN\}^{\perp} \\ & v & \mapsto & \frac{\mathrm{proj}_{\{N,JN\}^{\perp}}(v)}{||\mathrm{proj}_{\{N,JN\}^{\perp}}(v)||} \end{array}.$$

Note that $\Pi(e_a) = e_a$, with $a \in \{1, \overline{1}, \ldots, s-1, \overline{s-1}, s+1, \overline{s+1}, \ldots, n-1, \overline{n-1}\}$

Then, $V^{\perp}$ inside $\{N, JN\}^{\perp}$ is

$$E^{\perp} \cap \{N, JN\}^{\perp} = (\{N, JN, e_s, Je_s\}^{\perp})^{\perp} \cap \{N, JN\}^{\perp} = \{\Pi(e_s), J\Pi(e_s)\}. \tag{3.12}$$

As $\cos(\phi_{aj}) = \cos(e_a, JN, f_j)$ denotes the cosine of the spherical angle with vertex JN and points in e_a and f_j, by definition, it coincides with $\langle \Pi(e_a), \Pi(f_j)\rangle = \langle e_a, \Pi(f_j)\rangle$. Then, as $\Pi(f_j)$ is a unit vector contained in the vector subspace with basis $\{e_1, Je_1, \ldots, e_{s-1}, Je_{s-1}, \Pi(e_s), J\Pi(e_s)\}$ it is satisfied

$$\begin{aligned} 1 &= \langle \Pi(f_j), \Pi(f_j)\rangle^2 \\ &= \langle e_1, \Pi(f_j)\rangle^2 + \cdots + \langle Je_{s-1}, \Pi(f_j)\rangle^2 + \langle \Pi(e_s), \Pi(f_j)\rangle^2 + \langle J\Pi(e_s), \Pi(f_j)\rangle^2 \end{aligned}$$

and we get

$$\begin{aligned} &\int_{S^{2n-3}} (\cos^2\phi_{1j} + \cdots + \cos^2\phi_{\overline{s-1},j} + \cos^2\phi_{s+1,j} + \ldots + \cos^2\phi_{\overline{n-1},j}) dS \\ &= \int_{S^{2n-3}} (1 - \langle \Pi(e_s), \Pi(f_j)\rangle^2 - \langle J\Pi(e_s), \Pi(f_j)\rangle^2) dS. \end{aligned}$$

Now, we use polar coordinates θ_2, θ_3 with respect to $\Pi(f_j)$ such that

$$\langle \Pi(e_s), \Pi(f_j)\rangle = \cos\theta_2, \;\; \theta_2 \in (0, \pi),$$

and

$$\langle J\Pi(e_s), \Pi(f_j)\rangle = \sin(\Pi(e_s), \Pi(f_j)) \cos(\Pi(e_s), \Pi(f_j), J\Pi(f_j)) = \sin\theta_2 \cos\theta_3, \;\; \theta_3 \in (0, \pi).$$

By Lemma 3.1.2 and the relation

$$O_{2n-3} = O_{2n-5} \frac{\pi}{n-2}$$

we have

$$\begin{aligned} &\int_{S^{2n-5}} \int_0^{\pi} \int_0^{\pi} (1 - \cos^2\theta_2 - \sin^2\theta_2 \cos^2\theta_3) \sin^{2n-4}\theta_2 \sin^{2n-5}\theta_3 d\theta_3 d\theta_2 dS_1 \\ &= O_{2n-3} - \int_{S^{2n-4}} \int_0^{\pi} \cos^2\theta_2 \sin^{2n-4}\theta_2 - \int_{S^{2n-5}} \int_0^{\pi} \cos^2\theta_3 \sin^{2n-5}\theta_3 \int_0^{\pi} \sin^{2n-2}\theta_2 \\ &= O_{2n-3} - \frac{O_{2n-3}}{2n-2} - O_{2n-5} 2 \frac{\sqrt{\pi}\Gamma(n-2)}{4\Gamma(n-\frac{1}{2})} \frac{\sqrt{\pi}\Gamma(n-\frac{1}{2})}{\Gamma(n)} \\ &= O_{2n-5} \left(\frac{\pi}{n-2} - \frac{\pi}{2(n-1)(n-2)} - \frac{\pi}{2(n-1)(n-2)} \right) \\ &= \frac{O_{2n-5}\pi}{2(n-1)(n-2)} (2n-4) \\ &= \frac{O_{2n-5}\pi}{2(n-1)}. \end{aligned}$$

Thus,

$$J_E = \frac{O_{2n-3}\mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1})n(s-1)}{2s(2s-1)(n-s)(n-1)}\binom{n}{s}^{-1}\left((2n-1)M_1(S) - \int_S k_n(JN)\right)$$

and adding both expressions of J_E and J_s we get the result

$$\begin{aligned} J_E + J_s &= \frac{O_{2n-3}\mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1})}{4s(2s-1)(n-1)}\binom{n}{s}^{-1} \cdot \\ &\cdot\left(((2n-1)+(2n-1)\frac{n(s-1)}{n-s})M_1(S) + (\frac{2sn-n-s}{n-s} - \frac{n(s-1)}{n-s})\int_S k_n(JN)\right) \\ &= \frac{O_{2n-3}\mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1})}{4s(2s-1)(n-1)}\binom{n}{s}^{-1} \cdot \\ &\cdot\left(\frac{2n-1}{n-s}(n-s+ns-n)M_1(S) + \frac{2sn-n-s-ns+n}{n-s}\int_S k_n(JN)\right) \\ &= \frac{O_{2n-3}\mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1})}{4s(2s-1)(n-1)}\binom{n}{s}^{-1}\left(\frac{s(2n-1)(n-1)}{n-s}M_1(S) + \int_S k_n(JN)\right). \end{aligned}$$

□

It is natural to ask which functionals we have to integrate over the space of complex s-planes to obtain the mean curvature integral of the initial hypersurface.

Theorem 3.3.3. *Let $S \subset \mathbb{CK}^n(\epsilon)$ be a compact (possibly with boundary) oriented hypersurface of class $\mathcal{C}^2$. If we define*

$$\nu(S) = (2ns-n-s)M_1(S) - \frac{n-s}{2s-1}\int_S k_n(JN)dx,$$

then

$$\int_{\mathcal{L}^{\mathbb{C}}_r} \nu(S\cap L_r)dL_r = \frac{\omega_{2n-2}\mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1})2(n-1)(2n-1)(s-1)}{(2s-1)}M_1(S).$$

Proof. The result follows straightforward from Theorems 3.3.1 and 3.3.2. □

3.4 Reproductive valuations

Definition 3.4.1. Suppose given, for each $s \in \mathbb{N}$, a valuation in $\mathbb{CK}^n(\epsilon)$, $\varphi^{(s)}$. It is said that the collection $\{\varphi^{(s)}\}$ satisfies the *reproductive property* if for any regular domain Ω,

$$\int_{\mathcal{L}^{\mathbb{C}}_s} \varphi^{(s)}(\Omega\cap L_s)dL_s = c_{n,s}\varphi(\Omega),$$

for some constant $c_{n,s}$ depending on n and s.

Remark 3.4.2. Recall that mean curvature integral for regular domains extend to all $\mathcal{K}(\mathbb{C}^n)$. Also $\int_{\partial\Omega} k_n(JN)$ extends to $\mathcal{K}(\mathbb{C}^n)$ since it coincides with $\Gamma_{2n-2,n-1}(\Omega)$ (cf. Example 2.4.20).

As neither the mean curvature integral $M_1^{(s)}(\partial\Omega\cap L_s)$, nor the integral of the normal curvature, $\int_{\partial\Omega\cap L_s} k_n(J\tilde{N})dx$, satisfy the reproductive property, it is natural to ask whether there exists some linear combination of these such that satisfies this property. We consider a linear combination since, in $\mathbb{C}^n$, they constitute a basis of $\mathrm{Val}^{U(n)}_{n-2}(\mathbb{C}^n)$.

Theorem 3.4.3. *Let $\Omega \subset \mathbb{CK}^n(\epsilon)$ be a regular domain, $s \in \{1, ..., n-1\}$. Consider the smooth valuations defined by*

$$\varphi_1(\Omega) = M_1(\partial\Omega) - \int_{\partial\Omega} k_n(JN)$$

and

$$\varphi_2(\Omega) = (2s-1)(2n-1)M_1(\partial\Omega) + \int_{\partial\Omega} k_n(JN).$$

Then,

$$\int_{\mathcal{L}_s^{\mathbb{C}}} \varphi_1(\Omega \cap L_s) dL_s = \frac{\omega_{2n-2} \mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1})(s-1)(2n-1)}{(2s-1)(n-s)} \binom{n}{s}^{-1} \varphi_1(\Omega)$$

and

$$\int_{\mathcal{L}_s^{\mathbb{C}}} \varphi_2(\Omega \cap L_s) dL_s = \frac{\omega_{2n-2} \mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1})}{(2s-1)} \binom{n-2}{s-1}^{-1} \varphi_2(\Omega).$$

In $\mathbb{C}^n$, each of $\varphi_1(\Omega)$ and $\varphi_2(\Omega)$ expands a 1-dimensional subspace of reproductive valuations of degree $2n-2$.

Proof. Let

$$\nu(\Omega) = aM_1(\partial\Omega) + b\int_{\partial\Omega} k_n(JN).$$

We look for relations between a and b to be ν a reproductive valuation, that is,

$$\int_{\mathcal{L}_s^{\mathbb{C}}} \nu(\Omega \cap L_s) dL_s = \lambda\nu(\Omega).$$

From Theorems 3.3.1 and 3.3.2 we have

$$\begin{aligned}
\int_{\mathcal{L}_s^{\mathbb{C}}} \nu(\Omega \cap L_s) dL_s &= \int_{\mathcal{L}_s^{\mathbb{C}}} \left(aM_1(\partial\Omega \cap L_s) + b\int_{\partial\Omega \cap L_s} k_n(J\tilde{N}) \right) \\
&= \frac{\omega_{2n-2} \mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1})}{2s(2s-1)} \binom{n}{s}^{-1} \Big(\Big(a + \frac{b(2s-1)(2sn-n-s)}{n-s} \Big) \int_{\partial\Omega} k_n(JN) + \\
&\quad + \Big(\frac{a(2ns-n-s)}{n-s} + b(2s-1) \Big)(2n-1)M_1(\partial\Omega) \Big).
\end{aligned}$$

Thus, ν is reproductive if and only if for some $\lambda \in \mathbb{R}$

$$\begin{aligned}
(2n-1)\left(\frac{a(2ns-n-s)}{n-s} + b(2s-1) \right) &= \lambda a, \\
a + \frac{b(2s-1)(2sn-n-s)}{n-s} &= \lambda b.
\end{aligned}$$

Solving this system we get two solutions, $a = -b$, $\lambda = \dfrac{2s(s-1)(2n-1)}{n-s}$ and $a = b(2s-1)(2n-1)$, $\lambda = \dfrac{2n(n-1)}{n-s}$. □

Remark 3.4.4. Last theorem gives all valuations in $\mathrm{Val}_{n-2}^{U(n)}(\mathbb{C}^n)$ such that they satisfy the reproductive property. Why are these valuations reproductive? Are they special in some sense? It shall be interesting to know the answer and also to have a geometric interpretation for these valuations.

3.5 Relation with some valuations defined by Alesker

In Section 2.3 we recalled the definition of valuations $U_{k,p}$ in $\mathbb{C}^n$. They constitute a basis for $\mathrm{Val}^{U(n)}(\mathbb{C}^n)$. From this basis it can be established the following theorem by Alesker

Theorem 3.5.1. *(Theorem 3.1.2 [Ale03]) Let Ω be a regular domain in $\mathbb{C}^n$. Let $0 < q < n$, $0 < 2p < k < 2q$. Then,*

$$\int_{\mathcal{L}_q^{\mathbb{C}}} U_{k,p}(\Omega \cap L_q) dL_q = \sum_{p=0}^{[k/2]+n-q} \gamma_p \cdot U_{k+2(n-q),p}(\Omega),$$

where constants γ_p depend only on n, q and p.

In the following theorem we give the constants γ_p with arbitrary n, q and $k = 2q-2$.

Theorem 3.5.2. *Let Ω be a regular domain and $0 < q < n$. Then,*

$$\int_{\mathcal{L}_q^{\mathbb{C}}} U_{2q-2,p}(\Omega \cap L_q) dL_q = \frac{\omega_{2q-2}\omega_{2n-2}\mathrm{vol}(G^{\mathbb{C}}_{n-2,q-1})\mathrm{vol}(G^{\mathbb{C}}_{q-2,q-p-1})}{(q-p)(2q-2p-1)\binom{n-2}{q-1}\binom{q-2}{q-p-1}} \cdot$$
$$\cdot \left(\frac{(2n-3)(n-1)(n-q+p)}{\omega_{2n-2}} U_{2n-2,1}(\Omega) - (2n-1)n(n-q+p-1)U_{2n-2,0}(\Omega) \right).$$

First, we express $\int_{\partial\Omega} k_n(JN)$ (a translation invariant continuous valuation) in terms of $\{U_{k,p}\}$.

Proposition 3.5.3. *Let Ω be a regular domain in $\mathbb{C}^n$. Then*

$$\int_{\partial\Omega} k_n(JN) dp = \frac{n(2n-3)(2n-2)^2\omega_2}{\omega_{2n-2}} U_{2n-2,1}(\Omega) - 2n(2n-1)(2n^2-4n+1)\omega_2 U_{2n-2,0}(\Omega).$$

Proof. From the relations among valuations $\{U_{k,p}\}$ and mean curvature integrals (see (2.3)) we have

$$U_{2n-2,0}(\Omega) = \frac{1}{2n\omega_2} M_1(\partial\Omega) \tag{3.13}$$

and

$$U_{2n-2,1}(\Omega) = \frac{1}{(2n-2)\omega_2} \int_{\mathcal{L}^{\mathbb{C}}_{n-1}} M_1(\partial\Omega \cap L_{n-1}) dL_{n-1}.$$

Using Proposition 3.3.2 with $s = n-1$, we get the result. □

Proof of the theorem 3.5.2. From the definition of $U_{k,p}$ we have

$$U_{k,p}(\Omega) = \frac{1}{2(n-p)\omega_{2n-k}} \int_{G^{\mathbb{C}}_{n,n-p}} M_{k-2p}(\Omega \cap L_{n-p}) dL_{n-p}$$

and from Theorem 3.3.2

$$U_{2q-2,p}(\Omega \cap L_q) = \frac{1}{2(q-p)\omega_2} \int_{\mathcal{L}^{\mathbb{C}}_{q-p}} M_1((\partial\Omega \cap L_q) \cap L_{q-p}) dL_{q-p}$$
$$= \frac{O_{2q-3}\mathrm{vol}(G^{\mathbb{C}}_{q-2,q-p-1})}{8(q-p)^2(q-1)(2q-2p-1)\omega_2} \binom{q}{q-p}^{-1} \cdot$$
$$\cdot \left((2q-1)\frac{2q(q-p)-2q+p}{p} M_1(\partial\Omega \cap L_q) + \int_{\partial\Omega \cap L_q} k_n(J\tilde{N}) \right)$$

where $\tilde{N}$ denotes the normal inward vector field to $\partial\Omega \cap L_q$ as a hypersurface in L_q. Using again Theorems 3.3.1 and 3.3.2, we express the integrals over $\mathcal{L}_q^{\mathbb{C}}$ as an integral oer $\partial\Omega$. Finally, from the relation in Proposition 3.5.3 and (3.13) we get the result. □

3.6 Example: sphere in $\mathbb{CK}^3(\epsilon)$

In this section we check Theorem 3.3.2 in the case of a sphere of radius R in $\mathbb{CK}^3(\epsilon)$.

On page 18 we give an expression for the principal curvatures of a sphere of radius R in $\mathbb{CK}^n(\epsilon)$. Using this expression we get

$$\begin{aligned}\int_{\partial B_R} k_n(JN) &= 2\cot_\epsilon(2R)V(\partial B_R) = 2\frac{\cos_\epsilon^2(R)+\sin_\epsilon^2(R)}{2\sin_\epsilon(R)\cos_\epsilon(R)}\frac{\pi^3}{3!}6\sin_\epsilon^5(R)\cos_\epsilon(R)\\ &= \pi^3(\cos_\epsilon^2(R)+\sin_\epsilon^2(R))\sin_\epsilon^4(R) = \pi^3(2\sin_\epsilon^6(R)+\sin_\epsilon^4(R))\\ &= \pi^3(2\cos_\epsilon^6(R)-5\cos_\epsilon^4(R)+4\cos_\epsilon^2(R)-1)\end{aligned}$$

and

$$\begin{aligned}M_1(\partial B_R) &= \frac{\pi^3 6}{3!4}(5\sin_\epsilon^4(R)\cos_\epsilon^2(R)+\sin_\epsilon^6(R)) = \frac{\pi^3}{4}(6\sin_\epsilon^6(R)+5\sin_\epsilon^4(R))\\ &= \pi^3(\frac{6}{5}\cos_\epsilon^6(R)-\frac{13}{5}\cos_\epsilon^4(R)+\frac{8}{5}\cos_\epsilon^2(R)-\frac{1}{5}).\end{aligned}$$

Thus, the right hand side of Theorem 3.3.2 is

$$\begin{aligned}&\frac{O_3\pi^3}{144}((42+2)\cos_\epsilon^6(R)-(13\cdot 7+5)\cos_\epsilon^4(R)+(56+4)\cos_\epsilon^2(R)-(7+1))\\ &= 2\pi^5\left(\frac{11}{36}\cos_\epsilon^6(R)-\frac{2}{3}\cos_\epsilon^4(R)+\frac{5}{12}\cos_\epsilon^2(R)-\frac{1}{18}\right).\end{aligned}$$

The left hand side of Theorem 3.3.2 is

$$\int_{\mathcal{L}_2^{\mathbb{C}}} M_1^{(2)}(\partial B_R\cap L_2)dL_2.$$

Let us compute first $M_1^{(2)}(\partial B_R\cap L_2)$, for a fixed complex 2-plane, L_2. Recall that the intersection between a sphere and L_2 is a sphere inside L_2 with radius r satisfying $\cos_\epsilon(R) = \cos_\epsilon(r)\cos_\epsilon(\rho)$ where ρ is the distance from the origin of the sphere B_R to the plane L_2 (cf. [Gol99, Lemma 3.2.13]).

$$\begin{aligned}M_1^{(2)}(\partial B_R\cap L_2) &= \frac{1}{3}\int_{\partial B_R}(2\cot_\epsilon(r)+2\cot_\epsilon(2r))\\ &= \frac{2}{3}(\cot_\epsilon(r)+\cot_\epsilon(2r))\frac{4\pi^2}{2!}\sin_\epsilon^3(r)\cos_\epsilon(r)\\ &= \frac{1}{12}(4\cos_\epsilon^4(r)-5\cos_\epsilon^2(r)+1)\\ &= \frac{1}{12}\frac{1}{\cos_\epsilon^4(R)}(4\cos_\epsilon^4(R)-5\cos_\epsilon^2(R)\cos_\epsilon^2(\rho)+\cos_\epsilon^4(\rho)).\end{aligned}$$

Thus,

$$\begin{aligned}&\int_{\mathcal{L}_2^{\mathbb{C}}} M_1^{(2)}(\partial B_R\cap L_2)dL_2\\ &= \frac{1}{12}\int_{G_{3,1}^{\mathbb{C}}}\int_{S^1}\int_0^R \frac{\cos_\epsilon^4(\rho)}{\cos_\epsilon^4(\rho)}(4\cos_\epsilon^4(R)-5\cos_\epsilon^2(R)\cos_\epsilon^2(\rho)+\cos_\epsilon^4(\rho))2\cos_\epsilon(\rho)\sin_\epsilon(\rho)\\ &= \frac{2\pi V(G_{3,1}^{\mathbb{C}})2}{12}\int_0^R(4\cos_\epsilon^4(R)-5\cos_\epsilon^2(R)\cos_\epsilon^2(\rho)+\cos_\epsilon^4(\rho))2\cos_\epsilon(\rho)\sin_\epsilon(\rho)\\ &= \frac{\pi V(G_{3,1}^{\mathbb{C}})}{3}\left(\frac{11}{12}\cos_\epsilon^6(R)-2\cos_\epsilon^4(R)+\frac{5}{4}\cos_\epsilon^2(R)-\frac{1}{6}\right)\end{aligned}$$

and we get the same result in both side of the expression in Theorem 3.3.2.

Chapter 4

Gauss-Bonnet Theorem and Crofton formulas for complex planes

In this chapter we obtain an expression for the measure of complex r-planes intersecting a compact domain in $\mathbb{CK}^n(\epsilon)$. That is, we give an expression of

$$\int_{\mathcal{L}_r^{\mathbb{C}}} \chi(\Omega \cap L_r) dL_r \tag{4.1}$$

for a regular domain $\Omega \subset \mathbb{CK}^n(\epsilon)$ as a linear combination of the so-called *Hermitic intrinsic volumes* valuations in $\mathbb{CK}^n(\epsilon)$ (cf. Definition 2.4.11). The method we use consists on computing the variation, when the domain moves along the flow induced by a smooth vector field, of the measure of complex r-planes intersecting the convex domain. From the theory of valuations on $\mathbb{C}^n$, we know that the expression is a linear combination of certain valuations. Thus, computing also the variation of these valuations and then comparing both results we shall deduce the final expression.

Using the same method we obtain an expression (cf. Theorem 4.4.1) for the Euler characteristic of a compact domain in terms of its Gauss curvature of the boundary, its volume and others Hermitian intrinsic volumes. This expression is analogous to the one obtained in [San04, page 309] for real space forms.

Relating these two results we shall obtain another expression for the Euler characteristic. This one involves the measure of complex hyperplanes intersecting the regular domain (cf. Theorem 4.4.5).

4.1 Variation of the Hermitian intrinsic volumes

The study of the variation of a valuation when the domain moves along the flow of a smooth vector field will be useful to deduce some properties of the valuation. In [BF08] it is given the variation of some valuations on $\mathbb{C}^n$ and this variation is used to characterize monotone valuations. In this work, we give the variation of the Hermitian intrinsic volumes (cf. Definition 2.4.11) on $\mathbb{CK}^n(\epsilon)$ and we use it to deduce expression (4.1) in terms of these valuations.

In order to obtain the variation of Hermitian intrinsic volumes on $\mathbb{CK}^n(\epsilon)$ we follow the same method as in the proof of Corollary 2.6 in [BF08]. First, we recall the definition of the Rumin derivative, introduced in [Rum94].

Definition 4.1.1. Let $\mu \in \Omega^{2n-1}(S(\mathbb{CK}^n(\epsilon)))$, let α be the contact form of $S(\mathbb{CK}^n(\epsilon))$ and let $\alpha \wedge \xi \in \Omega^{2n-1}(S(\mathbb{CK}^n(\epsilon)))$ be the unique (cf. [Rum94]) form such that $d(\mu + \alpha \wedge \xi)$ is multiple of α. Then, the *Rumin operator* D is defined as

$$D\mu := d(\mu + \alpha \wedge \xi).$$

Let us recall the definition of the Reeb vector field over a contact manifold.

Definition 4.1.2. Let M be a contact manifold with contact form α. The Reeb vector field T is the only vector field over M such that

$$\begin{cases} i_T\alpha = 1, \\ \mathcal{L}_T\alpha = 0. \end{cases} \tag{4.2}$$

If the contact manifold is the fiber tangent bundle of a Riemann manifold, then the Reeb vector field coincides with the geodesic flow (cf. [Bla76, page 17]). This is the situation in this work, we consider the unit tangent bundle of $\mathbb{CK}^n(\epsilon)$ (cf. Lemma 1.3.7 and Remark 1.3.8).

Note that the condition $\mathcal{L}_T\alpha = 0$ is equivalent to $i_T d\alpha = 0$. Indeed,

$$\mathcal{L}_T\alpha = i_T d\alpha + d(i_T\alpha) = i_T d\alpha = d\alpha(T).$$

The following lemma contains the value of the contraction of T with α, β, γ and θ_i defined in Section 2.4.2.

Lemma 4.1.3. *In $S(\mathbb{CK}^n(\epsilon))$ it is satisfied*

$$\begin{array}{ll} i_T\alpha = 1, & i_T\theta_1 = \gamma, \\ i_T\theta_2 = \beta, & i_T\beta = i_T\gamma = i_T\theta_0 = i_T\theta_s = 0. \end{array}$$

Proof. The first equality is a characterization of the Reeb vector field. Moreover, $i_T\theta_s = -i_t d\alpha = di_T\alpha - \mathcal{L}_T\alpha = 0$.

As T is the geodesic flow, it satisfies $\alpha_i(T) = \beta_i(T) = 0$ and $\alpha_{1i} = \beta_{1i} = 0$, $i \in \{2, ..., n\}$. We get the result using Definition (2.6) extended to $\mathbb{CK}^n(\epsilon)$. $\square$

In [BF08] it is proved the following lemma, which allow to calculate the variation of a valuation defined from an invariant smooth form. The result is proved in $\mathbb{C}^n$ but the same remains true for $\epsilon \neq 0$, and for any Riemann manifold. Here we repeat the proof in detail for $\mathbb{CK}^n(\epsilon)$.

Lemma 4.1.4 ([BF08] Lemma 2.5)**.** *Suppose that $\Omega \subset \mathbb{CK}^n(\epsilon)$ is a regular domain, N the outward unit vector field to $\partial\Omega$, X is a smooth vector field on $\mathbb{CK}^n(\epsilon)$ with flow F_t and μ a smooth valuation given by a $(2n-1)$-form ρ in $S(\mathbb{CK}^n(\epsilon))$. Then*

$$\left.\frac{d}{dt}\right|_{t=0} \mu(F_t(\Omega)) = \delta_X\mu(\Omega) = \int_{N(\Omega)} \langle X, N\rangle\, i_T(D\beta)$$

where T is the Reeb vector field of $S(\mathbb{CK}^n(\epsilon))$ and $D\rho$ is the Rumin operator of ρ.

Proof. Let $\tilde{X}$ be a lift of X at $S(\mathbb{CK}^n(\epsilon))$ such that it preserves α, i.e. $\mathcal{L}_{\tilde{X}}(\alpha) = 0$. Then, if $\tilde{F}$ denotes the flow of $\tilde{X}$

$$
\begin{aligned}
\delta_X \mu(\Omega) &= \frac{d}{dt}\bigg|_{t=0} \left(\int_{N(F_t(\Omega))} \beta \right) = \frac{d}{dt}\bigg|_{t=0} \left(\int_{N(\Omega)} \tilde{F_t}^* \beta \right) \\
&= \int_{N(\Omega)} \mathcal{L}_{\tilde{X}} \beta \\
&\overset{(1)}{=} \int_{N(\Omega)} i_{\tilde{X}} d\beta + d(i_{\tilde{X}} \beta) \overset{(2)}{=} \int_{N(\Omega)} i_{\tilde{X}} d\beta \\
&\overset{(3)}{=} \int_{N(\Omega)} i_{\tilde{X}} D\beta - i_{\tilde{X}} d(\alpha \wedge \eta) \\
&\overset{(4)}{=} \int_{N(\Omega)} i_{\tilde{X}} D\beta \\
&\overset{(5)}{=} \int_{N(\Omega)} i_{\tilde{X}} (\alpha \wedge \rho) \\
&\overset{(6)}{=} \int_{N(\Omega)} (i_{\tilde{X}} \alpha) \rho \\
&\overset{(7)}{=} \int_{N(\Omega)} \alpha(\tilde{X}) i_T (\alpha \wedge \rho) \\
&= \int_{N(\Omega)} \langle X, N \rangle i_T D\beta
\end{aligned}
$$

First, note that we can change the variation of $F_t(\Omega)$ in the first integral for the Lie derivative of the integrated form since $N(F_t(\Omega)) = \tilde{F}_t(N(\Omega))$.

For (1) and (2), we use the following property of the Lie derivative, $\mathcal{L}_{\tilde{X}} \beta = i_{\tilde{X}} d\beta + d(i_{\tilde{X}} \beta)$, and that the second term is an exact form, thus the integral vanishes.

Equality (3) follows directly from the definition of the Rumin operator.

For (4) we use

$$i_{\tilde{X}} d(\alpha \wedge \eta) = \mathcal{L}_{\tilde{X}} (\alpha \wedge \eta) - d(i_{\tilde{X}} (\alpha \wedge \eta))$$

and that the second term is an exact form. The first term can be rewritten as

$$\mathcal{L}_{\tilde{X}} (\alpha \wedge \eta) = (\mathcal{L}_{\tilde{X}} \alpha) \wedge \eta + \alpha \wedge \mathcal{L}_{\tilde{X}} \eta,$$

and so, its integral vanishes since $\tilde{X}$ preserves α, which vanishes over the normal fiber bundle (cf. Lemma 1.3.12).

As the Rumin operator is, by definition, a $2n$-form multiple of α, and it is defined on the normal fiber bundle, we get (5).

For (6), using the notion of contraction we get

$$i_{\tilde{X}} (\alpha \wedge \rho) = (i_{\tilde{X}} \alpha) \wedge \rho + \alpha \wedge (i_{\tilde{X}} \rho).$$

The second term vanishes over $N(\Omega)$.

To get equality (7), we repeat the same argument as in (4) in order to obtain the form $\alpha \wedge \rho$, which is the Rumin operator of β.

Finally, we recall the definition of α and that the integral is over the unit fiber normal bundle, so that the points are (x, N) with $x \in \partial\Omega$ and N the unit normal vector on $\partial\Omega$ at x. $\square$

The previous lemma allows us to compute the variation of any valuation given by a form, once we know its Rumin operator. In this chapter we give the variation of the Hermitian intrinsic volumes in $\mathbb{CK}^n(\epsilon)$ (in [BF08] is given for $\epsilon = 0$).

In the following lemma we give the derivative $\beta_{k,q}$ and $\gamma_{k,q}$ (defined in 2.4.6) using Lemma 2.4.8. They will be used for the computation of the Rumin operator.

Lemma 4.1.5. *In* $\mathbb{CK}^n(\epsilon)$

$$d\beta_{k,q} = c_{n,k,q}(\theta_0^{n-k+q} \wedge \theta_1^{k-2q} \wedge \theta_2^q - \epsilon(n-k+q)\alpha \wedge \beta \wedge \theta_0^{n-k+q-1} \wedge \theta_1^{k-2q} \wedge \theta_2^q)$$

and

$$\begin{aligned} d\gamma_{k,q} =& c_{n,k,q}(\theta_0^{n-k+q} \wedge \theta_1^{k-2q} \wedge \theta_2^q - \epsilon\theta_0^{n-k+q-1} \wedge \theta_1^{k-2q} \wedge \theta_2^{q+1} - \epsilon\alpha \wedge \beta \wedge \theta_0^{n-k+q-1} \wedge \theta_1^{k-2q} \wedge \theta_2^q \\ &- \epsilon\frac{(n-k+q-1)}{2}\alpha \wedge \gamma \wedge \theta_0^{n-k+q-2} \wedge \theta_1^{k-2q+1} \wedge \theta_2^q \\ &- \epsilon\frac{(n-k+q-1)}{2}\beta \wedge \gamma \wedge \theta_{01} \wedge \theta_0^{n-k+q-2} \wedge \theta_1^{k-2q} \wedge \theta_2^q). \end{aligned}$$

From the previous lemma and following the method used by [BF08] we can compute the variation of $B_{k,q}$ and $\Gamma_{k,q}$ in $\mathbb{CK}^n(\epsilon)$.

Notation 4.1.6. We denote

$$\tilde{B}_{k,q} = \tilde{B}_{k,q}(\Omega) := \int_{\partial\Omega} \langle X, N\rangle \beta_{k,q} \quad \text{and} \quad \tilde{\Gamma}_{k,q} = \tilde{\Gamma}_{k,q}(\Omega) := \int_{\partial\Omega} \langle X, N\rangle \gamma_{k,q}.$$

Proposition 4.1.7. *Let X be a smooth vector field defined on $\mathbb{CK}^n(\epsilon)$ and $\Omega \subset \mathbb{CK}^n(\epsilon)$ be a regular domain. The variation in $\mathbb{CK}^n(\epsilon)$ of valuations $B_{k,q}$ and $\Gamma_{k,q}$ with respect to X is given by*

$$\begin{aligned} \delta_X B_{k,q}(\Omega) = & 2c_{n,k,q}(c^{-1}_{n,k-1,q}(k-2q)^2\tilde{\Gamma}_{k-1,q} - c^{-1}_{n,k-1,q-1}(n+q-k)q\tilde{\Gamma}_{k-1,q-1} \\ &+ c^{-1}_{n,k-1,q-1}(n+q-k+\frac{1}{2})q\tilde{B}_{k-1,q-1} - c^{-1}_{n,k-1,q}(k-2q)(k-2q-1)\tilde{B}_{k-1,q} \\ &+ \epsilon(c^{-1}_{n,k+1,q+1}(k-2q)(k-2q-1)\tilde{B}_{k+1,q+1} - c^{-1}_{n,k+1,q}(n-k+q)(q+\frac{1}{2})\tilde{B}_{k+1,q})) \end{aligned}$$

and

$$\begin{aligned} \delta_X \Gamma_{k,q}(\Omega) = & 2c_{n,k,q}\Big(c^{-1}_{n,k-1,q}(k-2q)^2\tilde{\Gamma}_{k-1,q} - c^{-1}_{n,k-1,q-1}(n+q-k)q\tilde{\Gamma}_{k-1,q-1} \\ &+ c^{-1}_{n,k-1,q-1}(n+q-k+\frac{1}{2})q\tilde{B}_{k-1,q-1} - c^{-1}_{n,k-1,q}(k-2q)(k-2q-1)\tilde{B}_{k-1,q} \\ &+ \epsilon\big(c^{-1}_{n,k+1,q+1}2(k-2q)(k-2q-1)\tilde{B}_{k+1,q+1} - c^{-1}_{n,k+1,q}((n-k+q)(2q+\frac{3}{2}) - \frac{1}{2}(q+1))\tilde{B}_{k+1,q} \\ &- c^{-1}_{n,k+1,q+1}(k-2q)^2\tilde{\Gamma}_{k+1,q+1} + c^{-1}_{n,k+1,q}(n-k+q-1)(q+1)\tilde{\Gamma}_{k+1,q} \\ &- \epsilon(c^{-1}_{n,k+3,q+2}(k-2q)(k-2q-1)\tilde{B}_{k+3,q+2} - c^{-1}_{n,k+3,q+1}(n-k+q-1)(q+\frac{3}{2})\tilde{B}_{k+3,q+1}))\Big). \end{aligned}$$

Proof. We first study the valuation given by $\beta_{k,q}$.

Lemma 4.1.4 provides an expression for the variation of a smooth valuation. In order to use this lemma, it is enough to find $i_T D\beta_{k,q}$ and $i_T\gamma_{k,q}$ modulo α and $d\alpha$ since the latter forms vanish over $N(\Omega)$ (cf. Lemma 1.3.12).

We will use the following fact from the proof of Proposition 4.6 in [BF08]: for $\max\{0, k-n\} \le q \le k/2 < n$ there exists an invariant form $\xi_{k,q} \in \Omega^{2n-1}(S(\mathbb{C}^n))$ such that

$$d\alpha \wedge \xi_{k,q} \equiv -\theta_0^{n-k+q}\theta_1^{k-2q}\theta_2^q \qquad \mathrm{mod}(\alpha), \tag{4.3}$$

and

$$\begin{aligned} \xi_{k,q} \equiv & \beta\gamma\theta_0^{n+q-k-1}\theta_1^{k-2q-2}\theta_2^{q-1} \\ &\wedge \big((n+q-k)q\theta_1^2 - (k-2q)(k-2q-1)\theta_0\theta_2\big) \qquad \mathrm{mod}(\alpha, d\alpha). \end{aligned} \tag{4.4}$$

In order to find $\delta_X B_{k,q}$ for general ϵ, we take a form $\xi^\epsilon \in \Omega^{2n-1}(S(\mathbb{CK}^n(\epsilon)))$ such that $\xi^\epsilon_{(p,v)} \equiv \xi_{(p',v')}$ when we identify $T_{(p,v)}S(\mathbb{CK}^n(\epsilon))$ and $T_{(p',v')}\mathbb{C}^n$, for every $(p,v) \in S(\mathbb{CK}^n(\epsilon)), (p',v') \in S(\mathbb{C}^n)$. That is, as ξ^ϵ is an invariant form, it can be expressed as a linear combination of products with the forms α, β, θ_0, θ_1, θ_2 and θ_s. We take this expression as a definition of ξ^ϵ. From Lemma 4.1.5 and (4.3) we have that $d(\beta_{k,q} + c_{n,k,q}\alpha \wedge \xi^\epsilon) \equiv 0$ modulo α.

By Lemma 2.4.8, the exterior differential of ξ^ϵ is

$$\begin{aligned} d\xi^\epsilon \equiv\ & \theta_0^{n+q-k-1}\theta_1^{k-2q-2}\theta_2^{q-1}((n-k+q)q\theta_1^2 - (k-2q)(k-2q-1)\theta_0\theta_2) \\ & \wedge (\gamma\theta_1 - 2\beta\theta_0 + 2\epsilon\beta\theta_2) \quad \mathrm{mod}(\alpha, d\alpha) \end{aligned}$$

and the contraction of $d\beta_{k,q}$ with respect to the vector field T, by Lemma 4.1.3, is

$$\begin{aligned} i_T d\beta_{k,q} \equiv\ & c_{n,k,q}\theta_0^{n+q-k-1}\theta_1^{k-2q-1}\theta_2^{q-1} \\ & \wedge ((k-2q)\gamma\theta_0\theta_2 + q\beta\theta_0\theta_1 - \epsilon(n-k+q)\beta\theta_1\theta_2) \quad \mathrm{mod}(\alpha). \end{aligned}$$

By substituting the last expressions in $i_T D\beta_{k,q} \equiv i_T d\beta_{k,q} - c_{n,k,q}d\xi$ (mod α, $d\alpha$), we get the result.

The variation of $\Gamma_{k,q}$ with $k \neq 2q$ can be obtained using the relation among $\Gamma_{k,q}$ and $B_{k,q}$ given in Proposition 2.4.8 and the variation of $B_{k,q}$.

To compute $\delta_X \Gamma_{2q,q}$, note that $d\gamma_{2q,q}$ has 3 terms not multiple of α (cf. Lemma 4.1.5). As before we take $\xi_1^\epsilon, \xi_2^\epsilon \in \Omega^{2n-1}(S(\mathbb{CK}^n(\epsilon)))$ corresponding to $\xi_{2q,q}$, and $\xi_{2q+2,q+1}$ respectively. Let us consider also

$$\xi_3^\epsilon = \frac{n-q-1}{2}\beta\gamma\theta_0^{n-q-2}\theta_2^q. \tag{4.5}$$

Then the Rumin differential of $\gamma_{2q,q}$ is given by $D\gamma_{2q,q} = d(\gamma_{2q,q} + c_{n,2q,q}\alpha \wedge (\xi_1^\epsilon - \epsilon\xi_2^\epsilon - \epsilon\xi_3^\epsilon))$. Indeed, $d\alpha \wedge \xi_1^\epsilon$ cancels the first term of $d\gamma_{2q,q}$ modulo α, and $d\alpha \wedge \xi_2^\epsilon$ cancels the second one. The third term is canceled exactly by $d\alpha \wedge \xi_3^\epsilon$.

Now, using Lemmas 4.1.5 and 4.1.3 we get

$$i_T d\gamma_{2q,q} \equiv q\beta\theta_0^{n-q}\theta_2^{q-1} - \epsilon(q+2)\beta\theta_0^{n-q-1}\theta_2^q - \epsilon\frac{n-q-1}{2}\gamma\theta_0^{n-q-2}\theta_1\theta_2^q \quad \mathrm{mod}(\alpha, d\alpha),$$

and from (4.4) and (4.5)

$$\begin{aligned} d\xi_1^\epsilon &\equiv (n-q)q\theta_0^{n-q-1}\theta_2^{q-1}(\gamma\theta_1 - 2\beta\theta_0 + 2\epsilon\beta\theta_2) \qquad \mathrm{mod}(\alpha, d\alpha). \\ d\xi_2^\epsilon &\equiv (n-q-1)(q+1)\theta_0^{n-q-2}\theta_2^q(\gamma\theta_1 - 2\beta\theta_0 + 2\epsilon\beta\theta_2) \qquad \mathrm{mod}(\alpha, d\alpha). \\ d\xi_3^\epsilon &\equiv \frac{n-q-1}{2}\theta_0^{n-q-2}\theta_2^q(\gamma\theta_1 - 2\beta\theta_0 + 2\epsilon\beta\theta_2) \qquad \mathrm{mod}(\alpha, d\alpha). \end{aligned}$$

Plugging this into $i_T D\gamma_{2q,q} \equiv i_T d\gamma_{2q,q} - c_{n,2q,q}(d\xi_1^\epsilon - \epsilon d\xi_2^\epsilon - \epsilon d\xi_3^\epsilon)$ mod $(\alpha, d\alpha)$ gives the result. □

Remark 4.1.8. For $\epsilon = 0$ the variation of $B_{k,q}$ coincides with the variation of $\Gamma_{k,q}$ and we get the result of Proposition 4.6 in [BF08].

From the previous proposition we can obtain easily the variation of the Gauss curvature integral. We know that this variation vanishes in $\mathbb{C}^n$, for the Gauss-Bonnet theorem, but not in the other complex space forms.

Corollary 4.1.9. *In $\mathbb{CK}^n(\epsilon)$ the variation of the Gauss curvature integral is*

$$\delta_X M_{2n-1}(\partial\Omega) = 2\epsilon\omega_{2n-1}(2(n-1)\tilde{\Gamma}_{1,0} - (3n-1)\tilde{B}_{1,0} + \frac{3}{2\pi}\epsilon(2n-1)\tilde{B}_{3,1}).$$

Proof. First, we relate the Gauss curvature integral with $\Gamma_{0,0}(\Omega)$ from Example 2.4.20.1

$$M_{2n-1}(\partial\Omega) = \frac{2c_{n,0,0}^{-1}}{(n-1)!}\Gamma_{0,0}(\Omega) = 2n\omega_{2n}\Gamma_{0,0}(\Omega). \tag{4.6}$$

Thus, from Proposition 4.1.7 and using that $c_{n,1,0}^{-1} = (n-1)!\omega_{2n-1}$, $c_{n,3,1}^{-1} = (n-2)!\omega_{2n-3}$ and $\omega_{2n-1} = (2n-1)\omega_{2n-3}/2\pi$ we obtain the result:

$$\begin{aligned}\delta_X M_{2n-1}(\partial\Omega) &= \frac{2\epsilon}{(n-1)!}(-c_{n,1,0}^{-1}(3n-1)\tilde{B}_{1,0} + 2c_{n,1,0}^{-1}(n-1)\tilde{\Gamma}_{1,0} + c_{n,3,1}^{-1}3\epsilon(n-1)\tilde{B}_{3,1})\\ &= 2\epsilon(-\omega_{2n-1}(3n-1)\tilde{B}_{1,0} + 2\omega_{2n-1}(n-1)\tilde{\Gamma}_{1,0} + 3\epsilon\omega_{2n-3}\tilde{B}_{3,1})\\ &= 2\epsilon\omega_{2n-1}(-(3n-1)\tilde{B}_{1,0} + 2(n-1)\tilde{\Gamma}_{1,0} + \frac{3}{2\pi}\epsilon(2n-1)\tilde{B}_{3,1}).\end{aligned}$$

□

4.2 Variation of the measure of complex r-planes intersecting a regular domain

Proposition 4.2.1. *Let $\Omega \subset \mathbb{CK}^n(\epsilon)$ be a regular domain, X a smooth vector field on $\mathbb{CK}^n(\epsilon)$ with flow ϕ_t and $\Omega_t = \phi_t(\Omega)$. Then*

$$\left.\frac{d}{dt}\right|_{t=0}\int_{\mathcal{L}_r^{\mathbb{C}}}\chi(\Omega_t\cap L_r)dL_r = \int_{\partial\Omega}\langle\partial\phi/\partial t, N\rangle\left(\int_{G_{n,r}^{\mathbb{C}}(\mathcal{D}_p)}\sigma_{2r}(\mathrm{II}|_V)dV\right)dx$$

where N is the outward normal field, $\mathcal{D}$ is the tangent distribution to $\partial\Omega$ and orthogonal to JN and $\sigma_{2r}(\mathrm{II}|_V)$ denotes the $2r$-th symmetric elementary function of II *restricted to $V \in G_{n-1,r}^{\mathbb{C}}(\mathcal{D}_p)$.*

Proof. The proof is similar to the one in [Sol06, Theorem 4] for real space forms.

Denote $G_{n-1,r}^{\mathbb{C}}(\mathcal{D}) = \{(p,l)\,|\, l \subseteq T_p\partial\Omega, \dim_{\mathbb{R}} l = 2r \text{ and } Jl = l\} = \bigcup_{p\in\partial\Omega} G_{n-1,r}^{\mathbb{C}}(\mathcal{D}_p)$.

For each $V \in G_{n-1,r}^{\mathbb{C}}(\mathcal{D}_p)$, we take the parallel translation V_t of V along $\phi_t(x)$. Recall that parallel translation preserves the complex structure (cf. [O'N83, page 326]). Then we project V_t orthogonally to $\mathcal{D}_{\phi_t(x)}$, obtaining a complex r-plane V_t' (at least for small values of t). We define

$$\begin{array}{rccc}\gamma: & G_{n-1,r}^{\mathbb{C}}(\mathcal{D})\times(-\epsilon,\epsilon) & \longrightarrow & \mathcal{L}_r^{\mathbb{C}}\\ & ((x,V),t) & \mapsto & \exp_{\phi_t(x)} V_t'.\end{array} \tag{4.7}$$

Proposition 3 in [Sol06] remains true, without change, in complex space forms. From this proposition and using a similar argument as in [Sol06, teorema 4] we get

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0}\int_{\mathcal{L}_r^{\mathbb{C}}}\chi(\Omega_t\cap L_r)dL_r &= \lim_{h\to 0}\frac{1}{h}\int_{G_{n-1,r}^{\mathbb{C}}(\mathcal{D})\times(0,h)}\sum \operatorname{sign}\langle\frac{\partial\phi}{\partial t}, N\rangle \operatorname{sign}(\sigma_{2r}(\mathrm{II}|_V))\gamma^* dL_r\\ &= \int_{G_{n-1,r}^{\mathbb{C}}(\mathcal{D})}\langle\frac{\partial\phi}{\partial t}, N\rangle \operatorname{sign}(\sigma_{2r}(\mathrm{II}|_V))\gamma_0^*(\iota_{d\phi\partial t}dL_r)\end{aligned}$$

where the sum on the second integral runs over the tangencies of L_r with the hypersurfaces $\partial\Omega_t$ with $0 < t < h$.

Consider a J-moving frame $\{g; g_1, Jg_1, ..., g_n, Jg_n\}$ such that $g((p,l),t) = \phi(p,t)$, $\gamma = \langle g; g_1, Jg_1, ..., g_r, Jg_r\rangle \cap \mathbb{CK}^n(\epsilon)$ and $Jg_n((p,l),t) = N_t$ (outward unit vector to $\partial\Omega_t$ at $\phi(p,t)$). We may assume that the moving frame is defined in a neighborhood of $\mathcal{L}_r^{\mathbb{C}}$ since we are only interested in regular points of γ.

Considerem the curve $L_r(t)$ given by the parallel translation of L_r along the geodesic given by N, the outward normal vector to $\partial\Omega_0$. Recall that parallel translations preserves the complex structure (cf. [O'N83, page 326]). If $P \in T_{L_r}\mathcal{L}_r^{\mathbb{C}}$ denotes the tangent vector to $L_r(t)$ at $t=0$, then

$$\begin{aligned}
\omega_i(P) &= \langle dg(P), g_i\rangle = \langle \frac{d}{dt} g(L_r(t)), g_i\rangle = 0, \quad i \in \{r+1, \overline{r+1}, ..., n-1, \overline{n-1}\},\\
\omega_{\overline{n}}(P) &= \langle dg(P), N\rangle = 1, \qquad\qquad (4.8)\\
\omega_{ij}(P) &= \langle \nabla g_i(P), g_j\rangle = \langle \frac{D}{dt} g_i(L_r(t)), g_j\rangle = 0, \quad j \in \{r+1, \overline{r+1}, ..., n, \overline{n}\},\ i \in \{1, \overline{1}, ..., r, \overline{r}\}.
\end{aligned}$$

The measure of complex r-planes in $\mathbb{CK}^n(\epsilon)$ is (cf. Proposition 1.5.5)

$$dL_r = \left| \bigwedge_{i=r+1}^{n} \omega_i \wedge \omega_{\overline{i}} \bigwedge_{\substack{i=1,...,r\\ j=r+1,...,n}} \omega_{ij}\omega_{\overline{ij}} \right|.$$

From (4.8), we get

$$dL_r = |\omega_{\overline{n}}| \iota_P dL_r$$

since $\iota_P dL_r = |\bigwedge_{h=r+1}^{n-1} \omega_h \wedge \overline{\omega_h} \wedge \omega_n \bigwedge \omega_{ij}|$.Thus,

$$\iota_{d\gamma\partial t} dL_r = |\omega_{\overline{n}}(d\gamma\partial t)| \iota_P dL_r + |\omega_{\overline{n}}| \iota_{d\gamma\partial t}\iota_P dL_r$$

with

$$\begin{aligned}
\omega_{\overline{n}}(d\gamma\partial t) &= \langle dg(d\gamma\partial t), N\rangle = \langle \frac{\partial\phi}{\partial t}, N\rangle,\\
\gamma_0^*(\omega_{\overline{n}})(v) &= \langle dg(d\gamma_0(v)), N\rangle = 0 \quad \forall v \in T_{(p,l)} G_{n-1,r}^{\mathbb{C}}(T_p\partial\Omega_0),
\end{aligned}$$

and we get

$$\gamma_0^*(\iota_{d\gamma\partial t} dL_r) = |\langle \frac{\partial\phi}{\partial t}, N\rangle| \gamma_0^*(\iota_P dL_r).$$

Finally, using that $\psi_0^*(\iota_P dL_r) = |\sigma_{2r}(\mathrm{II}|_V)| dV dx$, we get the result. □

Remark 4.2.2. The integral

$$\int_{G_{n,r}^{\mathbb{C}}} \sigma_{2r}(\mathrm{II}|_V) dV \qquad (4.9)$$

seems difficult to compute directly. However, we will find it by an indirect method. Recall that the analogous integral in real space forms is a multiple of an elementary symmetric function of the principal curvatures.

For $r = n-1$, the integral (4.9) can be easily computed in $\mathbb{CK}^n(\epsilon)$.

Corollary 4.2.3. *Let $\Omega \subset \mathbb{CK}^n(\epsilon)$ be a regular domain, X a smooth vector field on $\mathbb{CK}^n(\epsilon)$ with flow ϕ_t, and $\Omega_t = \phi_t(\Omega)$. Then,*

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\mathcal{L}_{n-1}^{\mathbb{C}}} \chi(L_{n-1} \cap \Omega_t) dL_{n-1} = \omega_{2n-1} \tilde{B}_{1,0}(\Omega).$$

Proof. From Proposition 4.2.1 we get the result since there is just one complex hyperplane tangent to a point in $\partial\Omega$. Thus,

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0}\int_{\mathcal{L}^{\mathbb{C}}_{n-1}}\chi(\Omega_t\cap L_{n-1})dL_{n-1} &= \int_{\partial\Omega_0}\langle\partial\phi/\partial t,N\rangle\int_{G^{\mathbb{C}}_{n-1,n-1}}\sigma_{2n-2}(\mathrm{II}|_V)dVdx\\
&= \int_{\partial\Omega_0}\langle\partial\phi/\partial t,N\rangle\sigma_{2n-2}(\mathrm{II}|_{\mathcal{D}})dx\\
&= \int_{\partial\Omega_0}\langle\partial\phi/\partial t,N\rangle\frac{\beta\wedge\theta_0^{n-1}}{(n-1)!}\\
&= \frac{c^{-1}_{n,1,0}}{(n-1)!}\tilde{B}_{1,0}(\Omega)\\
&= \omega_{2n-1}\tilde{B}_{1,0}(\Omega).
\end{aligned}$$

□

4.3 Measure of complex r-planes meeting a regular domain

4.3.1 In the standard Hermitian space

Using that the measure of complex r-planes in $\mathbb{C}^n$ meeting a regular domain is a linear combination of the Hermitian intrinsic volumes, and Propositions 4.2.1 and 4.1.7 we find explicitly the coefficients of this linear combination.

Theorem 4.3.1. *Let $\Omega\subset\mathbb{C}^n$ be a convex domain, X a smooth vector field over $\mathbb{C}^n$, ϕ_t the flow associated to X and $\Omega_t=\phi_t(\Omega)$. Then*

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0}\int_{\mathcal{L}^{\mathbb{C}}_r}\chi(\Omega_t\cap L_r)dL_r &= \mathrm{vol}(G^{\mathbb{C}}_{n-1,r})\omega_{2r+1}(r+1)\binom{n-1}{r}^{-1}\binom{n}{r}^{-1}\cdot\\
&\cdot\left(\sum_{q=\max\{0,n-2r-1\}}^{n-r-1}\binom{2n-2r-2q-1}{n-r-q}\frac{1}{4^{n-r-q-1}}\tilde{B}_{2n-2r-1,q}(\Omega)\right),
\end{aligned}\tag{4.10}$$

and

$$\begin{aligned}
\int_{\mathcal{L}^{\mathbb{C}}_r}\chi(\Omega\cap L_r)dL_r &= \mathrm{vol}(G^{\mathbb{C}}_{n-1,r})\omega_{2r}\binom{n-1}{r}^{-1}\binom{n}{r}^{-1}\cdot\\
&\cdot\left(\sum_{q=\max\{0,n-2r\}}^{n-r}\frac{1}{4^{n-r-q}}\binom{2n-2r-2q}{n-r-q}\mu_{2n-2r,q}(\Omega)\right).
\end{aligned}\tag{4.11}$$

Proof. In order to simplify the following computations, we consider

$$B'_{k,q}=B'_{k,q}(\Omega):=c^{-1}_{n,k,q}\mu_{k,q}(\Omega),\quad \Gamma'_{2q,q}=\Gamma'_{2q,q}(\Omega):=2c^{-1}_{n,2q,q}\mu_{2q,q}(\Omega).\tag{4.12}$$

and

$$\tilde{B}'_{k,q}=c^{-1}_{n,k,q}\tilde{B}_{k,q},\quad \tilde{\Gamma}'_{k,q}=2c^{-1}_{n,k,q}\tilde{\Gamma}_{k,q}.\tag{4.13}$$

The functional $\int_{\mathcal{L}^{\mathbb{C}}_r}\chi(\Omega\cap L_r)dL_r$ is a valuation on $\mathbb{C}^n$ with degree of homogeneity $2n-2r$. Thus, it can be expressed as a linear combination of the elements of degree $2n-2r$, $\{\mu_{2n-2r,q}|\max\{0,n-2r\}\leq q\leq n-r\}$ (cf. Definition 2.4.11). Then, by Remark 2.4.12 and (2.8), we have

$$\int_{\mathcal{L}^{\mathbb{C}}_r}\chi(\Omega\cap L_r)dL_r=\sum_{q=\max\{0,n-2r\}}^{n-r-1}C_qB'_{2n-2r,q}+D\Gamma'_{2n-2r,n-r}\tag{4.14}$$

for certain constants C_q, D which we wish to determine. This will be done by comparing the variation of both sides of this equality.

From here on we assume $2r < n$. The case $2r \geq n$ can be treated in the same way (cf. Remark 4.3.2).

By Proposition 4.1.7, the variation of the right hand side of (4.14) is a linear combination of the following type

$$\sum_{q=n-2r-1}^{n-r-1} c_q \tilde{B}'_{2n-2r-1,q} + \sum_{q=n-2r}^{n-r-1} d_q \tilde{\Gamma}'_{2n-2r-1,q} \tag{4.15}$$

where the coefficients c_q and d_q can be expressed in terms of a linear combination with known coefficients of the variables C_q and D, that still remain unknown.

The variation of the left hand side of (4.14), by Proposition 4.2.1 is

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\mathcal{L}_r^{\mathbb{C}}} \chi(\Omega_t \cap L_r) dL_r = \int_{\partial\Omega} \langle \partial\phi/\partial t, N\rangle \int_{G^{\mathbb{C}}_{n-1,r}} \sigma_{2r}(\mathrm{II}|_V) dV dx. \tag{4.16}$$

From Lemma 2.4.18 when pulling-back the form $\gamma_{k,q}$ from $N(\Omega)$ to $\partial\Omega$, one gets a polynomial expression $P_{k,q}$ of degree $2n-k-1$ in the coefficients h_{ij} of II with $i,j \in \{\overline{1}, 2, \overline{2}, \ldots, n, \overline{n}\}$. Moreover, for each q the monomials in $P_{k,q}$ containing only entries of the form h_{ii} contain the factor $h_{\overline{1}\overline{1}} = \mathrm{II}(JN, JN)$ and do not appear in any other $P_{k,q'}$ with $q' \neq q$. Therefore, every non-trivial linear combination of $\{P_{k,q}\}_q$ must contain the variable $h_{\overline{1}\overline{1}}$. On the other hand, the integral $\int_{G^{\mathbb{C}}_{n-1,r}} \sigma_{2r}(\mathrm{II}|_V) dV$ is a polynomial of the second fundamental form II restricted to the distribution $\mathcal{D} = \langle N, JN\rangle^{\perp}$, hence a polynomial not involving $h_{\overline{1}\overline{1}}$. Comparing the expressions of (4.15) and (4.16), it follows that $d_q = 0$ for all $q \in \{n-2r, \ldots, n-r-1\}$.

As c_q and d_q depend on C_q and D, we will obtain the value of c_q once we know the value of C_q and D. We will get their value from the equalities $\{d_q = 0\}$. Note that this gives r equations, since q runs from $n-2r$ to $n-r-1$ in (4.15). As for the unknowns, we need to find r constants C_q plus the constant D in (4.14).

We will get an extra equation by taking $\mathrm{II}|_{\mathcal{D}} = \mathrm{Id}$ and equating (4.16) to (4.15). Then, for any pair (n,r) we have a compatible linear system since constants in (4.14) exist. Next we find the solution.

Let us relate explicitly the coefficients $\{c_q\}$ and $\{d_q\}$ in (4.15) with C_q and D in (4.14). To simplify the range of the subscripts, we denote d_{n-r-a} with $a = 1, \ldots, r$ and c_{n-r-a} with $a = 1, \ldots, r+1$.

Coefficient d_{n-r-1}. From the variation of $B'_{k,q}$ in $\mathbb{C}^n$ (Proposition 4.1.7), the coefficient of $\tilde{\Gamma}'_{2n-2r-1,n-r-1}$ comes from the variation of $B'_{2n-2r,n-r-1}$ and $\Gamma'_{2n-2r,n-r}$. Then,

$$\begin{aligned} d_{n-r-1} &= -2r(n-r)D + (2n-2r-2(n-r-1))^2 C_{n-r-1} \\ &= 4C_{n-r-1} - 2r(n-r)D. \end{aligned} \tag{4.17}$$

Coefficient d_{n-r-a}, $a = 2, \ldots, r$. The coefficient of $\tilde{\Gamma}'_{2n-2r-1,n-r-a}$ comes from the variation of $B'_{2n-2r,n-r-a}$ and $B'_{2n-2r,n-r-a+1}$. Then,

$$\begin{aligned} d_{n-r-a} &= (2n-2r-2(n-r-a))^2 C_{n-r-a} - (2r+n-r-a+1-n)(n-r-a+1)C_{n-r-a+1} \\ &= 4a^2 C_{n-r-a} - (r-a+1)(n-r-a+1)C_{n-r-a+1}. \end{aligned} \tag{4.18}$$

Coefficient c_{n-r-1}. The coefficient of $\tilde{B}'_{2n-2r-1,n-r-1}$ comes from the variation of $B'_{2n-2r,n-r-1}$ and $\Gamma'_{2n-2r,n-r}$. Then,

$$\begin{aligned} c_{n-r-1} &= 4(r+1/2)(n-r)D - 4C_{n-r-1} \\ &= 2(2r+1)(n-r)D - 4C_{n-r-1}. \end{aligned} \tag{4.19}$$

Coefficient c_{n-r-a}, $a = 2,\ldots,r-2$. The coefficient of $\tilde{B}'_{2n-2r-1,n-r-a}$ comes from the variation of $B'_{2n-2r,n-r-a}$ and $B'_{2n-2r,n-r-a+1}$. Then,

$$\begin{aligned} c_{n-r-a} &= -2(2a)(2a-1)C_{n-r-a} + 2(r-a+3/2)(n-r-a+1)C_{n-r-a+1} \\ &= -4a(2a-1)C_{n-r-a} + (2r-2a+3)(n-r-a+1)C_{n-r-a+1}. \end{aligned} \tag{4.20}$$

Coefficient c_{n-2r-1}. The coefficient of $\tilde{B}'_{2n-2r-1,n-2r-1}$ comes from the variation of $B'_{2n-2r,n-2r}$. Then,

$$\begin{aligned} c_{n-2r-1} &= (2r-2(r+1)+3)(n-r-(r+1)+1)C_{n-2r} \\ &= (n-2r)C_{n-2r}. \end{aligned} \tag{4.21}$$

Now, we solve the linear system given by $\{d_{n-r-a} = 0\}$ where $a \in \{1,\ldots,r\}$. From equations (4.17) and (4.18) the system is given by

$$\left\{ \begin{array}{rcl} r(n-r)D &=& 2C_{n-r-1} \\ 4a^2 C_{n-r-a} &=& (n-r-a+1)(r-a+1)C_{n-r-a+1}. \end{array} \right.$$

Thus,

$$\begin{aligned} C_{n-r-a} &= \frac{(n-r-a+1)\cdot\cdots\cdot(n-r)(r-a+1)\cdot\cdots\cdot r}{2\cdot 4^{a-1}a^2(a-1)^2\cdot\cdots\cdot 1^2} D \\ &= \frac{(n-r)!r!}{2^{2a-1}(n-r-a)!(r-a)!a!a!} D \\ &= \frac{D}{2^{2a-1}}\binom{n-r}{a}\binom{r}{a}. \end{aligned} \tag{4.22}$$

To obtain the value of D, we compute $\int_{G^{\mathbb{C}}_{n-1,r}} \sigma_{2r}(p)dV$ and $\beta'_{2n-2r-1,n-r-a}$ in case $\mathrm{II}|_{\mathcal{D}}(p) = \lambda Id$ for $\lambda > 0$, which occurs when Ω is a geodesic ball. On one hand, we have

$$\int_{G^{\mathbb{C}}_{n-1,r}} \sigma_{2r}(p)(\lambda \mathrm{Id}|_V)dV = \lambda^{2r}\mathrm{vol}(G^{\mathbb{C}}_{n-1,r}).$$

On the other hand, if $\mathrm{II}|_{\mathcal{D}} = \lambda\mathrm{Id}$, then the connection forms satisfy $\alpha_{1i} = \lambda\omega_i$ and $\beta_{1i} = \lambda\omega_i$. Thus, $\theta_1 = 2\lambda\theta_2$ and $\theta_0 = \lambda^2\theta_2$ and we obtain

$$\begin{aligned} \beta'_{2n-2r-1,n-r-a}(p) &= \lambda^{2r}(\beta\wedge\theta_0^{r-a+1}\wedge\theta_1^{2a-2}\wedge\theta_2^{n-r-a})(p) \\ &= 2^{2a-2}\lambda^{2r}(\beta\wedge\theta_2^{n-1})(p) = 2^{2a-2}\lambda^{2r}(n-1)!. \end{aligned}$$

So, the equation

$$\mathrm{vol}(G^{\mathbb{C}}_{n-1,r}) = \sum_{a=1}^{r+1} c_{n-r-a}2^{2a-2}(n-1)!$$

must be satisfied.

Substituting equations (4.19), (4.20) and (4.21) in the last equation gives

$$\begin{aligned}
\frac{\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{(n-1)!} &= (2(2r+1)(n-r)D - 4C_{n-r-1}) \\
&\quad + \sum_{a=2}^{r} 2^{2a-2}((2r-2a+3)(n-r+a+1)C_{n-r-a+1} - 4a(2a-1)C_{n-r-a}) \\
&\quad + 2^{2r}(n-2r)C_{n-2r} \\
&= 2(2r+1)(n-r)D + 4((2r-1)(n-r-1)-1)C_{n-r-1} \\
&\quad + \sum_{a=2}^{r-1}(-2^{2a-2}4a(2a-1) + 2^{2a}(2r-2a+1)(n-r-a))C_{n-r-a} \\
&\quad + (2^{2r}(n-2r) - 2^{2r-2}4r(2r-1))C_{n-2r} \\
&= 2(2r+1)(n-r)D + \sum_{a=1}^{r} 2^{2a}((2r-2a+1)(n-r-a) - a(2a-1))C_{n-r-a} \\
&\overset{(4.22)}{=} D\left(2(n-r)!r!\sum_{a=0}^{r}\frac{(2r-2a+1)(n-r-a)-a(2a-1)}{(n-r-a)!(r-a)!a!a!}\right) \\
&= D\frac{2\,n!}{r!(n-r-1)!}
\end{aligned}$$

Thus,

$$\begin{aligned}
D &= \frac{\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{2\,n!}\binom{n-1}{r}^{-1}, \\
C_{n-r-a} &= \frac{\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{4^a n!}\binom{n-1}{r}^{-1}\binom{n-r}{a}\binom{r}{a}
\end{aligned}$$

and, for $2r < n$, we have

$$\begin{aligned}
\int_{\mathcal{L}^{\mathbb{C}}_r}\chi(\Omega\cap L_r)dL_r &= \sum_{a=1}^{r} C_{n-r-a}B'_{2n-2r,n-r-a} + D\Gamma'_{2n-2r,n-r} \\
&= \frac{\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{2\,n!}\binom{n-1}{r}^{-1}\left(\sum_{a=1}^{r}\binom{n-r}{a}\binom{r}{a}2^{-2a+1}B'_{2n-2r,n-r-a} + \Gamma'_{2n-2r,n-r}\right)
\end{aligned}$$

and

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0}\int_{\mathcal{L}^{\mathbb{C}}_r}\chi(\Omega_t\cap L_r)dL_r &= (2(2r+1)(n-r)D - 4C_{n-r-1})B'_{2n-2r-1,n-r-1} \\
&\quad + \sum_{a=2}^{r}((2r-2a+3)(n-r+a+1)C_{n-r-a+1} - 4a(2a-1)C_{n-r-a})B'_{2n-2r-1,n-r-a} \\
&\quad + (n-2r)C_{n-2r}B'_{2n-2r-1,n-2r-1} \\
&= \frac{\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{n!}\binom{n-1}{r}^{-1}\left(\sum_{a=1}^{r+1}\binom{n-r}{a}\binom{r+1}{a}\frac{a}{4^{a-1}}\tilde{B}'_{2n-2r-1,n-r-a}\right). \qquad (4.23)
\end{aligned}$$

Finally, we use the relation in (4.12) and (2.8) to obtain the result. □

Remark 4.3.2. If $2r \geq n$, then formula (4.10) follows directly from the relations among the different bases of valuations on $\mathbb{C}^n$ given in [BF08] and the following relation in [Ale03]

$$\int_{\mathcal{L}^{\mathbb{C}}_r}\chi(\Omega\cap L_r)dL_r = \frac{1}{O_{2r-1}}\int_{\mathcal{L}^{\mathbb{C}}_r}M_{2r-1}(\partial\Omega\cap L_r)dL_r = cU_{2(n-r),n-r}$$

for a certain constant c coming from the different normalizations in dL_r.

Corollary 4.3.3. *Let $\Omega \subset \mathbb{CK}^n(\epsilon)$ be a regular domain, X a smooth vector field over $\mathbb{CK}^n(\epsilon)$, ϕ_t the flow associated to X and $\Omega_t = \phi_t(\Omega)$. Then*

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\mathcal{L}_r^{\mathbb{C}}} \chi(\Omega_t \cap L_r) dL_r = \operatorname{vol}(G^{\mathbb{C}}_{n-1,r}) \omega_{2r+1}(r+1) \binom{n-1}{r}^{-1} \binom{n}{r}^{-1} \cdot$$
$$\cdot \left(\sum_{q=\max\{0,n-2r-1\}}^{n-r-1} \binom{2n-2r-2q-1}{n-r-q} \frac{1}{4^{n-r-q-1}} \tilde{B}_{2n-2r-1,q}(\Omega) \right). \tag{4.24}$$

Proof. Comparing equation (4.10) and Proposition 4.2.1 in case $\epsilon = 0$ shows that if Ω is a regular convex domain, then

$$\int_{\partial\Omega} \langle X, N \rangle \left(\int_{G^{\mathbb{C}}_{n,r}} \sigma_{2r}(\mathrm{II}|V) dV \right) dx$$

equals the right hand side of equation above. By taking a vector field X that vanishes outside an arbitrarily small neighborhood of a fixed $x \in \partial\Omega$, we deduce the following equality between forms

$$\left(\int_{G^{\mathbb{C}}_{n-1,r}(T_x\partial\Omega)} \sigma_{2r}(\mathrm{II}|_V) dV \right) dx = \frac{\omega_{2r+1}}{\binom{n-1}{r}\binom{n}{r}} \operatorname{vol}(G^{\mathbb{C}}_{n-1,r})(r+1) \cdot \tag{4.25}$$
$$\cdot \sum_{q=\max\{0,n-2r-1\}}^{n-r-1} \binom{2n-2r-2q-1}{n-r-q} \frac{c_{n,2n-2r-1,n-r-q}}{4^{n-r-q-1}} \beta \wedge \theta_0^{r-q+1} \wedge \theta_1^{2q-2} \wedge \theta_2^{n-r-q}.$$

This equation can be written as $P(\mathrm{II})dx = Q(\mathrm{II})dx$ where P and Q are polyomials with entries in the second fundamental form. These polynomials concide for any positive defined bilineal form. Thus, $P = Q$ and (4.25) holds for regular domains (not necessarily convex domains). Moreover, it is valid in $\mathbb{CK}^n(\epsilon)$ for any ϵ. Applying Proposition 4.2.1 we get the result. $\square$

Corollary 4.3.4. *Equation (4.11) holds for any regular domain not necessarily convex.*

Proof. Consider $\Omega_t = \phi_t(\Omega)$ with ϕ_t a given flow.

From the last corollary, it is known the variation of the left hand side of (4.11).

By Proposition 4.1.7, the variation of the right hand side is a linear combination of $\{B_{k,q}, \Gamma_{k,q}\}$. By Theorem 4.3.1 this linear combination coincides with the right hand side of (4.24).

Thus, the variation of both sides of (4.11) coincides. So, the difference between both members of (4.11) is constant.

Take ϕ_t such that $\phi_t(\Omega)$ converges to a point for $t \to \infty$. Both sides of (4.11) tend to zero when $t \to \infty$, thus their difference vanishes for all t. $\square$

4.3.2 In complex space forms

Theorem 4.3.5. *Let Ω be a regular domain in $\mathbb{CK}^n(\epsilon)$. Then*

$$\int_{\mathcal{L}_r^{\mathbb{C}}} \chi(\Omega \cap L_r) dL_r = \operatorname{vol}(G^{\mathbb{C}}_{n-1,r}) \binom{n-1}{r}^{-1} \cdot \tag{4.26}$$
$$\cdot \Big(\sum_{k=n-r}^{n-1} \epsilon^{k-(n-r)} \omega_{2n-2k} \binom{n}{k}^{-1} \left(\sum_{q=\max\{0,2k-n\}}^{k-1} \frac{1}{4^{k-q}} \binom{2k-2q}{k-q} \mu_{2k,q}(\Omega) + (k+r-n+1)\mu_{2k,k}(\Omega) \right)$$
$$+ \epsilon^r (r+1) \operatorname{vol}(\Omega) \Big).$$

Proof. We will show that both sides have the same variation δ_X with respect to any vector field X. This implies the result: one can take a deformation Ω_t of Ω such that Ω_t converges to a point. Then both sides of (4.26) have the same derivative, and both vanish in the limit.

The variation of the left hand side of (4.26) is given by Corollary 4.3.3. The variation of the right hand side can be computed by using Proposition 4.1.7, and $\delta_X V = 2\tilde{B}_{2n-1,n-1}$. In order to simplify the computations we rewrite the right hand side of (4.26) as

$$\mathcal{C}_r(\Omega) := \frac{\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{n!}\binom{n-1}{r}^{-1}\{\epsilon^r(r+1)n!V$$

$$+\sum_{j=n-r}^{n-1}\epsilon^{j-n+r}\left(\frac{j-n+r+1}{2}\Gamma'_{2j,j}+\sum_{q=\max(0,2j-n)}^{j-1}\frac{1}{4^{j-q}}\binom{n-j}{j-q}\binom{j}{q}B'_{2j,q}\right)\}.$$

By Proposition 3.8 we have

$$\delta_X\mathcal{C}_r(\Omega) = \frac{\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{n!}\binom{n-1}{r}^{-1}[\epsilon^r n(r+1)\delta_x\tilde{B}'_{2n-1,n-1} \tag{4.27}$$

$$+\sum_{j=n-r}^{n-1}\epsilon^{j-n+r}\frac{j-n+r+1}{2}\{-2(n-j)j\tilde{\Gamma}'_{2j-1,j-1}+2\epsilon(n-j-1)(j+1)\tilde{\Gamma}'_{2j+1,j}$$

$$+4(n-j+\frac{1}{2})j\tilde{B}'_{2j-1,j-1}+4\epsilon\left(\frac{j+1}{2}-(n-j)(2j+\frac{3}{2})\right)\tilde{B}'_{2j+1,j}+4\epsilon^2(n-j-1)(j+\frac{3}{2})\tilde{B}'_{2j+3,j+1}\}]$$

$$+\sum_{j=n-r}^{n-1}\sum_{q=\max\{0,2j-n\}}^{j-1}\frac{\epsilon^{j-n+r}}{4^{j-q}}\binom{n-j}{j-q}\binom{j}{q}\{(2j-2q)^2\tilde{\Gamma}'_{2j-1,q}$$

$$-(n+q-2j)q\tilde{\Gamma}'_{2j-1,q-1}+2(n+q-2j+\frac{1}{2})q\tilde{B}'_{2j-1,q-1}-2(2j-2q)(2j-2q-1)\tilde{B}'_{2j-1,q}$$

$$+2\epsilon(2j-2q)(2j-2q-1)\tilde{B}'_{2j+1,q+1}-2\epsilon(n-2j+q)(q+\frac{1}{2})\tilde{B}'_{2j+1,q}\}.$$

We will show that the previous expression is independent of ϵ; i.e. all the terms containing ϵ cancel out. Hence, $\delta_X\mathcal{C}_r(\Omega)$ coincides with (4.24) since we know this happens for $\epsilon=0$. This will finish the proof.

We concentrate first on the terms with $\tilde{B}'_{k,q}$. By putting together similar terms, the third line of (4.27) is

$$\sum_{h=n-r+1}^{n-1}\epsilon^{h-n+r}2\{(h-n+r+1)(n-h+\frac{1}{2})h+(h-n+r)(\frac{h}{2}-(n-h+1)(2h-\frac{1}{2})) \tag{4.28}$$

$$+(h-n+r+1)(n-h+1)(h-\frac{1}{2})\}\tilde{B}'_{2h-1,h-1}$$

$$-\epsilon^r\{(r+2)n-1\}\tilde{B}'_{2n-1,n-1}+(2r+1)(n-r)\tilde{B}'_{2n-2r-1,n-r-1}.$$

By putting together similar terms, the double sum in (4.27) (forgetting for the moment the terms with $\tilde{\Gamma}'_{k,q}$) becomes

$$\sum_{h=n-r}^{n-1}\sum_{a=\max(-1,2h-n-1)}^{h-2}\frac{\epsilon^{h-n+r}}{4^{h-a-1}}\binom{n-h}{h-a-1}\binom{h}{a+1}2(n+a-2h+\frac{3}{2})(a+1)\tilde{B}'_{2h-1,a}$$

$$-\sum_{h=n-r}^{n-1}\sum_{a=\max(0,2h-n)}^{h-1}\frac{\epsilon^{h-n+r}}{4^{h-a}}\binom{n-h}{h-a}\binom{h}{a}2(2h-2a)(2h-2a-1)\tilde{B}'_{2h-1,a}$$

$$+\sum_{h=n-r+1}^{n}\sum_{a=\max(1,2h-n-1)}^{h-1}\frac{\epsilon^{h-n+r}}{4^{h-a}}\binom{n-h+1}{h-a}\binom{h-1}{a-1}2(2h-2a)(2h-2a-1)\tilde{B}'_{2h-1,a}$$
$$-\sum_{h=n-r+1}^{n}\sum_{a=\max(0,2h-n-2)}^{h-2}\frac{\epsilon^{h-n+r}}{4^{h-a-1}}\binom{n-h+1}{h-a-1}\binom{h-1}{a}2(n-2h+a+2)(a+\frac{1}{2})\tilde{B}'_{2h-1,a}.$$

Note that the terms with $a=-1$ or $a=2h-n-2$ vanish, if they occur. Then, one checks that all the terms in the above expression cancel out except those with $h=n-r,n$, and those with $a=h-1$. Clearly the terms corresponding to $h=n-r$ are independent of ϵ. The terms with $h=n$ sum up $\epsilon^r(n-1)\tilde{B}'_{2n-1,n-1}$, and together with the similar term appearing in (4.28) cancel out the first term in (4.27). Finally, the terms with $a=h-1$ are cancelled with the sum in (4.28).

With a similar but shorter analysis one checks that the multiples of $\tilde{\Gamma}'_{k,q}$ cancel out completely. This shows that (4.27) is independent of ϵ, and finishes the proof. □

Remark 4.3.6. The coefficients of $\mu_{k,q}$ and vol in (4.26) were found by solving a linear system of equations, which we write down in the appendix.

4.4 Gauss-Bonnet formula in $\mathbb{CK}^n(\epsilon)$

Theorem 4.4.1. *Let Ω be a regular domain in $\mathbb{CK}^n(\epsilon)$. Then*

$$\omega_{2n}\chi(\Omega)=(n+1)\epsilon^n\mathrm{vol}(\Omega)+ \tag{4.29}$$
$$+\sum_{c=0}^{n-1}\frac{(n-c)\omega_{2n-2c}\epsilon^c}{n\binom{n-1}{c}}\left(\sum_{q=\max\{0,2c-n\}}^{c-1}\frac{1}{4^{c-q}}\binom{2c-2q}{c-q}\mu_{2c,q}(\Omega)+(c+1)\mu_{2c,c}(\Omega)\right).$$

Remark 4.4.2. For $\epsilon=0$ we have the Gauss-Bonnet formula in $\mathbb{C}^n\cong\mathbb{R}^{2n}$, where it is known

$$\chi(\Omega)=\frac{1}{2n\omega_{2n}}M_{2n-1}(\partial\Omega)=\mu_{0,0}(\Omega),$$

which coincides with the expression in the previous result.

Here we prove the certainty of (4.29) but in the appendix we give a constructive proof of the result.

Proof. We proceed analogously to the proof of Theorem 4.3.5. In fact, the same computations of the previous proof show (in case $r=n$) that the right hand side of (4.29) has null variation.

For $\epsilon=0$ equation (4.29) is the well know Gauss-Bonnet formula in $\mathbb{C}^n\cong\mathbb{R}^{2n}$. For $\epsilon\neq 0$, we take a smooth deformation of Ω to get a domain contained in a ball of radius r. Under this deformation, the right hand side of (4.29) remains constant. By taking r small enough, the difference between both sides can be made arbitarily small. Hence, they coincide. □

Although in (4.29) does not appear the Gauss curvature, we can easily get the following expression.

Corollary 4.4.3. *Let $\Omega\subset\mathbb{CK}^n(\epsilon)$ be a regular domain. Then,*

$$2n\omega_{2n}\chi(\Omega)=M_{2n-1}(\partial\Omega)+2n(n+1)\epsilon^n\mathrm{vol}(\Omega)+$$
$$+\sum_{c=1}^{n-1}2n\omega_{2n-2c}\epsilon^c\binom{n}{c}^{-1}\left(\sum_{q=\max\{0,2c-n\}}^{c-1}\frac{1}{4^{c-q}}\binom{2c-2q}{c-q}\mu_{2c,q}(\Omega)+(c+1)\mu_{2c,c}(\Omega)\right).$$

Proof. Apply relation (4.6) in (4.29). □

Remarks 4.4.4.

1. The Gauss-Bonnet-Chern formula in spaces of constant sectional curvature k and even dimension, for a regular domain Ω is given by

$$O_{2m-1}\chi(\Omega) = M_{2n-1}(\partial\Omega) + c_{n-3}M_{2n-3}(\partial\Omega) + \cdots + c_1 M_1(\partial\Omega) + (|k|)^{n/2}\mathrm{vol}(\Omega),$$

where c_j are known and depend on the curvature k.

Note that in the previous expression appear all mean curvature integrals with odd subscript and the volume. In formula (4.29) in $\mathbb{CK}^n(\epsilon)$, $\epsilon \neq 0$, also appear all the Hermitian intrinsic volumes in $\mathbb{CK}^n(\epsilon)$ with the first subscript odd.

2. In [Sol06] it is given an expression of the Gauss-Bonnet-Chern formula in space of constant sectional curvature k using the measure of planes of codimension 2 meeting the domain. The obtained formula for $\Omega \subset \mathbb{RK}^n(\epsilon)$ is

$$n\omega_n\chi(\Omega) = M_{n-1}(\partial\Omega) + \frac{2k}{\omega_{n-1}}\int_{\mathcal{L}_{n-2}} \chi(\Omega\cap L_{n-2})dL_{n-2}. \tag{4.30}$$

A natural question is whether in complex space forms, there exists a similar expression relating the Gauss curvature integral with the Euler characteristic and the measure of some complex planes meeting the domain

$$c_0\chi(\Omega) \overset{?}{=} M_{2n-1}(\partial\Omega) + \sum_{q=1}^{n-1} c_q \int_{\mathcal{L}_q^{\mathbb{C}}} \chi(\Omega\cap L_q)dL_q$$

or

$$c_0\chi(\Omega) \overset{?}{=} M_{2n-1}(\partial\Omega) + \sum_{q=0}^{n-1} M_{2q+1}(\partial\Omega) + \sum_{q=1}^{n-1} c_q \int_{\mathcal{L}_q^{\mathbb{C}}} \chi(\Omega\cap L_q)dL_q.$$

Taking variation on both sides in these expressions, we get that these expressions cannot hold in general (for $n = 2$ and $n = 3$ we can choose constants satisfying them). Anyway, in Theorem 4.4.5 we give a similar expression. Perhaps, if we knew a formula for the measure of totally real planes meeting a domain we could find a more similar expression.

3. For $n = 2$, the Gauss-Bonnet-Chern formula was already known in $\mathbb{CK}^n(\epsilon)$. It was given in [Par02]. From Theorem 4.4.1 we get the same expression, which can be written as

$$\chi(\Omega) = \frac{1}{\pi^2}\left(\frac{1}{2}\Gamma'_{0,0} + \epsilon\left(\frac{1}{4}B'_{2,0} + \Gamma'_{2,1}\right) + 6\epsilon^2\mathrm{vol}(\Omega)\right).$$

This expression can also be stated as

$$\chi(\Omega) = \frac{1}{2\pi^2}\left(M_3(\partial\Omega) + \frac{3\epsilon}{2}\left(M_1(\partial\Omega) + \int_{\partial\Omega} k_n(JN)\right) + 12\epsilon^2\mathrm{vol}(\Omega)\right) \tag{4.31}$$

and

$$\chi(\Omega) = \frac{1}{2\pi^2}\left(M_3(\partial\Omega) + 2\epsilon\int_{\mathcal{L}_1^{\mathbb{C}}}\chi(\partial\Omega\cap L_1)dL_1 + \frac{\epsilon}{2}\int_{\partial\Omega} k_n(JN) + 12\epsilon^2\mathrm{vol}(\Omega)\right).$$

In the following result we express the Euler characteristic in terms of the Gauss curvature integral, the volume, the measure of complex hyperplanes meeting a domain and the valuations $\mu_{2c,c}$. This formula generalizes (4.30) in complex space forms.

Theorem 4.4.5. *Let $\Omega \subset \mathbb{CK}^n(\epsilon)$ be a regular domain $\mathbb{CK}^n(\epsilon)$. Then,*

$$\begin{aligned}\omega_{2n}\chi(\Omega) &= \epsilon\int_{\mathcal{L}^{\mathbb{C}}_{n-1}}\chi(\Omega\cap L_{n-1})dL_{n-1} + \sum_{c=0}^{n}\frac{\epsilon^c\omega_{2n}}{\omega_{2c}}\mu_{2c,c}(\Omega)\\ &= \frac{1}{2n}M_{2n-1}(\partial\Omega) + \epsilon\int_{\mathcal{L}^{\mathbb{C}}_{n-1}}\chi(\Omega\cap L_{n-1})dL_{n-1} + \sum_{c=1}^{n}\frac{\epsilon^c\omega_{2n}}{\omega_{2c}}\mu_{2c,c}(\Omega).\end{aligned}$$

Proof. From Theorems 4.3.5 and 4.4.1 we get the stated formula

$$\begin{aligned}\chi(\Omega) &= \sum_{c=0}^{n-1}\frac{\epsilon^c\, c!}{\pi^c}\left(\sum_{q=\max\{0,2c-n\}}^{n-1}\frac{1}{4^{c-q}}\binom{2c-2q}{c-q}B_{2c,q}(\Omega) + (c+1)\Gamma_{2c,c}(\Omega)\right) + \frac{\epsilon^n(n+1)!}{\pi^n}\mathrm{vol}(\Omega)\\ &= \Gamma_{0,0}(\Omega) + \sum_{c=1}^{n-1}\frac{\epsilon^c\, c!}{\pi^c}\left(\sum_{q=\max\{0,2c-n\}}^{n-1}\frac{1}{4^{c-q}}\binom{2c-2q}{c-q}B_{2c,c}(\Omega) + (c+1)\Gamma_{2c,c}(\Omega)\right) + \frac{\epsilon^n(n+1)!}{\pi^n}\mathrm{vol}(\Omega)\\ &= \Gamma_{0,0}(\Omega) + \frac{\epsilon\, n!}{\pi^n}\sum_{c=1}^{n-1}\frac{\epsilon^{c-1}\, c!\pi^{n-c}}{n!}\left(\sum_{q=\max\{0,2c-n\}}^{n-1}\frac{1}{4^{c-q}}\binom{2c-2q}{c-q}B_{2c,c}(\Omega) + c\Gamma_{2c,c}(\Omega)\right) +\\ &\quad + \frac{\epsilon\, n!}{\pi^n}\sum_{c=1}^{n-1}\frac{\epsilon^{c-1}c!\pi^{n-c}}{n!}\Gamma_{2c,c}(\Omega) + \frac{\epsilon^n(n+1)!}{\pi^n}\mathrm{vol}(\Omega)\\ &= \Gamma_{0,0}(\Omega) + \frac{\epsilon\, n!}{\pi^n}\int_{\mathcal{L}^{\mathbb{C}}_{n-1}}\chi(\Omega\cap L_{n-1})dL_{n-1} + \sum_{c=1}^{n-1}\frac{\epsilon^c\, c!}{\pi^c}\Gamma_{2c,c}(\Omega) + \left(\frac{\epsilon^n(n+1)!}{\pi^n} - \frac{\epsilon^n\, n!n}{\pi^n}\right)\mathrm{vol}(\Omega).\end{aligned}$$

$\square$

4.5 Another method to compute the measure of complex lines meeting a regular domain

From Theorem 4.3.5 we can give an expression of the measure of complex lines meeting a regular domain (just taking $r = 1$). Here, we give another method to obtain this expression, using the results in Chapter 3.

4.5.1 Measure of complex lines meeting a regular domain in $\mathbb{C}^n$

Proposition 4.5.1. *Let $\Omega \subset \mathbb{C}^n$ be a regular domain. Then,*

$$\int_{\mathcal{L}^{\mathbb{C}}_1}\chi(\Omega\cap L_1)dL_1 = \frac{\omega_{2n-4}}{4n(n-1)}\left((2n-1)M_1(\partial\Omega) + \int_{\partial\Omega}k_n(Jn)\right).$$

Proof. Recall that each complex line is isometric to $\mathbb{C}$. Gauss-Bonnet formula in $\mathbb{C}^n$ for hypersurfaces $\partial\Omega$ states

$$M_{2n-1}(\partial\Omega) = 2n\omega_{2n}\chi(\Omega).$$

Applying Gauss-Bonnet formula in $\mathbb{C}$ and Proposition 3.3.2 with $s = 1$ we get the result

$$\int_{\mathcal{L}^{\mathbb{C}}_1}\chi(\Omega\cap L_1)dL_1 = \frac{1}{2\pi}\int_{\mathcal{L}^{\mathbb{C}}_1}\int_{\partial\Omega\cap L_1}k_g dp dL_1 = \frac{\omega_{2n-2}}{4n\omega_2}\left((2n-1)M_1(\partial\Omega) + \int_{\partial\Omega}k_n(Jn)\right).$$

$\square$

Although Gauss-Bonnet formula is known in $\mathbb{C}^n$ for $n \geq 1$, we cannot apply the same method to give the expression of the measure of s-planes meeting a regular domain since the integral $\int_{\mathcal{L}_r^{\mathbb{C}}} M_{2r-1}(\partial\Omega \cap L_r)dL_r$ is not in general known. In the next section we get an expression for this integral using the Gauss-Bonnet formula and the measure of complex r-planes meeting a regular domain.

4.5.2 Measure of complex lines meeting a regular domain in $\mathbb{CP}^n$ and $\mathbb{CH}^n$

The following result is given, for instance, in [ÁPF04].

Proposition 4.5.2 ([ÁPF04])**.** *Let Ω be a regular domain in $\mathbb{CP}^n$ or $\mathbb{CH}^n$. Then,*

$$\int_{\mathcal{L}_s^{\mathbb{C}}} \mathrm{vol}_{2s}(\Omega \cap L_s)dL_s = C\mathrm{vol}_{2n}(\Omega).$$

The value of the constant C, it is not known, but now we shall need it explicitly.

Proposition 4.5.3. *Let Ω be a regular domain in $\mathbb{CP}^n$ or $\mathbb{CH}^n$. Then,*

$$\int_{\mathcal{L}_s^{\mathbb{C}}} \mathrm{vol}_{2s}(\Omega \cap L_s)dL_s = \mathrm{vol}(G^{\mathbb{C}}_{n,n-s})\mathrm{vol}_{2n}(\Omega).$$

Proof. In order to find C we apply last proposition to a ball of radius R. Let L_s be a complex s-plane meeting B_R at a distance ρ from the center of the ball. From Lemma 3.2.13 in [Gol99], we have that the intersection $B_R \cap L_s$ is a ball of complex dimension s and radius r such that

$$\cos_\epsilon(R) = \cos_\epsilon(r)\cos_\epsilon(\rho).$$

The expression of the volume of a geodesic ball of radius R in $\mathbb{CK}^n(\epsilon)$ is (cf. [Gra73])

$$\mathrm{vol}_{2n}(B_R) = \frac{\pi^n}{|\epsilon|^n\, n!}\sin_\epsilon^{2n}(R).$$

Using this expression we get

$$\mathrm{vol}_{2s}(L_s \cap B_R) = \left(\frac{\pi}{|\epsilon|}\right)^s \frac{1}{s!}\left(\frac{\cos_\epsilon^2(R) - \cos_\epsilon^2(\rho)}{\cos_\epsilon^2(\rho)}\right)^s.$$

On the other hand, the Jacobian of the change of variables to polar coordinates is given by (cf. [Gra73])

$$\frac{\cos_\epsilon(R)\sin_\epsilon^{2n-1}(R)}{|\epsilon|^{n-1/2}}.$$

Then, using Proposition 1.5.8, we get

$$\begin{aligned}\int_{\mathcal{L}_s^{\mathbb{C}}} \mathrm{vol}_{2s}(L_s \cap B_R)dL_s &= \frac{\pi^s \mathrm{vol}(G^{\mathbb{C}}_{n,n-s})O_{2(n-s)-1}}{|\epsilon|^{n-1/2}s!}\sum_{i=0}^{s}(-1)^{i+1}\binom{s}{i}\cdot \\ &\quad \cdot \int_0^R \sin_\epsilon^{2(n-s)-1}(\rho)\cos_\epsilon^{2i}(R)\cos_\epsilon^{2(s-i)+1}(\rho)d\rho \\ &= \frac{\pi^s \mathrm{vol}(G^{\mathbb{C}}_{n,n-s})O_{2(n-s)-1}}{|\epsilon|^{n-1/2}s!}\cdot \\ &\quad \cdot \sum_{i=0}^{s}\sum_{j=0}^{s-i}\sum_{k=0}^{i}(-1)^{i+1}\binom{s}{i}\binom{s-i}{j}\binom{i}{k}\frac{\sin_\epsilon^{2(n-s+k+j)}(R)}{\sqrt{\epsilon}(n-s+j)}.\end{aligned}$$

From Proposition 4.5.2, this expression is a multiple of

$$\mathrm{vol}_{2n}(B_R) = \frac{\pi^n}{|\epsilon|^n n!}\sin_\epsilon^{2n}(R).$$

Thus, all terms in the sum are zero except for $2(n-s+k+j) = 2n$, i.e. $k+j=s$, which together with $j \le s-i$, $k \le i$ implies $j = s-i$, $k = i$, and we get

$$\begin{aligned}
&\int_{\mathcal{L}_s^{\mathbb{C}}} \mathrm{vol}_{2s}(L_s \cap B_R)dL_s \\
&\quad = \frac{\pi^s \mathrm{vol}(G^{\mathbb{C}}_{n,n-s})O_{2(n-s)-1}}{|\epsilon|^n s!}\sum_{i=0}^{s}(-1)^{i+1}\binom{s}{i}\frac{\sin_\epsilon^{2n}(R)}{2(n-i)} \\
&\quad = \frac{\pi^s \mathrm{vol}(G^{\mathbb{C}}_{n,n-s})O_{2(n-s)-1}}{2|\epsilon|^n}\sin_\epsilon^{2n}(R)\sum_{i=0}^{s}\frac{(-1)^{i+1}}{i!(s-i)!(n-i)} \\
&\quad = \frac{\pi^s \mathrm{vol}(G^{\mathbb{C}}_{n,n-s})O_{2(n-s)-1}(n-s-1)!}{2|\epsilon|^n n!}\sin_\epsilon^{2n}(R).
\end{aligned}$$

Finally, from equality

$$C\frac{\pi^n}{|\epsilon|^n n!} = \frac{\pi^s \mathrm{vol}(G^{\mathbb{C}}_{n,n-s})O_{2(n-s)-1}(n-s-1)!}{2|\epsilon|^n n!},$$

and using $O_{2(n-s)-1} = 2(n-s)\omega_{2(n-s)} = 2\frac{\pi^{n-s}}{(n-s-1)!}$, we get the value of the constant C. □

Corollary 4.5.4. *Let $\Omega \subset \mathbb{CK}^n(\epsilon)$ be a regular domain. Then,*

$$\int_{\mathcal{L}_1^{\mathbb{C}}} \chi(\Omega \cap L_1)dL_1 = \frac{\omega_{2n-4}}{4n(n-1)}\left((2n-1)M_1(\partial\Omega) + \int_{\partial\Omega} k_n(JN) + 8n\epsilon\mathrm{vol}(\Omega)\right).$$

where $k_n(JN)$ denotes the normal curvature in the direction JN.

Proof. Using Gauss-Bonnet formula in $\mathbb{H}^2(-4)$ we have (cf. [San04, page 309])

$$\int_{\mathcal{L}_1^{\mathbb{C}}} \chi(\Omega \cap L_1)dL_1 = \frac{1}{2\pi}\int_{\mathcal{L}_1^{\mathbb{C}}} M_1(\partial\Omega \cap L_1)dL_1 - \frac{2}{\pi}\int_{\mathcal{L}_1^{\mathbb{C}}} \mathrm{vol}(\Omega \cap L_1)dL_1,$$

and using Proposition 4.5.3 with $s=1$, and Proposition 3.3.2 we get the result. □

Corollary 4.5.5. *If Ω is a regular domain in $\mathbb{CK}^2(\epsilon)$, $\epsilon \neq 0$, then*

$$\int_{\Omega \cap L_1 \neq \emptyset} \chi(\partial\Omega \cap L_1)dL_1 = \frac{1}{4}\left(M_1(\partial\Omega) - \frac{1}{3\epsilon}M_3(\partial\Omega) + 4\epsilon\mathrm{vol}(\Omega) + \frac{2\pi^2}{3\epsilon}\chi(\Omega)\right).$$

Proof. From previous corollary, with $n=2$, we have

$$\int_{\mathcal{L}_1^{\mathbb{C}}} \chi(\Omega \cap L_1)dL_1 = \frac{1}{8}\left(3M_1(\partial\Omega) + \int_{\partial\Omega} k_n(JN) + 16\epsilon\mathrm{vol}(\Omega)\right).$$

Isolating $\int_{\partial\Omega} k_n(JN)$ in expression (4.31) we get the stated result. □

Note that the previous corollary cannot be extended to $\epsilon = 0$ since the expression (4.31) does not contain the term $\int_{\partial\Omega} k_n(JN)$.

4.6 Total Gauss curvature integral $\mathbb{C}^n$

Theorem 4.6.1. *If $\Omega \subset \mathbb{C}^n$ is a regular domain, then*

$$\int_{\mathcal{L}_r^{\mathbb{C}}} M_{2r-1}(\partial\Omega\cap L_r)dL_r = 2r\omega_{2r}^2 \mathrm{vol}(G_{n-1,r}^{\mathbb{C}})\binom{n-1}{r}^{-1}\binom{n}{r}^{-1}\cdot$$
$$\cdot\left(\sum_{q=\max\{0,n-2r\}}^{n-r}\frac{1}{4^{n-r-q}}\binom{2n-2r-2q}{n-r-q}\mu_{2n-2r,q}(\Omega)\right).$$

Proof. On one hand, by Gauss-Bonnet formula in $\mathbb{C}^n$ and the relation (4.6), we have

$$\int_{\mathcal{L}_r^{\mathbb{C}}} M_{2r-1}(\partial\Omega\cap L_r)dL_r = 2r\omega_r\int_{\mathcal{L}_r^{\mathbb{C}}}\chi(\Omega\cap L_r)dL_r = 2r\omega_{2r}\int_{\mathcal{L}_r^{\mathbb{C}}}\mu_{0,0}(\Omega\cap L_r)dL_r. \tag{4.32}$$

On the other hand, by Theorem 4.3.1, we have

$$\int_{\mathcal{L}_r^{\mathbb{C}}}\chi(\partial\Omega\cap L_r)dL_r = \mathrm{vol}(G_{n-1,r}^{\mathbb{C}})\omega_{2r}\binom{n-1}{r}^{-1}\binom{n}{r}^{-1}$$
$$\cdot\left(\sum_{q=\max\{0,n-2r\}}^{n-r}\frac{1}{4^{n-r-q}}\binom{2n-2r-2q}{n-r-q}\mu_{2n-2r,q}(\Omega)\right).$$

If we equate both expressions and we use the relation (4.6), we get the result. □

Chapter 5

Other Crofton formulas

In the previous chapter we give an expression for the measure of complex planes intersecting a regular domain in a complex space form. Complex planes in $\mathbb{CK}^n(\epsilon)$ are totally geodesic submanifolds, but, by Theorem 1.4.6, there are other totally geodesic submanifolds. Totally real planes are also totally geodesic submanifolds in $\mathbb{CK}^n(\epsilon)$ for any ϵ (cf. Theorem 1.4.6). Moreover, for $\epsilon = 0$, all submanifolds generated by the exponential map of a vector subspace holomorphically isometric to $\mathbb{C}^k \oplus \mathbb{R}^{k-2p}$ are totally geodesic. Note that complex planes and totally real planes are particular cases of these submanifolds, for $(k,p) = (2p,p)$ and $(k,p) = (k,0)$, respectively.

In this chapter we obtain an expression for the measure of planes of type $(2n-p, n-p)$, the so-called *coisotropic planes*, intersecting a domain in $\mathbb{C}^n$, and an expression for the measure of Lagrangian planes in $\mathbb{CK}^n(\epsilon)$.

5.1 Space of (k,p)-planes

First, we recall the definition of (k,p)-plane in $\mathbb{C}^n$, as it is given in [BF08].

Definition 5.1.1. Suppose that V is a real vector space and $\mathcal{L}_k^n(V)$ denote the space of all affine subspaces of dimension k in V. If $V = \mathbb{C}^n$, considered as a real vector space, then the *space of* (k,p)*-planes*, $\mathcal{L}_{k,p}(\mathbb{C}^n) \subset \mathcal{L}_k^n(\mathbb{C}^n)$ is defined as the subset of all subspace of (real) dimension k that can be expressed as the orthogonal direct sum of a complex subspace of complex dimension p and a totally real subspace of (real) dimension $(k-2p)$.

We denote the elements of $\mathcal{L}_{k,p}(\mathbb{C}^n)$ by $L_{k,p}$ and the Grassmannian of all (k,p)-planes through the origin in a vector space V by $G_{n,k,p}(V)$.

From the previous definition, $\mathcal{L}_{k,p}(\mathbb{C}^n)$ is the orbit of $\mathbb{C}^p \oplus \mathbb{R}^{k-2p}$ under the action of $\mathbb{C}^n \rtimes U(n)$ (which are the holomorphic isometries of $\mathbb{C}^n$).

The notion of (k,p)-plane was extended to $\mathbb{CH}^n$. In [Gol99] and [Hsi98], they are defined as particular cases of the so-called linear submanifolds.

Definition 5.1.2. The image of the exponential map from a point $x \in \mathbb{CH}^n$ of a vector subspace in $T_x\mathbb{CH}^n$ is called *linear submanifold.*

The image of the exponential map from a point $x \in \mathbb{CH}^n$ of a (k,p)-plane in $T_x\mathbb{CH}^n$ is called *linear* (k,p)*-plane.*

This definition could be also stated in $\mathbb{CP}^n$, but doing so, we obtain submanifolds with singularities except when the submanifold is totally geodesic, i.e. for complex planes (which correspond to $(2p,p)$-planes), and for totally real planes (which correspond to $(k,0)$-planes).

In $\mathbb{CH}^n$, linear submanifolds are not always totally geodesic submanifolds. They are totally geodesic just for complex planes and totally real planes (cf. Theorem 1.4.6).

5.1.1 Bisectors

As in complex hyperbolic space there are no totally geodesic real hypersurfaces, it is natural to look for real hypersurfaces with similar properties to those expected for a totally geodesic real hypersurface. In [Gol99, page 152] it is answered that they are the so-called *bisectors*, also denoted by *spinal superfaces.*

Definition 5.1.3. Let z_1, z_2 be two different points in $\mathbb{CH}^n$. The *bisector equidistant from* z_1 *and* z_2 is defined as

$$\mathfrak{E}(z_1, z_2) = \{z \in \mathbb{CH}^n \mid d(z_1, z) = d(z_2, z)\}$$

where $d(z, z_i)$ denotes the distance between points z_i and z at $\mathbb{CK}^n(\epsilon)$ (see Proposition 1.2.3).

Definition 5.1.4. Let z_1, z_2 be two different points in $\mathbb{CH}^n$.

- The complex geodesic Σ defined by z_1 and z_2 is the *complex spine of the bisector* $\mathfrak{E}(z_1, z_2)$.
- The *real spine of the bisector* $\mathfrak{E}(z_1, z_2)$, $\sigma(z_1, z_2)$ is the intersection between the bisector and the complex spine, i.e.

$$\sigma(z_1, z_2) = \mathfrak{E}(z_1, z_2) \cap \Sigma(z_1, z_2) = \{z \in \Sigma \mid d(z_1, z) = d(z_2, z)\}.$$

- A *slide of* $\mathfrak{E}$ is a complex hyperplane $\Pi_\Sigma^{-1}(s)$ where $\Pi : \mathbb{CH}^n \longrightarrow \Sigma$ denotes the orthogonal projection over Σ.

Remark 5.1.5. 1. The set of all slides in a bisector defines a foliation of the bisector by complex hyperplanes.

2. The real spine is a (real) geodesic in $\mathbb{CH}^n$ since Σ is totally geodesic and isometric to $\mathbb{H}^2$, and in the real hyperbolic space, the bisector line of two given points is a geodesic.

3. Each geodesic $\gamma \subset \mathbb{CH}^n$ is the real spine of a unique bisector. Indeed, take the complex line Σ containing γ and the orthogonal projection Π_Σ to Σ. Then, $\Pi_\Sigma^{-1}(\gamma)$ defines a bisector.

Example 5.1.6. In $\mathbb{CH}^2$ with the projective model, the bisector with respect to $z_1 = [(1, 0, i)]$ and $z_2 = [(1, 0, -i)]$ is

$$\mathfrak{E}(z_1, z_2) = \{[(1, z, t)] \in \mathbb{CH}^n \mid z \in \mathbb{C}, t \in \mathbb{R}\}.$$

This expression can be obtained directly using the formula for the distance between 2 points given at Proposition 1.2.3.

The complex spine is $\left\{\left[\left(1, 0, i\dfrac{\lambda - \mu}{\lambda + \mu}\right)\right] \text{ with } \lambda, \mu \in \mathbb{C} \text{ both nonzero}\right\}$ and the real spine is $\{[1, 0, t]\}$.

Proposition 5.1.7. *The isometries of* $\mathbb{CH}^n$ *act transitively over the space of bisectors.*

Proof. Using the correspondence between bisectors and real geodesics, we have that isometries act transitively over the space of bisectors, since they do so over the space of real geodesics. □

It is known that there are no non-trivial isometries which fix pointwise a bisector , since they are not totally geodesic hypersurfaces (if $\epsilon \neq 0$). Anyway, we can consider the reflection with respect to a slice S of the bisector. This reflexion fixes pointwise the slice S and lies the bisector invariant. Moreover, each of these reflexions is also a reflection with respect to the spine σ, thus, it fixes the points in $\sigma \cap S$.

Bisectors are *hypersuperfaces of cohomogenity one*. That is, the orbit, for the group of the isometries fixing a bisector, of a point (except for the points in the real spine, which are of null measure in the bisector) in a bisector is a submanifold of codimension 1 inside the bisector; a submanifold of codimension 2 in $\mathbb{CH}^n$ (cf. [GG00]).

Thus, bisectors are not *homogeneous hypersurfaces*. By definition a hypersurface is homogeneous if the orbit of each point (for the isometry group fixing the hypersurface) is all the hypersurface.

Let us study the orbit of a point in a bisector.

A geodesic $\gamma(t) \subset \mathbb{CH}^n$ uniquely determines a tube $T(r)$ of radius r and a bisector $\mathfrak{E}$ with real spine γ.

We study the relation between these two hypersurfaces, obtaining the orbit of a point in the bisector in terms of the tube containing the point.

Proposition 5.1.8. *Let $p \in \mathfrak{E}$. Consider r such that $p \in T(r) \cap \mathfrak{E}$ where $T(r)$ denotes the tube of radius r along the real spine γ of $\mathfrak{E}$. The orbit of p by the isometries fixing $\mathfrak{E}$, is given by $T(r) \cap \mathfrak{E}$.*

Proof. Each point of the orbit O_p of p belongs to $T(r)$ since the isometries fixing the bisector fix the spine, and they preserve distances. Then, $O_p \subset T(r) \cap \mathfrak{E}$.

Every point in $T(r) \cap \mathfrak{E}$ belongs to the orbit of p. Indeed, if $q \in T(r) \cap \mathfrak{E}$ then $d(q, \gamma(t)) = d(p, \gamma(t))$, a necessary condition to be q in the orbit of p. The projection of the points p, q to Σ can or cannot be the same point. Let us prove that in both cases there exists an isometry g such that fixes the bisector and $g(p) = q$.

Suppose that p and q project at Σ to the same point x. Then p and q belong to the same slide of $\mathfrak{E}$. Let us define an isometry g fixing x and the bisector. We denote by v the tangent vector to the real spine γ at x. As the isometries fixing the bisector also fix γ, g satisfies $dg(v) = \pm v$. Moreover, as isometries preserve the holomorphic angle $dg(Jv) = \pm Jv$. Thus, g fixes the complex spine Σ and its orthogonal complement at x, which is the slide containing p and q, and is isometric to $\mathbb{CH}^{n-1}$. Now, in $\mathbb{CH}^{n-1}$ there exists an isometry $\tilde{g}$ such that $\tilde{g}(p) = q$ (since $\mathbb{CH}^{n-1}$ is a homogeneous space). Therefore, g defined by $g(x) = x$, $dg(v) = \pm v$, $dg(Jv) = \pm Jv$ and $dg(u) = \tilde{dg}(u)$, for all $u \in \langle v, Jv \rangle^{\perp}$, gives an isometry of $\mathbb{CK}^n(\epsilon)$ fixing $\mathfrak{E}$ and such that $g(p) = q$.

Suppose that p and q do not project at Σ to the same point. Let $x = \Pi_\Sigma p$ and $y = \Pi_\Sigma q$. Note that x, $y \in \gamma$ since p and q are points in the bisector. Then, there exists a reflection ρ such that $\rho(x) = y$ and $\rho(\gamma) = \gamma$. Thus, $d\rho$ takes the orthogonal space of $\{\gamma'_x, J\gamma'_x\}$ to the orthogonal space of $\{\gamma'_y, J\gamma'_y\}$. Moreover, $\tilde{q} = \rho(q)$ satisfies $\Pi_\Sigma \tilde{q} = \Pi_\Sigma q$. If we consider the points $\tilde{q}$ and q, then we are in the previous case and we know that there exists an isometry g such that $g(\tilde{q}) = q$. □

From this proposition we have that the subset of bisectors containing a point is a non-compact set, in the space of bisectors. In the next proposition, we prove that the measure of bisectors meeting a regular domain is infinite.

Remark 5.1.9. Denote by dL the invariant density of the space of bisectors $\mathfrak{B}$ and by dL_1 the invariant density of the space of real geodesics in $\mathbb{CH}^n$. By the correspondence between geodesics and bisectors we have

$$dL = dL_1.$$

If Σ denotes the complex line containing a real geodesic γ, then the density of the space of real geodesics can be expressed as

$$dL_1 = dL_1^{\Sigma} d\Sigma,$$

where $d\Sigma$ denotes the invariant density of complex lines and dL_1^Σ the invariant density of the space of real geodesics contained in Σ. Using polar coordinates ρ, θ in Σ we get

$$dL_1^\Sigma d\Sigma = \cos_{4\epsilon}(\rho) d\rho d\theta d\Sigma. \tag{5.1}$$

Proposition 5.1.10. *The measure of bisectors meeting a regular domain in $\mathbb{CK}^n(\epsilon)$ with $\epsilon < 0$ is infinite.*

Proof. We prove that the measure of bisectors intersecting a ball B of radius R in $\mathbb{CK}^n(\epsilon)$ is infinite, i.e.

$$\int_{\mathfrak{B}} \chi(B \cap L) dL = +\infty.$$

Consider the expression for the density of bisectors in (5.1). Denote the volume element of $\mathbb{CK}^n(\epsilon)$ by dx. Fixed a bisector L by x, then dx can be expressed as $dx = dx_1 dx_L$ where dx_L denotes the volume element in the bisector and dx_1 the length element in the direction N_x orthogonal to the bisector at x.

If N_y is the normal vector to the bisector at $y = \Pi_\Sigma(x)$, then the plane spanned by N_y (which coincides with the normal vector to the real spine inside Σ) and the tangent vector u to the geodesic joining y and x is a totally real plane and contains N_x.

Thus, the plane $\exp_y(\mathrm{span}\{N_y, u\})$ is isometric to $\mathbb{H}^2(\epsilon)$. If r denotes the distance between y and x, then $dx_1 = \cos_\epsilon(r) dy_1$ where dy_1 denotes the length element in the direction N_y.

In the previous remark, we give an expression for dL_1. Now, we use it taking polar coordinates with respect to $y \in \Sigma$, so $\rho = 0$. Then, $dy_1 = d\rho$ and

$$dx_L dL_1 = dx_L dL_1^\Sigma d\Sigma = dx_L d\theta d\rho d\Sigma = d\theta dx_L dy_1 d\Sigma = \frac{1}{\cos_\epsilon(r)} d\theta d\Sigma dx.$$

On the other hand, fixed a regular domain $\Omega \subset \mathbb{CK}^n(\epsilon)$ it follows, for some constant $C > 0$, $\mathrm{vol}(\Omega) < \frac{1}{C}\chi(\Omega)$.

Then,

$$\begin{aligned}
\int_{\mathfrak{B}} \chi(B \cap L) dL &> C \int_{\mathfrak{B}} \mathrm{vol}(B \cap L) dL = C \int_{\mathfrak{B}} \int_{B \cap L} dx_L dL \\
&= C \int_B \int_{\mathcal{L}_1^{\mathbb{C}}} \int_0^{2\pi} \frac{1}{\cos_\epsilon(r)} d\theta d\Sigma dx = 2\pi C \int_B \int_{\mathcal{L}_1^{\mathbb{C}}} \frac{1}{\cos_\epsilon(r)} d\Sigma dx \\
&\overset{(1.16)}{=} 2\pi C \int_B \int_{G^C_{n,n-1}} \int_{L^{\mathbb{C}}_{(n-1)[x]}} \frac{\cos^2_\epsilon(r)}{\cos_\epsilon(r)} dp_{n-1} dG_{n,n-1} dx \\
&= 2\pi \mathrm{vol}(B) \int_{\mathbb{CH}^{n-1}(\epsilon)} \cos_\epsilon(r) dx = +\infty.
\end{aligned}$$

□

5.2 Variation of the measure of planes meeting a regular domain

At Chapter 4 we give an expression for the measure of complex r-planes meeting a regular domain in $\mathbb{CK}^n(\epsilon)$. Now, we give a generalization of this result for the space of (k,p)-planes in $\mathbb{C}^n$, and for the space of totally real k-planes in $\mathbb{CK}^n(\epsilon)$.

First, we need the expression of the density of the space of (k,p)-planes with respect to the forms ω_{ij} defined at (1.13).

Lemma 5.2.1. *1. In $\mathbb{C}^n$, the space $\mathcal{L}_{k,p}$ is a homogeneous space and*

$$\mathcal{L}_{k,p} \cong U(n) \ltimes \mathbb{C}^n/(U(p) \times O(k-2p) \times U(n-k+p) \ltimes \mathbb{R}^n).$$

Let $\{g; g_1, g_2, ..., g_{2n-1}, g_{2n}\}$ with $g_{2i} = Jg_{2i+1}$ be a J-moving frame adapted to a (k,p)-plane in g such that $\{g_1, Jg_1, ..., g_p, Jg_p, g_{2p+1}, ..., g_{2k-1}\}$ expand the tangent space to the (k,p)-plane. The invariant density of $\mathcal{L}_{k,p}$ is given by

$$dL_{k,p} = \left| \bigwedge_i \omega_i \bigwedge_{j,i} \omega_{ji} \right| \tag{5.2}$$

where $i \in \{2p+2, 2p+4, ..., 2k, 2k+1, 2k+2, ..., 2n\}$ and $j \in \{1, 3, ..., 2p-1, 2p+1, 2p+3, ..., 2k-1\}$.

2. In $\mathbb{CK}^n(\epsilon)$, $\epsilon \neq 0$, the space of complex p-planes $\mathcal{L}_p^{\mathbb{C}}$ and the space of totally real k-planes $\mathcal{L}_k^{\mathbb{R}}$ are homogeneous spaces and

$$\mathcal{L}_p^{\mathbb{C}} \cong U_\epsilon(n)/(U_\epsilon(p) \times U(n-p)),$$

$$\mathcal{L}_k^{\mathbb{R}} \cong U_\epsilon(n)/(O_\epsilon(k) \times U(n-k)),$$

where

$$U_\epsilon(n) = \begin{cases} U(1+n), & \text{if } \epsilon > 0, \\ U(1,n), & \text{if } \epsilon < 0. \end{cases}, \quad O_\epsilon(k) = \begin{cases} O(1+k), & \text{if } \epsilon > 0, \\ O(1,k), & \text{if } \epsilon < 0. \end{cases}$$

Moreover, fixed a J-moving frame as in the previous statement, the expression (5.2) remains true.

Proof. 1. By Lemma 1.5.1 we have that the isometry group of $\mathbb{C}^n$ acts transitively over J-basis. Thus, there exists an isometry that carries a fixed (k,p)-plane to another.

The isotropy group of a (k,p)-plane in $\mathbb{C}^n$ is isomorphic to $U(p) \times O(k-2p) \times U(n-k+p)$ since (k,p)-planes in $\mathbb{C}^n$ are totally geodesic submanifolds and the tangent space at each point is isometric to $\mathbb{C}^p \oplus \mathbb{R}^{k-2p}$.

The density can be obtained using the theory of moving frames that we have discussed in Section 1.3.

2. The arguments for the previous case are also valid, since we restrict to totally geodesic submanifolds.

□

The following proposition is a generalization of the Proposition 4.2.1 for any (k,p)-plane in $\mathbb{C}^n$.

Proposition 5.2.2. *Let $\Omega \subset \mathbb{C}^n$ be a regular domain, X a smooth vector field defined at $\mathbb{C}^n$ with ϕ_t the flow associated to X and $\Omega_t = \phi_t(\Omega)$. Then, in the space of (k,p)-planes $\mathcal{L}_{k,p}$ in $\mathbb{C}^n$, it is satisfied*

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\mathcal{L}_{k,p}} \chi(\Omega_t \cap L_{k,p}) dL_{k,p} = \int_{\partial\Omega} \langle \partial\phi/\partial t, N \rangle \int_{G_{n,k,p}(T_x\partial\Omega)} \sigma_k(II|_V) dV dx$$

where N is the outward normal field at $\partial\Omega$ and $\sigma_k(II|_V)$ denotes the k-th symmetric elementary function of II restricted to $V \in G_{n,k,p}(T_x\partial\Omega)$, the Grassmanian of the (k,p)-planes contained in the tangent space of $\partial\Omega$ at x.

It is also satisfied the following extension of Proposition 4.2.1 for totally real planes in $\mathbb{CK}^n(\epsilon)$.

Proposition 5.2.3. *Let $\Omega \subset \mathbb{CK}^n(\epsilon)$ be a regular domain, X a smooth vector field defined at $\mathbb{CK}^n(\epsilon)$ with ϕ_t the flow associated to X and $\Omega_t = \phi_t(\Omega)$. Then, in the space of totally real k-plane $\mathcal{L}_k^{\mathbb{R}}$ in $\mathbb{CK}^n(\epsilon)$, it is satisfied*

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\mathcal{L}_k^{\mathbb{R}}} \chi(\Omega_t \cap L_{k,p}) dL_{k,p} = \int_{\partial\Omega} \langle \partial\phi/\partial t, N\rangle \int_{G_{n,k,0}(T_x\partial\Omega)} \sigma_k(II|_V) dV dx$$

where N is the outward normal field at $\partial\Omega$ and $\sigma_k(II|_V)$ denotes the k-th symmetric elementary function of II restricted to $V \in G_{n,k,0}(T_x\partial\Omega)$, the Grassmanian of the totally real k-planes contained in the tangent space of $\partial\Omega$ at x.

Proof. This proof is analog to the proof of Proposition 4.2.1, since the expression for the density of the space of totally real planes in (5.2) holds. We just have to modify the construction of the map γ in (4.7).

For every $x \in \partial\Omega$ consider the curve $c(t) = \varphi_t(x)$. For every t, let $\mathcal{D}_{c(t)} = \langle N_{c(t)}, JN_{c(t)}\rangle^\perp \subset d\varphi_t(T_x\partial\Omega)$ the complex hyperplane tangent to $\varphi_t(\partial\Omega)$ at $c(t)$. If $\nabla_{\partial t}$ denotes the covariant derivative of $\mathbb{CK}^n(\epsilon)$ along $c(t)$, we define

$$\nabla^D_{\partial t} X(t) = \pi_t(\nabla_{\partial t} X(t))$$

where $\pi_t : T_{c(t)}\mathbb{CK}^n(\epsilon) \to \mathcal{D}_{c(t)}$ denotes the orthogonal projection. Given a vector $X \in T_x\partial\Omega$, there exists a unique vector field $X(t)$ defined along $c(t)$ such that $\nabla^D_{\partial t} X(t) = 0$ (it can be proved in the same way as the existence of the usual parallel translation). This define a linear map $\psi_t : \mathcal{D}_x \to \mathcal{D}_{c(t)}$, which preserves the complex structure J since

$$\nabla^D_{\partial t} JX(t) = \pi_t(\nabla_{\partial t} JX(t)) = \pi_t(J\nabla_{\partial t} X(t)) = J\pi_t(\nabla_{\partial t} X(t)).$$

Finally, we extend ψ_t linearly to $\psi_t : T_x\partial\Omega \to d\varphi_t(T_x\partial\Omega)$ such that $\psi_t(JN_x) = JN_{c(t)}$. This map takes totally real planes into totally real planes. So, we can define the new map γ as

$$\begin{array}{rccl} \gamma : & G_{n,k,p}(T\partial\Omega) \times (-\epsilon,\epsilon) & \longrightarrow & \mathcal{L}_{k,p} \\ & ((x,V),t) & \mapsto & \exp_{\phi_t(x)} \psi_t(V) \end{array}.$$

□

5.3 Measure of real geodesics in $\mathbb{CK}^n(\epsilon)$

The following result, obtained straightforward from the last proposition, states that in complex space forms, the measure of real geodesics meeting a regular domain is a multiple of the area of the domain (as in real space forms).

Theorem 5.3.1. *Let $\Omega \subset \mathbb{CK}^n(\epsilon)$ be a regular domain, let X be a smooth vector field over $\mathbb{CK}^n(\epsilon)$, let ϕ_t be the flow associated to X and let $\Omega_t = \phi_t(\Omega)$. Then*

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\mathcal{L}_1^{\mathbb{R}}} \chi(\Omega_t \cap L_1) dL_1^{\mathbb{R}} = O_{2n+1}(\tilde{B}_{2n-2,n-2}(\Omega) + \tilde{\Gamma}_{2n-2,n-1}(\Omega))$$

and

$$\int_{\mathcal{L}_1^{\mathbb{R}}} \chi(\Omega \cap L_1) dL_1^{\mathbb{R}} = \omega_{2n}\mu_{2n-1,n-1}(\Omega) = \frac{\omega_{2n}}{2}\mathrm{vol}(\partial\Omega).$$

Proof. From Proposition 5.2.2 we have

$$\left.\frac{d}{dt}\right|_{t=0}\int_{\mathcal{L}_1^{\mathbb{R}}}\chi(\Omega_t\cap L_1)dL_1^{\mathbb{R}}=\int_{\partial\Omega}\langle\partial\phi/\partial t,N\rangle\int_{G^{\mathbb{R}}_{n,1,0}(T_x\partial\Omega)}\sigma_1(\mathrm{II}|_V)dV dx.$$

Let us study the integral with respect to the Grassmanian of geodesics in the tangent space of each point $x\in\partial\Omega$. We denote by $\{f_1,\dots,f_{2n-1}\}$ the principal directions at x. Then,

$$\begin{aligned}\int_{G^{\mathbb{R}}_{n,1,0}(T_x\partial\Omega)}\sigma_1(\mathrm{II}|_V)dV&=\frac{1}{2}\int_{S^{2n-1}}k_n(v)dv=\frac{1}{2}\sum_{i=1}^{2n-1}\int_{S^{2n-1}}\langle v,f_i\rangle^2k_idv\\&=\frac{1}{2}\sum_{i=1}^{2n-1}k_i\int_{S^{2n-1}}\langle v,f_i\rangle^2dv\overset{(3.1.2)}{=}\frac{O_{2n-1}}{4n}\sum_{i=1}^{2n-1}k_i=\frac{O_{2n-1}(2n-1)}{4n}\mathrm{tr(II)}.\end{aligned}$$

Thus, by Examples 2.4.20.3 and 2.4.20.4

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0}\int_{\mathcal{L}_1^{\mathbb{R}}}\chi(\Omega_t\cap L_1)dL_1^{\mathbb{R}}&=\frac{O_{2n-1}(2n+1)}{4n}\int_{\partial\Omega}\langle X,N\rangle\mathrm{tr(II)}dx\\&=\frac{O_{2n-1}}{4n}\int_{N(\Omega)}\langle X,N\rangle\left(\frac{1}{(n-1)!}\gamma\wedge\theta_2^{n-1}+\frac{1}{(n-2)!}\beta\wedge\theta_1\wedge\theta_2^{n-2}\right)\\&=\frac{O_{2n-1}}{4\,n!}\left(\int_{N(\Omega)}\langle X,N\rangle\gamma\wedge\theta_2^{n-1}+(n-1)\int_{N(\Omega)}\langle X,N\rangle\beta\wedge\theta_1\wedge\theta_2^{n-2}\right)\\&=\frac{\omega_{2n}}{2\,(n-1)!}\left(\tilde{\Gamma}'_{2n-2,n-1}(\Omega)+(n-1)\tilde{B}'_{2n-2,n-2}(\Omega)\right)\\&=O_{2n+1}(\tilde{B}_{2n-2,n-2}(\Omega)+\tilde{\Gamma}_{2n-2,n-1}(\Omega)).\end{aligned}$$

In $\mathbb{C}^n$, the valuation $\int_{\mathcal{L}_1^{\mathbb{R}}}\chi(\partial\Omega\cap L_1)dL_1$ has degree $2n-1$, so, it is a multiple of $B_{2n-1,n-1}(\Omega)$, which has variation (cf. Proposition 4.1.7)

$$\begin{aligned}\delta_X\mu_{2n-1,n-1}(\Omega)&=c_{n,2n-1,n-1}(2c^{-1}_{n,2n-2,n-1}\tilde{\Gamma}_{2n-2,n-1}(\Omega)+c^{-1}_{n,2n-2,n-2}(n-1)\tilde{B}_{2n-2,n-2}(\Omega))\\&=c_{n,2n-2,n-1}(\tilde{\Gamma}'_{2n-2,n-1}(\Omega)+(n-1)\tilde{B}'_{2n-2,n-2}(\Omega))\\&=\frac{1}{(n-1)!\omega_1}(\tilde{\Gamma}'_{2n-2,n-1}(\Omega)+(n-1)\tilde{B}'_{2n-2,n-2}(\Omega)).\end{aligned}$$

Therefore, comparing both variations we get the stated result in $\mathbb{C}^n$. The same expression holds for $\epsilon\neq 0$ since the variation of $\mu_{2n-1,n-1}$ does not depend on ϵ.

The relation with the volume of the boundary of the domain is obtained from the relation of $\beta_{2n-1,n-1}$ with the second fundamental form given in Example 2.4.20.5. □

5.4 Measure of real hyperplanes in $\mathbb{C}^n$

The measure of real hyperplanes intersecting a regular domain in $\mathbb{C}^n$ also follows immediately from Proposition 5.2.2. This particular case has interest by its own since real hyperplanes are submanifolds of codimension 1.

Theorem 5.4.1. *Let $\Omega\subset\mathbb{C}^n$ be a regular domain, X a smooth vector field over $\mathbb{C}^n$, ϕ_t the flow associated to X and $\Omega_t=\phi_t(\Omega)$. Then*

$$\left.\frac{d}{dt}\right|_{t=0}\int_{\mathcal{L}_{2n-1,n-1}}\chi(\Omega_t\cap L_{2n-1,n-1})dL_{2n-1,n-1}=O_{2n+1}\tilde{\Gamma}_{0,0}(\Omega)$$

and

$$\int_{\mathcal{L}_{2n-1,n-1}}\chi(\Omega\cap L_{2n-1,n-1})dL_{2n-1,n-1}=\omega_{2n-1}\mu_{1,0}(\Omega).$$

Proof. From Proposition 4.2.1 we have

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0}\int_{\mathcal{L}_{2n-1,n-1}}\chi(\Omega_t\cap L_{2n-1,n-1})dL_{2n-1,n-1} &= \int_{\partial\Omega_0}\langle X,N\rangle\int_{G_{n,2n-1,n-1}(T_x\partial\Omega)}\sigma_{2n-1}(\mathrm{II}|_V)dVdx\\ &= \int_{\partial\Omega}\langle X,N\rangle\sigma_{2n-1}(\mathrm{II})dx\\ &= \int_{N(\Omega)}\langle X,N\rangle\frac{1}{(n-1)!}\gamma\wedge\theta_0^{n-1}\\ &= \frac{2c_{n,0,0,}^{-1}}{(n-1)!}\tilde{\Gamma}_{0,0}(\Omega).\end{aligned}$$

In $\mathbb{C}^n$, the valuation $\int_{\mathcal{L}_{2n-1,n-1}}\chi(\partial\Omega\cap L_{2n-1,n-1})dL_{2n-1,n-1}$ has degree 1, thus, it is a multiple of $\mu_{1,0}$, which has variation

$$\delta_X\mu_{1,0} = 2c_{n,1,0}c_{n,0,0}^{-1}\tilde{\Gamma}_{0,0}.$$

Then, comparing both expressions we obtain the result. □

Remark 5.4.2. From Example 2.4.20, it follows the equality

$$\mu_{1,0}(\Omega) = \frac{1}{\omega_{2n-1}}\int_{\partial\Omega}\det(\mathrm{II}|_{\mathcal{D}}) = \frac{1}{\omega_{2n-1}}M_{2n-2}^{\mathcal{D}}(\partial\Omega).$$

On the other hand, there is just one linearly independent valuation in the space of continuous translation and $U(n)$-invariant valuations in $\mathbb{C}^n$ of degree 1. Thus,

$$\mu_{1,0}(\Omega) = cM_{2n-2}(\partial\Omega),$$

and the measure of real hyperplanes in $\mathbb{C}^n$ meeting a regular domain is a multiple of the so-called "mean width", as in the Euclidean space.

5.5 Measure of coisotropic planes in $\mathbb{C}^n$

A subspaces of $\mathbb{C}^n$ is called *coisotropic* if its orthogonal is a totally real plane.

Lemma 5.5.1. *The $(2n-p,n-p)$-planes in $\mathbb{C}^n$ are the coisotropic planes.*

Proof. If $L\in\mathcal{L}_{2n-p,n-p}$, then $L^\perp$ has dimensió $2n-(2n-p)=p$. The dimension of the maximal complex subspace contained in $L^\perp$ is $n-(n-p)-p=0$. Thus, $L^\perp$ is a totally real plane.

Reciprocally, if $L^\perp$ is a totally real p-plane, then L has dimension $2n-p$ and the maximal complex subspace has dimension $n-p$. □

Lemma 5.5.2. *Let $S\subset\mathbb{C}^n$ be a hypersurface and $L\in\mathcal{L}_{2n-p,n-p}$, $p\in\{1,\ldots,n\}$, be a $(2n-p,n-p)$-plane tangent in S at p. If N denotes a normal vector to S at x, then $JN\in T_xL$.*

Proof. As L is a $(2n-p,n-p)$-plane, we can consider, at each point, a basis of its tangent space of the form $\{e_1,Je_1,\ldots,e_{n-p},Je_{n-p},e_{n-p+1},e_{n-p+2},\ldots,e_n\}$, in a way such that $Je_i\perp T_xL$ for $i\in\{n-p+1,\ldots,n\}$. Moreover, we can complete this basis to a basis of $T_x\mathbb{C}^n$ with the vectors $\{J_{e_{n-p+1}},\ldots,Je_n\}$.

On the other hand, at $x\in L\cap S$, is it satisfied $T_xL\subset T_xS$, thus, $N\perp T_xL$, i.e.

$$\begin{aligned}\langle N,e_i\rangle &= 0, \quad \forall\, i\in\{1,\ldots,n\},\\ \langle N,Je_j\rangle &= 0, \quad \forall\, j\in\{1,\ldots,n-p\}.\end{aligned} \tag{5.3}$$

Now, if $JN=\sum_{i=1}^n\alpha_ie_i+\sum_{i=1}^n\beta_iJe_i$, then $N=-\sum_{i=1}^n\alpha_iJe_i+\sum_{i=1}^n\beta_ie_i$. Using (5.3) we get $JN=\sum_{i=n-p+1}^n\alpha_ie_i$. Thus, $JN\in T_xL$. □

From this lemma, we can prove the following result.

Theorem 5.5.3. *Let $\Omega \subset \mathbb{C}^n$ be a regular domain, X a smooth vector field defined in $\mathbb{C}^n$, ϕ_t the flow associated to X and $\Omega_t = \phi_t(\Omega)$. Then,*

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\mathcal{L}_{2n-p,n-p}} \chi(\Omega_t \cap L)dL = \frac{\mathrm{vol}(G_{n,2n-p,n-p})\omega_{2n-p+1}}{(n-1)!}(2n-p+2)\binom{n}{p-1}^{-1} \cdot \tag{5.4}$$
$$\cdot \sum_{q=\max\{0,p-n-1\}}^{\lfloor \frac{p-1}{2} \rfloor} \frac{4^{q-p+1}}{(2n+2q-2p+3)}\binom{2n+2q-2p+1}{n+q-p+1}^{-1} \tilde{\Gamma}_{p-1,q}(\Omega),$$

and

$$\int_{\mathcal{L}_{2n-p,n-p}} \chi(\Omega \cap L)dL = \frac{\mathrm{vol}(G_{n,2n-p,n-p})\omega_{2n-p}}{(n-1)!}\binom{n}{p-1}^{-1} \cdot$$
$$\cdot \sum_{q=\max\{0,p-n\}}^{\lfloor \frac{p}{2} \rfloor} \binom{2n+2q-2p-1}{n+q-p-1}^{-1} \frac{4^{q-p}}{2n+2q-2p+1}\mu_{p,q}(\Omega).$$

Proof. First of all, we prove that for the space of coisotropics planes, the variation of the measure does not have contribution in $\tilde{B}_{p,q}$. From Proposition 5.2.2 we have

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\mathcal{L}_{2n-p,n-p}} \chi(\Omega_t \cap L)dL = \int_{\partial\Omega} \langle \partial\phi/\partial t, N\rangle \int_{G_{n,2n-p,n-p}(T_x\partial\Omega)} \sigma_{2n-p}(\mathrm{II}|_V)dVdx$$

but each $V \in G_{n,2n-p,n-p}(T_x\partial\Omega)$, by the previous lemma, contains the JN direction (with N the outward normal vector to $\partial\Omega$ at x), so that $\mathrm{II}|_V$ always contains the entry corresponding to the normal curvature of the direction JN. From Lemma 2.4.18 we have that only the polynomials obtained from $\phi^*(\gamma_{k,q})$ contain this entry of the second fundamental form.

In order to find the constants, we solve a linear system. First, note that the functional $\int_{\mathcal{L}_{2n-p,n-p}} \chi(\Omega \cap L)dL$ is a valuation in $\mathbb{C}^n$ with homogeneous degree p. Thus, it can be expressed as a linear combination of the Hermitian intrinsic volumes with the same degree

$$\int_{\mathcal{L}_{2n-p,n-p}} \chi(\Omega \cap L)dL = \sum_{q=\max\{0,p-n\}}^{\lfloor \frac{p-1}{2} \rfloor} A_{p,q}\mu_{p,q}(\Omega) \tag{5.5}$$

for some $A_{p,q}$, which we want to determine.

Taking the variation in both sides, we find the value of these constants. By Proposition 4.1.7, the variation on the right hand side of (5.5) is

$$\sum_{q=\max\{0,p-n-1\}}^{\lfloor \frac{p}{2} \rfloor -1} (A_{p,q}2c_{n,p,q}c^{-1}_{n,p-1,q}(p-2q)^2 \tag{5.6}$$
$$- A_{p,q+1}2c_{n,p,q+1}c^{-1}_{n,p-1,q}(n-p+q+1)(q+1))\tilde{\Gamma}_{p-1,q}$$
$$+ (A_{p,q+1}2c_{n,p,q+1}c^{-1}_{n,p-1,q}(n-p+q+3/2)(q+1)$$
$$- A_{p,q}2c_{n,p,q}c^{-1}_{n,p-1,q}(p-2q)(p-2q-1))\tilde{B}_{p-1,q}$$
$$+ A_{p,\lfloor \frac{p}{2} \rfloor}2c_{n,p,\lfloor \frac{p}{2} \rfloor}c_{n,p-1,\lfloor \frac{p}{2} \rfloor}(p-2\lfloor \frac{p}{2} \rfloor)^2\tilde{\Gamma}_{p-1,\lfloor \frac{p}{2} \rfloor}.$$

Imposing that the variation vanishes on $\tilde{B}_{p-1,q}$ we get some equations, from which we obtain the relations

$$A_{p,q+1} = \frac{n-p+q+1}{(n-p+q+3/2)}A_{p,q}. \tag{5.7}$$

So, each $A_{p,q+1}$ is a multiple of $A_{p,\max\{0,p-n\}}$. To find this value, we need another equation, obtained taking $\mathrm{II}|_{\mathcal{D}} = \lambda \mathrm{Id}$, $\lambda \in \mathbb{R}^+$, and equation the expression (5.6) with the variation of the Proposition 5.2.2. Then, for each (n,r) we have a compatible linear system, since constant in (5.5) exist. Moreover, by (5.7) they are unique. Doing so, we get, in the same way as in the proof of Proposition 4.3.1, the desired result. Finally, we get the variation substituting the obtained values of $A_{p,q}$ at (5.6). □

An interesting particular case of the last theorem is the case of Lagrangian planes. This is the case stated by Alesker [Ale03] as a remarkable case of study. From the previous theorem we can give explicitly the constant in the theorem of Alesker reproduced at 2.3.5, but with respect to the Hermitian intrinsic volumes defined by Bernig-Fu, and not directly by the bases defined by Alesker.

Corollary 5.5.4. *Let Ω be a regular domain in $\mathbb{C}^n$ with piecewise smooth boundary. Then,*

$$\int_{\mathcal{L}_n^{\mathbb{R}}} \chi(\Omega\cap L)dL = \frac{\mathrm{vol}(G_{n,n,0})\omega_n}{n!} \sum_{q=0}^{\lfloor\frac{n-1}{2}\rfloor} \binom{2q-1}{q-1}^{-1} \frac{4^{q-n}}{2q+1}\mu_{n,q}(\Omega).$$

where $\mathcal{L}_n^{\mathbb{R}}$ denotes the space of Lagrangian planes in $\mathbb{C}^n$.

5.6 Measure of Lagrangian planes in $\mathbb{CK}^n(\epsilon)$

Using the same techniques as in Chapter 4, it can be proved the following result.

Theorem 5.6.1. *Let $\Omega \subset \mathbb{CK}^n(\epsilon)$ be a regular domain, X a smooth vector field defined at $\mathbb{CK}^n(\epsilon)$, ϕ_t the flow associated to X and $\Omega_t = \phi_t(\Omega)$. Then,*

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\mathcal{L}_n^{\mathbb{R}}} \chi(\Omega_t\cap L)dL = \mathrm{vol}(G_{n,n,0})\omega_{n+1}\frac{(n+2)}{n!} \sum_{q=0}^{\lfloor\frac{n-1}{2}\rfloor} \frac{4^{q-n+1}}{2q+3}\binom{2q+1}{q+1}^{-1} \tilde{\Gamma}_{n-1,q}(\Omega),$$

and

if n is odd

$$\int_{\mathcal{L}_n^{\mathbb{R}}} \chi(\Omega\cap L)dL = \frac{\mathrm{vol}(G_{n,n,0})\omega_n}{n!} \sum_{q=0}^{\frac{n-1}{2}} \binom{2q-1}{q-1}^{-1} \frac{4^{q-n}}{2q+1}\mu_{n,q}(\Omega), \tag{5.8}$$

and if n is even

$$\int_{\mathcal{L}_n^{\mathbb{R}}} \chi(\Omega\cap L)dL = \frac{\mathrm{vol}(G_{n,n,0})}{n!}\cdot \tag{5.9}$$

$$\cdot\left(\sum_{q=0}^{\frac{n}{2}} \binom{2q-1}{q-1}^{-1} \frac{4^{q-n}\omega_n}{2q+1}\mu_{n,q}(\Omega) + \sum_{i=1}^{\frac{n}{2}} \epsilon^i \binom{n}{\frac{n}{2}+i}^{-1} \frac{2^{-n+1}\omega_{n-2i}}{n+1}\mu_{n+2i,\frac{n}{2}+i}(\Omega)\right).$$

Proof. In the same way as in the proof of Theorem 4.3.5, it is enough to prove that the variation in both sides coincides.

The variation on the left hand side of (5.9) and (5.8) coincides and is independent on ϵ. Thus, it coincides with the variation in (5.4).

We compute the variation on the right hand side by using Proposition 4.1.7. Here, we just reproduce the computations when n is odd. For n even, a similar, but longer study can be done to verify expression (5.9).

Denote by $\mathcal{E}_n(\Omega)$ the right hand side of (5.8). Then, by (5.4) we have

$$
\begin{aligned}
\delta_X\mathcal{E}_n(\Omega) = & \frac{\mathrm{vol}(G_{n,n,0})\omega_n}{n!}\sum_{q=0}^{\frac{n-1}{2}}\binom{2q-1}{q-1}^{-1}\frac{4^{q-n}}{2q+1}2c_{n,n,q}\cdot \\
&\cdot\Big(c_{n,n-1,q}^{-1}(n-2q)^2\tilde{\Gamma}_{n-1,q} - c_{n,n-1,q-1}^{-1}q^2\tilde{\Gamma}_{n-1,q-1} \\
&+ c_{n,n-1,q-1}^{-1}q(q+1/2)\tilde{B}_{n-1,q-1} - c_{n,n-1,q}(n-2q)(n-2q-1)\tilde{B}_{n-1,q} \\
&+ \epsilon(c_{n,n+1,q+1}^{-1}(n-2q)(n-2q-1)\tilde{B}_{n+1,q+1} - c_{n,n+1,q}^{-1}q(q+1/2)\tilde{B}_{n+1,q})\Big) \\
= & 2\frac{\mathrm{vol}(G_{n,n,0})\omega_n}{n!}\{\sum_{q=0}^{\frac{n-3}{2}}\{\left(\frac{c_{n,n,q}4^{q-n}(n-2q)^2}{(2q+1)\binom{2q-1}{q-1}} - \frac{c_{n,n,q+1}4^{q-n+1}(q+1)^2}{(2q+3)\binom{2q+1}{q}}\right)c_{n,n-1,q}^{-1}\tilde{\Gamma}_{n-1,q} \\
&+\left(\frac{c_{n,n,q+1}4^{q-n+1}(q+1)(q+3/2)}{(2q+3)\binom{2q+1}{q}} - \frac{c_{n,n,q}4^{q-n}(n-2q)(n-2q-1)}{(2q+1)\binom{2q-1}{q-1}}\right)c_{n,n-1,q}^{-1}\tilde{B}_{n-1,q}\} \\
&+\epsilon\sum_{q=1}^{\frac{n-1}{2}}\left(\frac{c_{n,n,q-1}4^{q-n-1}(n-2q+2)(n-2q+1)}{(2q-1)\binom{2q-3}{q-2}} - \frac{c_{n,n,q}4^{q-n}q(q+\frac{1}{2})}{(2q+1)\binom{2q-1}{q-1}}\right)c_{n,n+1,q}^{-1}\tilde{B}_{n+1,q}\}.
\end{aligned}
$$

In order to prove the result, it suffices to prove that this expression is independent on ϵ. As for $\epsilon = 0$ we know that $\delta_X\mathcal{E}_n(\Omega)$ coincides with (5.4), we get the result.

Now, to prove the independence of ϵ, we collect the coefficient for each $\tilde{B}_{n-1,q}$ and $\tilde{B}_{n+1,q}$, and we prove that they vanish. $\square$

Appendix

This appendix contains a constructive proof of Theorems 4.3.5 and 4.4.1.

Proof of Theorem 4.3.5

We prove that it is possible to find constants $\alpha_{k,q}$ such that

$$\int_{\mathcal{L}_r^{\mathbb{C}}} \chi(\Omega \cap L_r) dL_r = \sum_{k,q} \alpha_{k,q} B_{k,q}(\Omega) + \sum_{j=1}^{\lfloor n/2 \rfloor} \alpha_{2j,j} \Gamma_{2j,j}(\Omega) + \alpha_{2n,n} \mathrm{vol}(\Omega) \tag{A.10}$$

where $\max\{0, k-n\} \leq q < k/2 \leq n$.

For $\epsilon = 0$, the existence of these constants follows from the fact that Hermitian intrinsic volumes constitute a basis of smooth valuations. If $\epsilon \neq 0$, we cannot ensure this fact. Anyway, we find the value of the previous constants imposing that the variation in both sides of (A.10) coincides. This is enough to prove (A.10). Indeed, take a deformation Ω_t of Ω such that Ω_t converges to a point. Then, both sides of (A.10) have the same variation and it vanishes in the limit.

The variation of the left hand side of (A.10) is given in Corollary 4.3.3. The variation of the right hand side can be computed using Proposition 4.1.7 and $\delta_X \mathrm{vol} = 2\tilde{B}_{2n-1,n-1}$.

In the variation of the left hand side, just appear the terms $\{\tilde{B}_{2n-2r-1,q}\}_q$. Thus, the variation of the right hand side can only have these terms. On the other hand, the variation of a Hermitian intrinsic volume $B_{k,q}$ with k even (resp. odd) has only terms $\tilde{B}_{a,b}$ and $\tilde{\Gamma}_{a',b'}$ with a, a' odd (resp. even) (cf. Proposition 4.1.7). As the variation of the left hand side has only non-vanishing terms with odd subscript, we just consider the valuations with first even subscript. Doing also the change in (4.12), expression (A.10) reduces to

$$\int_{\mathcal{L}_r^{\mathbb{C}}} \chi(\Omega \cap L_r) dL_r = \sum_{k=1}^{n-1} (\sum_{q=\max\{0,2k-n\}}^{k-1} C_{2k,q} B'_{2k,q}(\Omega) + D_{2k,k} \Gamma'_{2k,k}(\Omega)) + d\mathrm{vol}(\Omega). \tag{A.11}$$

Now, we start the study to find constants $C_{k,q}$, $D_{2q,q}$, d such that

$$\begin{aligned}
&\sum_{k=1}^{n-1} (\sum_{q=\max\{0,2k-n\}}^{k-1} C_{2k,q} \delta B'_{2k,q}(\Omega) + D_{2k,k} \delta \Gamma'_{2k,k}(\Omega) + d\delta \mathrm{vol}(\Omega) \\
&= \frac{\mathrm{vol}(G^{\mathbb{C}}_{n-1,r}) \omega_{2r+1} (r+1)}{\binom{n-1}{r}\binom{n}{r}} \left(\sum_{q=\max\{0,n-2r-1\}}^{n-r-1} \binom{2n-2r-2q-1}{n-r-q} \frac{1}{4^{n-r-q-1}} \tilde{B}'_{2n-2r-1,q}(\Omega) \right).
\end{aligned}$$

By Proposition 4.1.7, this equation gives rise to a linear system. We write this linear system in matrix form $Ax = b$. Consider the vector of unknowns as

$$x^t = (C_{2,0}, D_{2,1}, C_{4,0}, C_{4,1}, D_{4,2}, \ldots, C_{2c,\max\{0,2c-n\}}, \ldots, D_{2c,c}, \ldots, C_{2n-2,n-2}, D_{2n-2,n-1}, d).$$

Vector b contains the coefficient of $\tilde{B}'_{k,q}$, $\tilde{\Gamma}'_{k,q}$ given in (4.23), that is

$$b^t = \frac{\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{n!\binom{n-1}{r}}\left(0,\dots,0,\binom{n-r}{1}\binom{r+1}{1},\binom{n-r}{2}\binom{r+1}{2}\frac{1}{2},\dots,\binom{n-r}{r+1}\binom{r+1}{r+1}\frac{r+1}{4^r},0,\dots,0\right).$$

Note that b has all entries null except the ones corresponding to $\tilde{B}'_{2n-2r-1,q}$.

The coefficients of the matrix A contain the variation of each $B'_{2k,q}$ and $\Gamma'_{2q,q}$ with respect to $\tilde{B}'_{r,s}$ and $\tilde{\Gamma}'_{r,s}$. We denote by $(\delta B'_{k,q}, \tilde{B}_{r,s})$, the coefficient of $\tilde{B}_{r,s}$ in the variation of the valuation $B'_{k,q}$. By Proposition 4.1.7

$$(\delta B'_{k,q}, \tilde{B}'_{r,s}) = \begin{cases} 2q(n+q-k+1/2), & \text{if } r=k-1, s=q-1 \\ -2(k-2q)(k-2q-1), & \text{if } r=k-1, s=q \\ 2\epsilon(k-2q)(k-2q-1), & \text{if } r=k+1, s=q+1 \\ -2\epsilon(n-k+q)(q+1/2), & \text{if } r=k+1, s=q \\ 0, & \text{otherwise.} \end{cases}$$

$$(\delta B'_{k,q}, \tilde{\Gamma}'_{r,s}) = \begin{cases} (k-2q)^2, & \text{if } r=k-1, s=q \\ -(n+q-k)q, & \text{if } r=k-1, s=q-1 \\ 0, & \text{otherwise.} \end{cases}$$

$$(\delta \Gamma'_{2q,q}, \tilde{B}'_{r,s}) = \begin{cases} 4q(n-q+1/2), & \text{if } r=2q-1, s=q-1 \\ -4\epsilon((n-q)(2q+3/2)-(q+1)/2), & \text{if } r=2q+1, s=q \\ 4\epsilon^2(n-q-1)(q+3/2), & \text{if } r=2q+3, s=q+1 \\ 0, & \text{otherwise.} \end{cases}$$

$$(\delta \Gamma'_{2q,q}, \tilde{\Gamma}'_{r,s}) = \begin{cases} -2(n-q)q, & \text{if } r=2q-1, s=q-1 \\ 2\epsilon(n-q-1)(q+1), & \text{if } r=2q+1, s=q \\ 0, & \text{otherwise.} \end{cases}$$

Each column of the matrix A contains the variation of a valuation $B'_{2k,q}$, $\Gamma'_{2q,q}$ or the volume. We take the valuations $B'_{2k,q}$, $\Gamma'_{2q,q}$ in the same order as in the vector b. (The volume corresponds to the last column.) That is, the columns of A contain the variation of the valuations in the following order

$$(\delta B'_{2,0}, \delta\Gamma'_{2,1}, \delta B'_{4,0}, \delta B'_{4,1}, \delta\Gamma'_{4,2}, \dots, \delta B'_{2n-2,n-2}, \delta\Gamma'_{2n-2,n-1}, \delta\mathrm{vol}).$$

We denote

$$\begin{aligned} \delta B'_{2k,\cdot} &= (\delta B'_{2k,\max\{0,2k-n\}}, \delta B'_{2k,\max\{0,2k-n\}+1}, \dots, \delta B'_{2k,k-1}), \\ \delta\Gamma'_{2k,\cdot} &= \delta\Gamma'_{2k,k}, \\ \tilde{B}'_{2k+1,\cdot} &= (\tilde{B}'_{2k+1,\max 0,2k-n+1}, \tilde{B}'_{2k+1,\max 0,2k-n+1+1}, \dots, \tilde{B}'_{2k+1,k})^t, \\ \tilde{\Gamma}'_{2k+1,\cdot} &= (\tilde{\Gamma}'_{2k+1,\max 0,2k-n+1}, \tilde{\Gamma}'_{2k+1,\max 0,2k-n+1+1}, \dots, \tilde{\Gamma}'_{2k+1,k})^t. \end{aligned}$$

Then, A has the following boxes structure

	$\delta B'_{2,\cdot}$	$\delta\Gamma'_{2,\cdot}$	$\delta B'_{4,\cdot}$	$\delta\Gamma'_{4,\cdot}$	$\delta B'_{6,\cdot}$	$\delta\Gamma'_{6,\cdot}$	$\delta B'_{8,\cdot}$	$\delta\Gamma'_{8,\cdot}$	$\cdots$	$\delta B'_{2n-4,\cdot}$	$\delta\Gamma'_{2n-4,\cdot}$	$\delta B'_{2n-2,\cdot}$	$\delta\Gamma'_{2n-2,\cdot}$	$\delta\mathrm{vol}$
$\tilde{B}'_{1,\cdot}$	$*$	$*$												
$\tilde{\Gamma}'_{1,\cdot}$	$*$	$*$												
$\tilde{B}'_{3,\cdot}$	$*_\epsilon$	$*_\epsilon$	$*$	$*$										
$\tilde{\Gamma}'_{3,\cdot}$		$*_\epsilon$	$*$	$*$										
$\tilde{B}'_{5,\cdot}$		$*_\epsilon$	$*_\epsilon$	$*_\epsilon$	$*$	$*$								
$\tilde{\Gamma}'_{5,\cdot}$				$*_\epsilon$	$*$	$*$								
$\tilde{B}'_{7,\cdot}$				$*_\epsilon$	$*_\epsilon$	$*_\epsilon$	$*$	$*$						
$\tilde{\Gamma}'_{7,\cdot}$						$*_\epsilon$	$*$	$*$						
$\tilde{B}'_{9,\cdot}$						$*_\epsilon$	$*_\epsilon$	$*_\epsilon$						
$\tilde{\Gamma}'_{9,\cdot}$								$*_\epsilon$						
$\vdots$														
$\tilde{B}'_{2n-3,\cdot}$										$*_\epsilon$	$*_\epsilon$	$*$	$*$	
$\tilde{\Gamma}'_{2n-3,\cdot}$											$*_\epsilon$	$*$	$*$	
$\tilde{B}'_{2n-1,\cdot}$											$*_\epsilon$	$*_\epsilon$	$*_\epsilon$	$*$

We denoted by $*$ the boxes of A with non-null coefficients and independent of ϵ, and by $*_\epsilon$ the boxes of A with non-null coefficients (for $\epsilon \neq 0$) and multiples of ϵ.

The structure by boxes of the linear system given by A suggests the method of resolution: we start with the top box, and we get the value of variables $C_{2,q}$ and $D_{2,1}$. Then we solve the next bloc with rows $\tilde{B}'_{3,\cdot}, \tilde{\Gamma}'_{3,\cdot}$, using the value of variables $C_{2,q}$ and $D_{2,1}$. We can continue this process, so that, once we know the value of variables $C_{2k,q}$ and $D_{2k,k}$, we substitute it on the equations given by the rows $\tilde{B}'_{2k+1,\cdot}, \tilde{\Gamma}'_{2k+1,\cdot}$.

Recall that the independent vector b has all terms null except the ones corresponding to $\tilde{B}'_{2n-2r-1,q}$. Thus, the linear system is homogeneous for the first equations until $\tilde{B}'_{2n-2r-2,q}$, and we can take $C_{k,q} = D_{2q,q} = 0$ whenever $k \leq 2n-2r-1$.

By Theorem 4.3.1 we have a solution for the system $Ax = b$ when $\epsilon = 0$. This solution has $C_{2n-2r,q}$ and $D_{2n-2r,n-r}$ as non-null terms, and satisfies the equations until rows $\tilde{B}'_{2n-2r,\cdot}, \tilde{\Gamma}'_{2n-2r,\cdot}$ also for $\epsilon \neq 0$.

So, we consider for all $\epsilon \in \mathbb{R}$ and for all $a \in \{1, ..., \min\{n-r, r\}\}$

$$\begin{aligned} C_{2n-2r,n-r-a} &= \frac{\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{4^a n!} \binom{n-1}{r}^{-1} \binom{n-r}{a} \binom{r}{a}, \\ D_{2n-2r,n-r} &= \frac{\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{2\, n!} \binom{n-1}{r}^{-1}. \end{aligned}$$

Now, we go on with the resolution of the linear system in $\mathbb{CK}^n(\epsilon)$. We study for each $c \in \{n-r+1, \ldots, n\}$ the submatrix of A with all rows $\tilde{B}'_{2c-1,q}$, $\tilde{\Gamma}'_{2c-1,q}$. This matrix has the following non-zero columns of A: $\delta\Gamma'_{2c-4,c-2}$, $\delta B'_{2c-2,\cdot}$, $\delta\Gamma'_{2c-2,c-1}$, $\delta B'_{2c,\cdot}$ and $\delta\Gamma'_{2c,c}$.

Suppose that we know the value of $D_{2c-4,c-2}$, $C_{2c-2,q}$ and $D_{2c-2,c-1}$. Then, we can substitute them in the equations given by the rows corresponding to $\tilde{B}_{2c-1,q}, \tilde{\Gamma}_{2c-1,q}$. We get equations with $C_{2c,q}$ and $D_{2c,c}$ as unknowns. If we denote $i = \max\{0, 2c-n\}$, the matrix of the coefficients for the obtained equations (which corresponds to a matrix bloc of A independent of ϵ) is

	$\delta B_{2c,i}$	$\delta B_{2c,i+1}$	$\delta B_{2c,i+2}$	$\cdots$	$\delta B_{2c,c-2}$	$\delta B_{2c,c-1}$	$\delta \Gamma_{2c,c}$
$\tilde{B}_{2c-1,i}$	$-4c(2c-1)$	$2(n-2c+3/2)$					
$\tilde{B}_{2c-1,i+1}$		$-2(2c-2)(2c-3)$	$4(n-2c+5/2)$				
$\tilde{B}_{2c-1,i+2}$			$-2(2c-4)(2c-5)$				
$\vdots$							
$\tilde{B}_{2c-1,c-2}$					-24	$2(c-1)(n-c-\frac{1}{2})$	
$\tilde{B}_{2c-1,c-1}$						-4	$4c(n-c+\frac{1}{2})$
$\tilde{\Gamma}_{2c-1,0}$	$(2c)^2$	$-(n-2c+1)$					
$\tilde{\Gamma}_{2c-1,1}$		$(2c-2)^2$	$-2(n-2c+2)$				
$\tilde{\Gamma}_{2c-1,2}$			$(2c-4)^2$				
$\vdots$							
$\tilde{\Gamma}_{2c-1,c-2}$					4^2	$-(c-1)(n-1)$	
$\tilde{\Gamma}_{2c-1,c-1}$						4	$-2c(n-c)$

The independent term is obtained from the initial independent term b (which in these cases is always zero) and from the part of the initial equation in which we substituted the value of $D_{2c-4,c-2}, C_{2c-2,q}, D_{2c-2,c-1}$. Comparing the box structure of A on page 95 and its coefficients on page 94, we obtain that the independent term of this new linear system has zero the terms $\tilde{\Gamma}'_{2c-1,q}$ with $q \in \{\max\{0, 2c-n\}, ..., c-2\}$, and the term $\tilde{\Gamma}_{2c-1,c-1}$ equals to $2\epsilon(n-c)cD_{2c-2,c-1}$.

Now, we consider the equations given by rows $\tilde{\Gamma}'_{2c-1,q}$ and $\tilde{B}_{2c-1,\max\{0,2c-n\}}$ in the previous matrix, which give a compatible linear system with one solution. The independent terms of the equation given by equation $\tilde{B}_{2c-1,\max\{0,2c-n\}}$ is $\epsilon(n-2c+2)C_{2c-2,\max\{0,2c-n-2\}}$. The other ones are zero. Solving this system we get the variables $C_{2c,\max\{0,2c-n\}}$ and $D_{2c,c}$ in terms of $C_{2c-2,\max\{0,2c-n-2\}}$ and $D_{2c-2,c-1}$, which we suppose known.

In order to avoid considering the maximum $\max\{0, 2c-n-2\}$ we distinguish two cases.

First stage: $2c \leq n$**.** This case appears if $2r > n$ (since $c \in \{n-r+1, \ldots, n\}$).

The linear system we have to solve is given by the augmented matrix

$$\left(\begin{array}{cccccc|c} -4c(2c-1) & 2(n-2c+3/2) & & & & & \epsilon(n-2c+2)C_{2c-2,0} \\ (2c)^2 & -(n-2c+1) & & & & & \\ & (2c-2)^2 & -2(n-2c+2) & & & & \\ & & & \ddots & & & \\ & & & & -(c-1)(n-c-1) & & \\ & & & & 4 & -2c(n-c) & -2\epsilon c(c-n)D_{2c-2,c-1} \end{array}\right)$$

with variables $\{C_{2c,0}, C_{2c,1}, \ldots, C_{2c,c-1}, D_{2c,c}\}$.

From the first two equations we obtain

$$C_{2c,0} = \frac{\epsilon(n-2c+2)(n-2c+1)}{4c(n-c+1)}C_{2c-2,0},$$
$$C_{2c,1} = \frac{\epsilon(n-2c+2)c}{(n-c+1)}C_{2c-2,0}.$$

For every $q \in \{0, ..., c-2\}$ the following relations are satisfied

$$(2c-2q)^2 C_{2c,q} = (q+1)(n-2c+q+1)C_{2c,q+1},$$
$$4C_{2c,c-1} - 2c(n-c)D_{2c,c} = -2\epsilon c(n-c)D_{2c-2,c-1},$$

and we get

$$C_{2c,q+1} = \frac{\epsilon 4^q (n-2c+2)!c!(c-1)!}{(n-c+1)(q+1)!(n-2c+q+1)!(c-q-1)!(c-q-1)!}C_{2c-2,0},$$
$$D_{2c,c} = \epsilon(D_{2c-2,c-1} + \frac{2(n-2c+2)!(c-1)!4^{c-2}}{(n-c+1)!}C_{2c-2,0}). \tag{A.12}$$

As the known constants are $C_{2n-2r,q}$ and $D_{2n-2r,n-r}$ we write the previous ones in terms of these, using the recurrence we just obtained.

$$
\begin{aligned}
C_{2c,0} &= \frac{\epsilon(n-2c+2)(n-2c+1)}{4c(n-c+1)} C_{2c-2,0} \\
&= \frac{\epsilon^{c-(n-r)}(n-2c+2)(n-2c+1)\cdot \ldots \cdot (n-2n+2r-2+2)(n-2n+2r-2+1)}{4^{c-(n-r)}c(c-1)\cdot \ldots \cdot (n-r+1)(n-c+1)(n-c+2)\cdot \ldots \cdot (n-(n-r))} C_{2n-2r,0} \qquad \text{(A.13)} \\
&= \frac{\epsilon^{c-(n-r)}(2r-n)!(n-r)!(n-c)!\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{4^{c-(n-r)}(n-2c)!c!r!4^{n-r}n!} \binom{n-1}{r}^{-1} \binom{r}{n-r} \\
&= \frac{\epsilon^{c-(n-r)}\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{4^{c}n!} \binom{n-1}{r}^{-1} \binom{n-c}{c},
\end{aligned}
$$

$$
\begin{aligned}
C_{2c,q+1} &= \frac{\epsilon(n-2c+2)4^{q}(n-2c+1)!c!(c-1)!}{(n-c+1)(q+1)!(n-2c+q+1)!(c-q-1)!(c-q-1)!} C_{2c-2,0} \qquad \text{(A.14)} \\
&= \frac{\epsilon(n-2c+2)4^{q}(n-2c+1)!c!(c-1)!\epsilon^{c-1-(n-r)}\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})(n-c+1)!}{(n-c+1)(q+1)!(n-2c+q+1)!((c-q-1)!)^{2}4^{c-1}n!(c-1)!(n-2c+2)!} \binom{n-1}{r}^{-1} \\
&= \frac{\epsilon^{c-(n-r)}c!(n-c)!\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{4^{c-q-1}(q+1)!(n-2c+q+1)!(c-q-1)!(c-q-1)!n!} \binom{n-1}{r}^{-1} \\
&= \frac{\epsilon^{c-(n-r)}\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{4^{c-q-1}n!} \binom{n-1}{r}^{-1} \binom{c}{q+1} \binom{n-c}{c-q-1},
\end{aligned}
$$

$$
\begin{aligned}
D_{2c,c} &= \epsilon \left(D_{2c-2,c-1} + \frac{2(n-2c+2)!(c-1)!4^{c-2}}{(n-c+1)!} C_{2c-2,0} \right) \\
&= \epsilon \left(D_{2c-2,c-1} + \frac{2(n-2c+2)!(c-1)!4^{c-2}}{(n-c+1)!} \frac{\epsilon^{c-1-(n-r)}\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})(n-c+1)!}{4^{c-1}n!(c-1)!(n-2c+2)!} \binom{n-1}{r}^{-1} \right) \\
&= \epsilon D_{2c-2,c-1} + \frac{\epsilon^{c-(n-r)}\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{2\,n!} \binom{n-1}{r}^{-1} \\
&= \epsilon^{c-(n-r)} D_{2n-2r,n-r} + (c-(n-r)) \frac{\epsilon^{c-(n-r)}\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{2\,n!} \binom{n-1}{r}^{-1} \qquad \text{(A.15)} \\
&= \frac{\epsilon^{c-(n-r)}\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{2\,n!} \binom{n-1}{r}^{-1} + (c-(n-r)) \frac{\epsilon^{c-(n-r)}\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{2\,n!} \binom{n-1}{r}^{-1} \\
&= \frac{\epsilon^{c-(n-r)}\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{2\,n!} (c+r-n+1) \binom{n-1}{r}^{-1}.
\end{aligned}
$$

Thus, we get the value of the unknowns (in the vector x) until position $D_{2\lfloor n/2\rfloor,\lfloor n/2\rfloor}$.

Second stage: $2c \geq n$**.** Note that $B'_{2c,q}$ is defined if $q \geq 2c-n > 0$. In this case, $2c \geq n$, and the system we have to solve has the same structure as in the previous case ($2c \leq n$) but with less equations and unknowns. Taking the same equations as in the previous case $2c \leq n$, we obtain, as augmented matrix,

$$
\left(\begin{array}{cccc|c}
(2c-n) & & & & \begin{array}{c} \epsilon((4c-2n-1)C_{2c-2,2c-n-1}- \\ -4(n-c+1)(2n-2c+1)C_{2c-2,2c-n-2}) \end{array} \\
(2n-2c)^{2} & -(2c-n-1) & & & \\
 & & 4 & -2c(n-c) & -2\epsilon c(c-n)D_{2c-2,c-1}
\end{array} \right).
$$

From the first equation, it follows

$$C_{2c,2c-n} = \frac{\epsilon}{2c-n}((4c-2n-1)C_{2c-2,2c-n-1} - 2(2n-2c+2)(2n-2c+1)C_{2c-2,2c-n-2}). \quad \text{(A.16)}$$

For $a \in \{0, ..., n-c\}$, by relation

$$(2n-2c-2a+2)^2 C_{2c,2c-n+a-1} = a(2c-n+a)C_{2c,2c-n+a}$$

we get

$$\begin{aligned} C_{2c,2c-n+a} &= \frac{4(n-c-a+1)^2}{a(2c-n+a)} C_{2c,2c-n+a-1} \\ &= \frac{4^a(n-c-a+1)^2(n-c-a+2)^2 \dots (n-c)^2}{a(a-1)\dots 2(2c-n+a)(2c-n+a-1)\dots(2c-n+1)} C_{2c,2c-n} \\ &= \frac{4^a(n-c)!(n-c)!(2c-n)!}{a!(n-c-a)!(n-c-a)!(2c-n+a)!} C_{2c,2c-n} \end{aligned} \quad \text{(A.17)}$$

and from

$$4C_{2c,c-1} - 2(n-c)cD_{2c,c} = -2\epsilon(n-c)(c)D_{2c-2,c-1}$$

and (A.17) we get

$$\begin{aligned} D_{2c,c} &= \epsilon D_{2c-2,c-1} + \frac{2}{c(n-c)} C_{2c,c-1} \\ &= \epsilon D_{2c-2,c-1} + \frac{2 \cdot 4^{n-c-1}(n-c)!(n-c)!(2c-n)!}{c(n-c)(n-c-1)!(c-1)!} C_{2c,2c-n} \\ &= \epsilon D_{2c-2,c-1} + 2\frac{4^{n-c-1}(n-c)!(2c-n)!}{c!} C_{2c,2c-n}. \end{aligned} \quad \text{(A.18)}$$

In order to obtain the value of $C_{2c,2c-n}$ we use the value of $C_{2c-2,2c-n-1}$ and $C_{2c-2,2c-n-2}$ if $c \in \{n-r, ..., \lfloor n/2 \rfloor\}$. We consider $c_0 = \lfloor \frac{n+2}{2} \rfloor$.

From the previous case $2c \leq n$ we know the value of the unknowns $C_{2c_0-2,2c_0-n-1}$, $C_{2c_0-2,2c_0-n-2}$ and D_{2c_0-2,c_0-1}. For n even we have (we omit the analogous computation for n odd)

$$\begin{aligned} C_{2c_0-2,2c_0-n-1} = C_{n,1} &= \frac{\epsilon^{n/2-(n-r)}\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})(n/2)!(n/2)!}{4^{n/2-1}n!((n-2)/2)!((n-2)/2)!} \binom{n-1}{r}^{-1} \\ &= \frac{\epsilon^{r-n/2}\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{2^n n!} n^2 \binom{n-1}{r}^{-1}, \end{aligned} \quad \text{(A.19)}$$

$$C_{2c_0-2,2c_0-n-2} = C_{n,0} = \frac{\epsilon^{r-n/2}\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{2^n n!} \binom{n-1}{r}^{-1},$$

$$D_{2c_0-2,c_0-1} = D_{n,n/2} = \frac{\epsilon^{n/2-(n-r)}\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{2\, n!} (r - \frac{n}{2} + 1) \binom{n-1}{r}^{-1}.$$

Then

$$\begin{aligned} C_{n+2,2} &= \frac{\epsilon}{2c-n}((4c-2n-1)C_{2c-2,2c-n-1} - 2(2n-2c+2)(2n-2c+1)C_{2c-2,2c-n-2}) \\ &= \frac{\epsilon^{r-n/2+1}\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{2^{n+1}n!}(3n^2 - 2n(n-1)) \binom{n-1}{r}^{-1} \\ &= \frac{\epsilon^{r-n/2+1}\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{2^{n+1}n!} n(n+2) \binom{n-1}{r}^{-1}. \end{aligned}$$

Once we know the expression of $C_{n+2,2}$ we find the value of $C_{2c,2c-n}$, for every $c \in \{\lfloor n/2 \rfloor, \dots, n\}$, using the recurrence for $C_{2c,2c-n}$ and $C_{2c-2,2c-n-1}$. First, we have

$$\begin{aligned}
C_{2c,2c-n} &\overset{(A.16)}{=} \frac{\epsilon}{2c-n}((4c-2n-1)C_{2c-2,2c-n-1} - 2(2n-2c+2)(2n-2c+1)C_{2c-2,2c-n-2}) \\
&\overset{(A.17)}{=} \frac{\epsilon C_{2c-2,2c-n-2}}{2c-n} \cdot \\
&\quad \cdot \left(\frac{(4c-2n-1)4(n-c+1)!(n-c+1)!(2c-n-2)!}{(n-c)!(n-c)!(2c-n-1)!} - 4(n-c+1)(2n-2c+1) \right) \\
&= \frac{4\epsilon(n-c+1)}{2c-n} \left(\frac{(4c-2n-1)(n-c+1)-(2n-2c+1)(2c-n-1)}{(2c-n-1)} \right) C_{2c-2,2c-n-2} \\
&= \frac{4\epsilon(n-c+1)c}{(2c-n)(2c-n-1)} C_{2c-2,2c-n-2}.
\end{aligned}$$

We go on with this recurrence until $C_{*,*-n}$ with $* \leq \frac{n+2}{2}$. In this case we know the value of the constants, and we can find the value of $C_{2c,2c-n}$.

$$\begin{aligned}
C_{2c,2c-n} &= \frac{(4\epsilon)^{c-(n+2)/2} c(c-1) \cdot \ldots \cdot ((n+4)/2)(n-c+1)(n-c+2) \cdot \ldots \cdot (n/2-1)}{(2c-n)(2c-n-1) \cdot \ldots \cdot 4 \cdot 3} C_{n+2,2} \\
&= \frac{(4\epsilon)^{c-(n+2)/2} c!((n-2)/2)!2}{((n+2)/2)!(n-c)!(2c-n)!} C_{n+2,2} \\
&= \frac{(4\epsilon)^{c-(n+2)/2} 2^3}{(n+2)n} \binom{c}{2c-n} \frac{\epsilon^{r-n/2+1} \mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{2^{n+1} n!} n(n+2) \binom{n-1}{r}^{-1} \\
&= \frac{\epsilon^{c-(n-r)} \mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{4^{n-c} n!} \binom{n-1}{r}^{-1} \binom{c}{2c-n}. \qquad \text{(A.20)}
\end{aligned}$$

Finally,

$$\begin{aligned}
C_{2c,2c-n+a} &= \frac{4^a (n-c)!(n-c)!(2c-n)!}{a!(n-c-a)!(n-c-a)!(2c-n+a)!} \frac{\epsilon^{c-(n-r)} c! \mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{4^{n-c}(2c-n)!(n-c)!n!} \binom{n-1}{r}^{-1} \\
&= \frac{\epsilon^{c-(n-r)} \mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{4^{n-c-a} n!} \binom{n-1}{r}^{-1} \binom{n-c}{n-c-a} \binom{c}{2c-n+a}
\end{aligned}$$

and

$$\begin{aligned}
D_{2c,c} &\overset{(A.17) \text{ and } (A.20)}{=} \epsilon D_{2c-2,c-1} + 2 \frac{4^{n-c-1}(n-c)!(2c-n)!}{c!} \frac{\epsilon^{c-(n-r)} c! \mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{4^{n-c}(2c-n)!(n-c)!n!} \binom{n-1}{r}^{-1} \\
&= \epsilon D_{2c-2,c-1} + \frac{\epsilon^{c-(n-r)} \mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{2\,n!} \binom{n-1}{r}^{-1} \\
&= \epsilon^{c-n/2} D_{n,n/2} + (c-n/2) \frac{\epsilon^{c-(n-r)} \mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{2\,n!} \binom{n-1}{r}^{-1} \\
&= \frac{\epsilon^{c-(n-r)} \mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{2\,n!} (c+r-n+1) \binom{n-1}{r}^{-1}.
\end{aligned}$$

To determine the value of d, the coefficient of $\delta\mathrm{vol}$, we consider the last equation of the initial linear system

$$\begin{aligned}\frac{d}{(n-1)!} &= -2\epsilon^2(2n-1)D_{2n-4,n-2} - 4\epsilon C_{2n-2,n-2} + 2\epsilon(3n-1)D_{2n-2,n-1} \\ &= 2\epsilon^r\left(-(2n-1)(r-1)-(n-1)+(3n-1)r\right)\frac{\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{2\,n!}\binom{n-1}{r}^{-1} \\ &= \frac{\epsilon^r\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{n!}n(r+1)\binom{n-1}{r}^{-1}.\end{aligned}$$

So,

$$d = \epsilon^r\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})(r+1)\binom{n-1}{r}^{-1}.$$

Now, we have to prove that the given solution satisfies all the equations we did not use to solve the system. This is because we cannot ensure that the equation (A.10) has solution.

Let us study first the case $2c \leq n$. Consider the matrix on page 96. The rows we did not use correspond to $\tilde{B}'_{2c-1,q}$, $q \in \{1,...,c-1\}$. Suppose $q \neq c-1$. The equations given by this row are $\tilde{B}'_{2c-1,q}$

$$\begin{aligned}&-(2c-2q)(2c-2q-q)C_{2c,q} + (q+1)(n-2c+q+3/2)C_{2c,q+1} \\ &\qquad = -\epsilon(2c-2q)(2c-2q-1)C_{2c-2,q-1} + \epsilon(n-2c+q+2)(q+1/2)C_{2c-2,q}.\end{aligned}$$

For $q = c-1$, the equation is

$$\begin{aligned}&2\epsilon^2(n-c+1)(c-1/2)D_{2c-4,c-2} + 2\epsilon C_{2c-2,c-2} - 2\epsilon((n-c+1)(2c-1/2)-c/2)D_{2c-2,c-1} \\ &\qquad = -2C_{2c,c-1} + 2c(n-2+1/2)D_{2c,c}.\end{aligned}$$

Substituting the value of each $C_{*,\cdot}$ and $D_{*,\cdot}$ given on page 97 we prove that the equations are satisfied.

In the same way, we can prove that all equations appearing in the case $2c \geq n$ are also satisfied.

Finally, using again the relation in (4.12), we get the result with respect to $\{B_{k,q}, \Gamma_{k,q}\}$.

Proof of Theorem 4.4.1

The idea of the proof of this theorem is the same as for Theorem 4.3.5. In the same way, if the variation δ_X is the same in both sides, for all differentiable vector field X, then the expression holds.

From the Gauss-Bonnet-Chern formula, we know that $\chi(\Omega)$ can be written as the integral over $N(\Omega)$ of a differential form $O(2n)$-invariant, and also $U(n)$-invariant. Thus, by Proposition 2.4.5 there exist constants $C_{k,q}$, $D_{k,q}$, d such that

$$\chi(\Omega) = \sum_{k,q} C_{k,q}B'_{k,q}(\Omega) + \sum_{j=1}^{\lfloor\frac{n}{2}\rfloor} D_{2j,j}\Gamma'_{2j,j}(\Omega) + d\mathrm{vol}(\Omega) \tag{A.21}$$

where $\max\{0, k-n\} \leq q < k/2 \leq n$, and $B'_{k,q}$ and $\Gamma'_{k,q}$ are the valuations defined in (4.12). Taking the variation in both sides of the previous equality we have

$$0 = \sum_{k,q}(c_{k,q}\tilde{B}'_{k,q}(\Omega) + d_{2q,q}\tilde{\Gamma}'_{2q,q}(\Omega))$$

with $c_{k,q}$ and $d_{k,q}$ linear combination of $C_{k,q}$ and $D_{2q,q}$.

Thus, we have to impose $c_{k,q} = d_{k,q} = 0$.

The variation of $\Gamma'_{0,0}$ in $\mathbb{CK}^n(\epsilon)$ is (cf. Corollary 4.1.9)

$$\delta\Gamma'_{0,0}(\Omega) = 2\epsilon(-(3n-1)\tilde{B}'_{1,0}(\Omega) + (n-1)\tilde{\Gamma}'_{1,0}(\Omega) + 3\epsilon(n-1)\tilde{B}'_{3,1}(\Omega)).$$

It is necessary to cancel the variation of the terms $\tilde{B}'_{1,0}$, $\tilde{B}'_{3,1}$ and $\tilde{\Gamma}'_{1,0}$. By Proposition 4.1.7 we have that the variation of a valuation $B'_{k,q}$ in $\mathbb{CK}^n(\epsilon)$ with k even (resp. odd) has only terms $\tilde{B}'_{k',q'}$ and $\tilde{\Gamma}'_{k',q'}$ with k' odd (resp. even). Thus, in the expresssion (A.21) we can restrict the value of k to k even, and (A.21) can be reduced to

$$\chi(\Omega) = \sum_{k=0}^{n-1}\left(\sum_{q=\max\{0,2k-n\}}^{k-1} C_{2k,q}B'_{2k,q}(\Omega) + D_{2k,k}\Gamma'_{2k,k}(\Omega)\right) + d\mathrm{vol}(\Omega). \tag{A.22}$$

The right hand side in the previous equality coincides with the right hand side of (A.11) plus the term $D_{0,0}\Gamma'_{0,0}$. Thus, the variation is very similar and the linear system we have to solve will be also very similar to the one solved in Theorem 4.3.5. The only different equations are the ones given by $c_{1,0} = 0$, $d_{1,0} = 0$ and $c_{3,1} = 0$, that is

$$\begin{aligned} -\epsilon(3n-1)D_{0,0} - 2C_{2,0} + 2(n-1/2)D_{2,1} &= 0,\\ \epsilon(n-1)D_{0,0} + 2C_{2,0} - (n-1)D_{2,1} &= 0,\\ 3\epsilon^2(n-1)D_{0,0} + 2\epsilon C_{2,0} - \epsilon(7n-9)D_{2,1} - 2C_{4,1} + 2(2n-3)D_{4,2} &= 0. \end{aligned} \tag{A.23}$$

We find the value of $D_{0,0}$ for $\epsilon = 0$, i.e. in $\mathbb{C}^n$, using the Gauss-Bonnet formula so that

$$D_{0,0} = \frac{1}{O_{2n-1}(n-1)!} = \frac{1}{2n\omega_{2n}(n-1)!} = \frac{n!}{2\,n!\pi^n} = \frac{1}{2\pi^n}.$$

The choice for the value of $D_{0,0}$ ensures that both sides in (A.21) coincide when Ω collapses to a point.

From the first two equation and the value of $D_{0,0}$ we get

$$\begin{aligned} C_{2,0} &= \frac{\epsilon}{2}(n-1)D_{0,0} = \frac{\epsilon(n-1)}{4\pi^n},\\ D_{2,1} &= 2\epsilon D_{0,0} = \frac{\epsilon}{\pi^n}. \end{aligned}$$

In order to find the value of $C_{4,1}$ and $D_{4,2}$ we consider the equations given by $\{c_{3,0} = 0, c_{3,1} = 0, d_{3,0} = 0, d_{3,1} = 0\}$. The equation $c_{3,1} = 0$ is the one given in (A.23), and the others are

$$\begin{aligned} -\epsilon(n-2)C_{2,0} - 24C_{4,0} + (2n-5)C_{4,1} &= 0,\\ 16C_{4,0} - (n-3)C_{4,1} &= 0,\\ \epsilon(n-2)D_{2,1} + C_{4,1} - (n-2)D_{4,2} &= 0. \end{aligned}$$

(Note that they coincide with the ones in Theorem 4.3.5.) Solving the system given by these 3 equations and (A.23), we get that it is compatible with solution

$$C_{4,0} = \frac{\epsilon^2}{32\pi^n}(n-2)(n-3), \quad C_{4,1} = \frac{\epsilon^2}{2\pi^n}(n-2), \quad D_{4,2} = \frac{3\epsilon^2}{2\pi^n}.$$

To find the value of the unknowns $C_{2c,q}$, $D_{2c,c}$ with $c \geq 3$, we have to solve the same equations as in the proof of Theorem 4.3.5. We can use the same relations if we first prove that $C_{4,0}$, $C_{4,1}$ and $D_{4,2}$ also satisfies (A.12). We have to check it because variables $C_{4,0}$, $C_{4,1}$

and $D_{4,2}$ here were obtained solving another linear system. But, it is straightforward verified. So, we get the same relation among the unknowns.

Thus, from the equalities (A.13), (A.14), (A.15), (A.17) and (A.18), with $r = n-1$, and the computation of $C_{n,1}$, $C_{n,0}$, $D_{n,\lfloor n/2\rfloor}$ in the same way as in (A.19) we have

$$\chi(\Omega) = \sum_{c=0}^{n-1} \frac{\epsilon^c}{\pi^n} \left(\sum_{q=\max\{0,2c-n\}}^{c-1} \frac{1}{4^{c-q}} \binom{c}{q} \binom{n-c}{c-q} B'_{2c,q}(\Omega) + \frac{c+1}{2} \Gamma'_{2c,c}(\Omega) \right) + \frac{\epsilon^n (n+1)!}{\pi^n} \mathrm{vol}(\Omega).$$

Using the relation in (4.12) we get the stated expression

$$\begin{aligned}
\chi(\Omega) &= \sum_{c=0}^{n-1} \frac{\epsilon^c}{\pi^n} \left(\sum_{q=\max\{0,2c-n\}}^{c-1} \frac{1}{4^{c-q}} \binom{c}{q} \binom{n-c}{c-q} B'_{2c,q}(\Omega) + \frac{c+1}{2} \Gamma'_{2c,c}(\Omega) \right) + \frac{\epsilon^n (n+1)!}{\pi^n} \mathrm{vol}(\Omega) \\
&= \sum_{c=0}^{n-1} \frac{\epsilon^c}{\pi^n} \Big(\sum_{q=\max\{0,2c-n\}}^{c-1} \frac{1}{4^{c-q}} \frac{c!(n-c)!q!(n-2c+q)!(2c-2q)!\omega_{2n-2c}}{q!(c-q)!(c-q)!(n-2c+q)!} B_{2c,q}(\Omega) + \\
&\quad + \frac{2(c+1)}{2} c!(n-2c+c)!(2c-2c)!\omega_{2n-2c}\Gamma_{2c,c}(\Omega) \Big) + \frac{\epsilon^n (n+1)!}{\pi^n} \mathrm{vol}(\Omega) \\
&= \sum_{c=0}^{n-1} \frac{\epsilon^c}{\pi^n} \Big(\sum_{q=\max\{0,2c-n\}}^{c-1} \frac{1}{4^{c-q}} \binom{2c-2q}{c-q} \frac{c!(n-c)!\pi^{n-c}}{(n-c)!} B_{2c,q}(\Omega) \\
&\quad + (c+1)!(n-c)! \frac{\pi^{n-c}}{(n-c)!} \Gamma_{2c,c}(\Omega) \Big) + \frac{\epsilon^n (n+1)!}{\pi^n} \mathrm{vol}(\Omega) \\
&= \sum_{c=0}^{n-1} \frac{\epsilon^c}{\pi^c} c! \left(\sum_{q=\max\{0,2c-n\}}^{c-1} \frac{1}{4^{c-q}} \binom{2c-2q}{c-q} B_{2c,q}(\Omega) + (c+1)\Gamma_{2c,c}(\Omega) \right) + \frac{\epsilon^n (n+1)!}{\pi^n} \mathrm{vol}(\Omega).
\end{aligned}$$

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Notation

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