LIE FLOWS OF CODIMENSION 3

E. GALLEGO AND A. REVENTÓS

ABSTRACT. We study the following realization problem: given a Lie algebra of dimension 3 and an integer q, $0 \leq q \leq 3$, is there a compact manifold endowed with a Lie flow transversely modeled on $\mathcal{G}$ and with structural Lie algebra of dimension q? We give here a quite complete answer to this problem but some questions remain still open (cf. §2).

0 Introduction

Among the class of foliations with a transverse structure Lie foliations stand out. These are foliations transversely modeled on Lie groups. They have been studied by several authors, mainly by Fedida (cf. [3]). Apart from its intrinsic interest the importance of this study is increased by the fact that they arise naturally in Molino's classification of Riemanian foliations ([6]).

To each Lie foliation are associated two Lie algebras, the Lie algebra $\mathcal{G}$ of the Lie group on which it is modeled and the structural Lie algebra $\mathcal{H}$. The latter algebra is the Lie algebra of the Lie foliation $\mathcal{F}$ restricted to the closure of any one of its leaves. In particular it is a subalgebra of $\mathcal{G}$. We remark that although $\mathcal{H}$ is canonically associated to $\mathcal{F}$, $\mathcal{G}$ is not.

Thus, one natural and interesting question is to know which pairs of Lie algebras $(\mathcal{G}, \mathcal{H})$, with $\mathcal{H}$ a subalgebra of $\mathcal{G}$, can arise as transverse algebra and structural Lie algebra repectively of a Lie foliation $\mathcal{F}$ on a *compact* manifold M.

We shall study here a particular but interesting case, namely given a Lie algebra of dimension 3 and an integer q, $0 \leq q \leq 3$, is there a compact manifold endowed with a Lie flow transversely modeled on $\mathcal{G}$ and with structural Lie algebra of dimension q? For simplicity's sake we shall say that the pair $(\mathcal{G}, q)$ is (or is not) realizable.

By using the classification of the 3–dimensional Lie algebras and the fact that the structural Lie algebra of a Lie flow is abelian (cf. [1]) it becomes apparent that certain pairs $(\mathcal{G}, q)$ are not realizable (for instance

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$(sl(2), 2)$ and $(so(3), 2)$ are not realizable because $sl(2)$ and $so(3)$ have no abelian subalgebras of dimension two).

Nevertheless in some cases the obstruction for certain pairs to be realizable is rooted in the compactness of M and not based on purely algebraic reasons (for instance the pair $(affine, 0)$ is not realizable (cf. Theorem 1)).

We classify the 3–dimenisonal Lie algebras in 6 algebras $\mathcal{G}_1, \ldots, \mathcal{G}_6$ and two families $\mathcal{G}_7$ (parametrized by $k \in \mathbf{R}$, $k \neq 0$) and $\mathcal{G}_8$ (parametrized by $h \in \mathbf{R}$, $h^2 < 4$) (cf. §1). We obtain

Theorem 1. *If the structural Lie algebra is zero, i.e. $\mathcal{F}$ is a compact foliation, then $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$ are realizable. $\mathcal{G}_5$ and $\mathcal{G}_6$ are not realizable. $\mathcal{G}_7$ is realizable if and only if $k = -1$ and $\mathcal{G}_8$ is realizable if and only if $h = 0$.*

Theorem 2. *If the structural Lie algebra has dimension 1, then $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$ and $\mathcal{G}_5$ are realizable. $\mathcal{G}_6$ and $\mathcal{G}_7$ are not realizable and $\mathcal{G}_8$ with $h = 0$ is realizable.*

We do not know any realization of $\mathcal{G}_8$ with $h \neq 0$ and 1–dimensional structural Lie algebra of dimension 1.

Finally it is remarkable that the realization of the pair $(\mathcal{G}_7, 2)$ depends on k. In fact we have

Theorem 3. *If the structural Lie algebra has dimension 2 then $\mathcal{G}_1$, $\mathcal{G}_5$ and $\mathcal{G}_8$ with $h = 0$ are realizable. $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$, $\mathcal{G}_6$ and $\mathcal{G}_7$ with $k \in \mathbf{Q}$ are not realizable.*

We give a realization of $\mathcal{G}_7$ with $k \notin \mathbf{Q}$. A characterization of those k for which $\mathcal{G}_7$ is realizable and the $\mathcal{G}_8$ case, are still open.

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1 Preliminary definitions and results

Let $\mathcal{F}$ be a smooth foliation of codimension n on a smooth manifold M given by an integrable subbundle $L \subset TM$. We denote by $\mathcal{L}(M, \mathcal{F})$ the Lie algebra of foliated vector fields, i.e. $X \in \mathcal{L}(M, \mathcal{F})$ if and only if $[X, Y] \in \Gamma L$ for all $Y \in \Gamma L$. Thus, the set of sections of L, ΓL, is an ideal of $\mathcal{L}(M, \mathcal{F})$. The elements of $\mathcal{X}(M, \mathcal{F})$ are called basic vector fields.

If there is a family $\{X_1, \ldots, X_n\}$ of foliated vector fields of M such that the correspondig family $\{\overline{X}_1, \ldots, \overline{X}_n\}$ of basic vector fields has rank n everywhere, the foliation is called transversely prallelizable and $\{\overline{X}_1, \ldots, \overline{X}_n\}$ a transvers parallelism. If the vector subspace $\mathcal{G}$ of

$\mathcal{X}(M, \mathcal{F})$ generated by $\{\overline{X}_1, \ldots, \overline{X}_n\}$ is a Lie subalgebra, the foliation is called a Lie foliation.

We shall use the following structure theorems (cf. [3] and [6]):

Theorem A. *Let $\mathcal{F}$ be a tansversally parallelizabe foliation on a compact manifold M of codimension n. Then:*

a) *There is a Lie algebra $\mathcal{H}$ of dimension $q \leq n$.*
b) *There is a locally trivial fibration $\pi : M \longrightarrow W$ with compact fibre F and $\dim W = n - q = m$.*
c) *There is a dense Lie $\mathcal{H}$–foliation on F such that:*
   (i) The fibres of π are the closures of the leaves of $\mathcal{F}$.
   (ii) The foliation induced by $\mathcal{F}$ on each fibre of π is isomorphic to the $\mathcal{H}$–foliation on F.

$\mathcal{H}$ is called the structural Lie algebra of $(M, \mathcal{F})$, π the basic fibration and W the basic manifold. The foliation given by the fibres of π is denoted by $\overline{\mathcal{F}}$. Note that $\operatorname{codim} \overline{\mathcal{F}} + q = \operatorname{codim} \mathcal{F}$.

Theorem B. *Let $\mathcal{F}$ be a $\mathcal{G}$–foliation on a compact manifold M and let G be the connected simply connected Lie group with Lie algebra $\mathcal{G}$. Let $p : \widetilde{M} \longrightarrow M$ be the universal covering of M. Then there is a locally trivial fibration $D : \widetilde{M} \longrightarrow G$ equivariant by $Aut(p)$ (i.e. if $D(x) = D(y)$ then $D(gx) = D(gy)$ for all $x, y \in \widetilde{M}$ and $g \in Aut(p)$) such that the foliation $\widetilde{\mathcal{F}} = p^*\mathcal{F}$ is given by the fibres of D.*

The natural morphism $h : \pi_1(M) \longrightarrow Diff(G)$ is such that $\Gamma = im(h) \subset G$, where the inclusion $G \subset Diff(G)$ is by left translations.

We shall also use some cohomological properties of the foliation. Recall that the basic forms complex is given by the forms $\alpha \in \Omega^*(M)$ such that $\mathcal{L}_X \alpha = 0$ and $i_X \alpha = 0$ for all $X \in \Gamma L$. The cohomology of this complex, $H^*(M, \mathcal{F})$, is the basic cohomology of the foliated manifold $(M, \mathcal{F})$. If $H^n(M, \mathcal{F}) \neq 0$ we say that $\mathcal{F}$ is homologically orientable or unimodular. We have (cf. [5]):

Theorem C. *Let $\mathcal{F}$ be an unimodular Lie $\mathcal{G}$–foliation on a compact manifold M. Then the Lie algebra $\mathcal{G}$ is unimodular.*

Finally we recall that the 3–dimesional Lie algebras can be classified in eight families

– $\mathcal{G}_1$ (Abelian):

$$[e_1, e_2] = [e_1, e_3] = [e_2, e_3] = 0$$

– $\mathcal{G}_2$ (Heisenberg):

$$[e_1, e_2] = [e_1, e_3] = 0, \quad [e_2, e_3] = e_1$$

– $\mathcal{G}_3$ $(so(3))$:

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2$$

– $\mathcal{G}_4$ $(sl(2))$:

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = -e_1, \quad [e_3, e_1] = e_2$$

– $\mathcal{G}_5$ (Affine):

$$[e_1, e_2] = e_1, \quad [e_1, e_3] = [e_2, e_3] = 0$$

– $\mathcal{G}_6$:

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = e_1 + e_2$$

– $\mathcal{G}_7$:

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = ke_2 \qquad k \neq 0$$

– $\mathcal{G}_8$:

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_2, \quad [e_2, e_3] = -e_1 + he_2 \qquad h^2 < 4$$

Notice that $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$ are unimodular, $\mathcal{G}_5$ and $\mathcal{G}_6$ are not unimodular, $\mathcal{G}_7$ is unimodular only if $k = -1$ and $\mathcal{G}_8$ only if $h = 0$.

Remark: We can think that $\mathcal{G}_7$ is parametrized by $k \in [-1, 0) \cup (0, 1]$. In fact two of these algebras are isomorphic if and only if $k \cdot k' = 1$.

2 Lie flows of codimension 3

Let $\mathcal{F}$ be a Lie flow of codimension 3 on a compact manifold M. Since the closures of the leaves of $\mathcal{F}$ are the fibres of a bundle (cf. Theorem A), there are four possible cases.

1 Case $\operatorname{codim} \overline{\mathcal{F}} = \mathbf{3}$**.**

In this case $\mathcal{F}$ is compact and the basic bundle is $M \longrightarrow M/\mathcal{F}$. Thus the basic cohomology coincides with the de Rham cohomology of the compact manifold $M/\mathcal{F}$ and hence $H^3(M/\mathcal{F}) \neq 0$. By Theorem C, if such a flow exists it is transversely modeled on a unimodular Lie algebra. So $\mathcal{G}_5$ and $\mathcal{G}_6$ are not realizables, $\mathcal{G}_7$ is realizable (a priori) only if $k = -1$ and $\mathcal{G}_8$ only if $h = 0$.

We give now examples for each one of the remainder algebras.

– $\mathcal{G}_1$: Just consider the trivial bundle $T^1 \times T^3 \longrightarrow T^3$.

– $\mathcal{G}_2$: Consider the trivial bundle $T^1 \times M \longrightarrow M$ where M is the homogeneous space N/Γ of the Heisenberg group

$$N = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} ; a, b, c \in \mathbf{R} \right\}$$

by the discret uniform subgroup Γ of N given by the matrices of N with integer coefficients.

– $\mathcal{G}_3$: Just consider the trivial bundle $T^1 \times S^3 \longrightarrow S^3$.

– $\mathcal{G}_4$: Consider the trivial bundle $T^1 \times T_1 W \longrightarrow T_1 W$ where $T_1 W$ is the unit sphere bundle of the two hole torus W. $T_1 W$ is the homogeneous space $PSL(2, \mathbf{R})/\pi_1(W)$ and therefore we have the desired example.

– $\mathcal{G}_7$ (with $k = -1$) : Let $A \in SL(2, \mathbf{Z})$ be a matrix with eigenvalues λ, $1/\lambda$ (being $\lambda > 0$ and $\lambda \neq 1$). We can give a solvable Lie group structure on $\mathbf{R}^3 = \mathbf{R} \times \mathbf{R}^2$ by

$$(t, u).(s, v) = (t + s, A^t \cdot v + u)$$

The Lie algebra of this group is $\mathcal{G}_7$ with $k = -1$ (cf.[4]). Moreover the points of $\mathbf{R}^3$ with integer coordinates constitute a uniform discret subgroup Γ of $\mathbf{R}^3$. The quotient is usually denoted by T^3_A. Then, one example of a Lie flow transversely modeled on $\mathcal{G}_7$, with $k = -1$, is given by the trivial bundle $T^1 \times T^3_A \longrightarrow T^3_A$.

– $\mathcal{G}_8$ (with $h = 0$) (P. Molino) : Let us consider the flow given by the fibres of the trivial bundle $T^1 \times T^3 \longrightarrow T^3$. Let $\theta^0, \theta^1, \theta^2, \theta^3$ denote the canonical coordinates in $T^1 \times T^3$. The parallelism given by $\partial/\partial\theta^1, \partial/\partial\theta^2, \partial/\partial\theta^3$ makes the fibres of the bundle an abelian Lie foliation. But we have basic functions enough to modify this parallelism. In fact, we can take

$$\left.\begin{aligned} e_1 &= \cos\theta^1 \cdot \partial/\partial\theta^2 + \sin\theta^1 \cdot \partial/\partial\theta^3 \\ e_2 &= -\sin\theta^1 \cdot \partial/\partial\theta^2 + \cos\theta^1 \cdot \partial/\partial\theta^3 \\ e_3 &= -\partial/\partial\theta^1 \end{aligned}\right\}$$

to obtain a new parallelism with $[e_1, e_2] = 0, [e_1, e_3] = e_2$, $[e_2, e_3] = -e_1$ i.e. the flow is also transversely modeled on $\mathcal{G}_8$ (with $h = 0$).

2 Case $\operatorname{codim} \overline{\mathcal{F}} = 2$.

In this case we give examples for $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$, $\mathcal{G}_5$ and $\mathcal{G}_8$ (with $h=0$). We also prove that $\mathcal{G}_6$ and $\mathcal{G}_7$ are not realizable.

– $\mathcal{G}_1$: One example is given by the flow $(X,0)$ on $T^2 \times T^2$ where X is a dense linear flow on T^2.
– $\mathcal{G}_2$: Let M be the homogeneous space of the Heisenberg group considered before. The flow on $M \times T^1$ whose integral curves are given by

$$\varphi_t(p) = (\begin{pmatrix} 1 & a & b+t \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, t+d)$$

where

$$p = (\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, d) \qquad \text{and} \qquad d \in \mathbf{R} \setminus \mathbf{Q}$$

is transverse to M and the closure of each leaf is T^2. Hence it is one example of a $\mathcal{G}_2$–Lie flow with codim $\overline{\mathcal{F}} = 2$.
– $\mathcal{G}_3$: As $S^3 = SU(3)$ an example can be constructed by suspending the representation

$$h : \pi_1(S^1) \longrightarrow \mathrm{Diff}(S^3)$$

given by

$$h(1) = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

where $\alpha \in \mathbf{R} \setminus \mathbf{Q}$.
– $\mathcal{G}_4$ (A. ElKacimi) : Let $\mathcal{F}_0$ be the transverse affine Lie flow on T^3_A (cf.[1]). Using the fact that the affine group GA can be considered, lifting the map

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \longrightarrow \frac{1}{\sqrt{a}} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

of GA in $SL(2,\mathbf{R})$, as a Lie subgroup of $SL(2,\mathbf{R})$ and using also that the unfolding diagram of $\mathcal{F}_0$ (cf. Theorem B), $D_0 : \widetilde{T}^3_A \longrightarrow GA$, $\rho_0 : \pi_1(T^3_A) \longrightarrow GA$ the desired foliation can be constructed as follows:

Let $\widetilde{M} = \widetilde{T}^3_A \times \mathbf{R}$ be the universal covering of $M = T^3_A \times S^1$ and define $D : \widetilde{M} \longrightarrow \widetilde{SL}(2,\mathbf{R})$, $\rho : \pi_1(M) \longrightarrow \widetilde{SL}(2,\mathbf{R})$

by $D(x,t) = D_0 x \cdot \widetilde{\varphi}(t)$ and $\rho(\gamma, n) = \rho_0(\gamma) \cdot \varphi(n)$ where $\widetilde{\varphi} : \mathbf{R} \longrightarrow \widetilde{SL}(2, \mathbf{R})$ is a lift of the uniparametric subgroup $\varphi : \mathbf{R} \longrightarrow SL(2, \mathbf{R})$ given by

$$\varphi = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

It turns out, using that $\widetilde{\varphi}(n)$ is in the center of $\widetilde{SL}(2, \mathbf{R})$, that ρ is an homomorphism and D is equivariant (i.e. $D((\gamma, n).(x,t)) = \rho(\gamma, n).D(x,t)$). Thus the fibres of D induce the desired Lie foliation on M (cf.[2] for details).

– $\mathcal{G}_5$: Let X be the generator of the transversely affine Lie flow on T^3_A. As we have that $\mathcal{G}_5 = \mathcal{A} + \mathbf{R}$, where $\mathcal{A}$ is the affine Lie algebra of dimension 2, the vector field $(X, 0)$ on $T^3_A \times S^1$ is transversely modeled on $\mathcal{G}_5$ and $\operatorname{codim} \overline{\mathcal{F}} = 2$.
– $\mathcal{G}_6$ and $\mathcal{G}_7$ are not realizables : Let $\mathcal{F}$ be a $\mathcal{G}_6$ or a $\mathcal{G}_7$ Lie flow on a compact manifold M. Fix a generator X of $\mathcal{F}$ and a transverse parallelism Y_1, Y_2, Y_3 such that $[Y_1, Y_2] = 0$, $[Y_1, Y_3] = Y_2$, $[Y_2, Y_3] = Y_1 + Y_2$ for $\mathcal{G}_6$ and $[Y_1, Y_2] = 0$, $[Y_1, Y_3] = Y_1$, $[Y_2, Y_3] = kY_2$ for $\mathcal{G}_7$. Let g be a Riemannian metric on M. Then we have the orthogonal decomposition $TM = T\overline{\mathcal{F}} + T\overline{\mathcal{F}}^{\perp}$ and we shall denote by Z^t and Z^n the tangent and the orthogonal parts of a vector field Z on M.

The set $T = \{p \in M; Y_1^n(p) = 0\}$ is open. In fact, if $p \in T$, Y_1 is tangent to $\overline{\mathcal{F}}$ in p therefore Y_2^n, Y_3^n are independent in p. Hence they are independent in an open neighborhood U of p and we can writte $Y_1^n = \lambda Y_2^n + \mu Y_3^n$ where λ, μ are basic functions on U. Computing now $[Y_1^n, Y_2^n]$ and $[Y_1^n, Y_3^n]$ we deduce the following system of differential equations:

$$\left.\begin{aligned} Y_2^n(\lambda) + \mu\lambda + \mu = 0 \\ Y_2^n(\mu) + \mu^2 = 0 \\ Y_3^n(\lambda) - \lambda^2 = 0 \\ Y_3^n(\mu) - \mu\lambda + \mu = 0 \end{aligned}\right\}$$

for $\mathcal{G}_6$ and

$$\left.\begin{aligned} Y_2^n(\lambda) + k\mu = 0 \\ Y_2^n(\mu) = 0 \\ Y_3^n(\lambda) + (1-k)\lambda = 0 \\ Y_3^n(\mu) + \mu = 0 \end{aligned}\right\}$$

for $\mathcal{G}_7$, with the initial conditions $\lambda(p) = \mu(p) = 0$.

This implies that $\mu = 0$ on the integral curves of Y_3 and Y_2. Due to transverse transitivity $\mu = 0$ on U. It follows in a similar way that $\lambda = 0$ on U. Thus $Y = 0$ on U and T is open.

As it is also closed and M is supposed to be connected, $T = \emptyset$ or $T = M$.

But if $T = M$ we arrive in both cases ($\mathcal{G}_6$ and $\mathcal{G}_7$) to a contradiction. In fact, if we denote by $\theta^0, \theta^1, \theta^2, \theta^3$ the dual basis of X, Y_1, Y_2, Y_3 we have $d\theta^2 = -\theta^2 \wedge \theta^3$ in $\mathcal{G}_6$ and $d\theta^2 = k\theta^2 \wedge \theta^3$ ($k \neq 0$) in $\mathcal{G}_7$. As $\theta^2(Z) = \theta^3(Z) = d\theta^2(Z, \cdot) = d\theta^3(Z, \cdot) = 0$ for each vector field Z tangent to $\overline{\mathcal{F}}$, the 1-forms θ^2 and θ^3 are projectable on the basic manifold $W = M/\overline{\mathcal{F}}$.

So we would have an exact volume element on the compact manifold W, which is a contradiction.

Therefore $T = \emptyset$.

Next we consider the set $Q = \bigcup_{a \in \mathbf{R}} Q_a$ where $Q_a = \{p \in M;\ Y_2^n(p) = aY_1^n(p)\}$.

Q is open: If $p \in Q$, there is $a \in \mathbf{R}$ such that $Y_2^n(p) = aY_3^n(p)$ and hence Y_3^n and Y_2^n are independent in p. So $Y_2^n = \lambda Y_1^n + \mu Y_2^n$ in an open neighborhood U of p with $\lambda(p) = a$ and $\mu(p) = 0$. Computing now $[Y_1, Y_2^n], [Y_3, Y_2^n]$ and considering their tangent and normal parts one obtains the equations:

$$\left.\begin{aligned} Y_1(\lambda) + \mu &= 0 \\ Y_1(\mu) &= 0 \\ Y_3(\lambda) + 1 &= 0 \\ Y_3(\mu) - \mu &= 0 \end{aligned}\right\}$$

As before, this yields $\mu = 0$ i.e. $Y_2^n = \lambda Y_1^n$ on U. Thus every point $x \in U$ is in $Q_{\lambda(x)} \subset Q$ and Q is open.

Q is closed: If $p \notin Q$, for each $a \in \mathbf{R}$, $Y_2^n(p) \neq aY_1^n(p)$. In particular $Y_2^n(p) = 0$. As we have proved that $Y_1^n \neq 0$, the vector fields Y_1, Y_2 are linearly independent on p. Hence they are independent in an open neighborhood U of p, i.e. $U \subset M \setminus Q$ and Q is closed.

As M is connected $Q = \emptyset$ or $Q = M$.

If $Q = \emptyset$, Y_1^n and Y_2^n are linearly independent in each point. So there are differentiable functions λ and μ globally defined on M, such that $Y_3^n = \lambda Y_1^n + \mu Y_2^n$. Computing now $[Y_1, Y_3^n]$ we obtain $Y_1(\lambda) = 1$, but as M is compact this is impossible.

If $Q = M$, for each $p \in M$ there is $a(p) \in \mathbf{R}$ such that $Y_2^n(p) = a(p)Y_1^n(p)$. This gives rise to a differentiable basic function a on

M with $Y_2^n = a\cdot Y_1^n$. Equivalently $Y_2 - a\cdot Y_1$ is everywhere tangent to $\overline{\mathcal{F}}$. Since $[Y_3, Y_2 - a\cdot Y_1]$ must be in $\overline{\mathcal{F}}$ we obtain $Y_3(a) = -1$ for $\mathcal{G}_6$, which is again a contradiction, and $Y_3(a) = (1-k)a$ for $\mathcal{G}_7$. If $k \neq 1$, the only possibility is $a = 0$ and so Y_2 is everywhere tangent to $\overline{\mathcal{F}}$. As before, this yields a contradiction because $d\theta^1 = -\theta^1 \wedge \theta^3$, with θ^1 and θ^3 projectables on $W = M/\overline{\mathcal{F}}$. If $k = 1$ it follows that a is constant over the integral curves of Y_1, Y_2, Y_3, i.e. a is constant. Beeing $\omega^0, \omega^1, \omega^2, \omega^3$ the dual basis of $X, Y_2 - aY_1, Y_1, Y_3$ we obtain $d\omega^2 = -\omega^2 \wedge \omega^3$ with ω^2, ω^3 projectables on W, again a contradiction. This proves that $\mathcal{G}_6$ and $\mathcal{G}_7$ are not realizables.

– $\mathcal{G}_8$ (with $h = 0$) : The same construction as before. If $\theta^0, \theta^1, \theta^2, \theta^3$ are the canonical coordinates on $T^2 \times T^2$, the vector field $X = \partial/\partial\theta^0 + \alpha\partial/\partial\theta^1$, $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ is transversely abelian for the parallelism $\partial/\partial\theta^1, \partial/\partial\theta^2, \partial/\partial\theta^3$ and has codim $\overline{\mathcal{F}} = 2$.

We modify this parallelism by taking

$$\left.\begin{aligned} e_1 &= \cos\theta^2 \cdot \partial/\partial\theta^1 + \sin\theta^2 \cdot \partial/\partial\theta^3 \\ e_2 &= -\sin\theta^2 \cdot \partial/\partial\theta^1 + \cos\theta^2 \cdot \partial/\partial\theta^3 \\ e_3 &= -\partial/\partial\theta^2 \end{aligned}\right\}$$

Thus X is also transversely modeled on $\mathcal{G}_8$ (with $h = 0$).

3 Case codim $\overline{\mathcal{F}}$ **= 1.**

In this case the structural Lie algebra has dimension 2. As this algebra is abelian (cf.[2]), $\mathcal{G}_3$ and $\mathcal{G}_4$ are not realizables because they do not have abelian subalgebras of dimension 2. Examples for the algebras $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_5$ and $\mathcal{G}_8$ ($h = 0$) are given. For the algebra $\mathcal{G}_7$ we prove that the only realizable cases are when $k \notin \mathbf{Q}$, an example will be given. We also prove that $\mathcal{G}_6$ is not realizable.

– $\mathcal{G}_1$: Consider the flow $(X, 0)$ on $T^3 \times T^1$ where X is a dense linear flow on T^3.

– $\mathcal{G}_2$ is not realizable : As $\mathcal{G}_2$ is unimodular and codim $\overline{\mathcal{F}} = 1$, $\mathcal{F}$ is unimodular (cf. [5]) and it follows, from the results by Molino (cf. [6]), that the central transvers sheaf $\mathcal{C}$ admits a global trivialization, i.e. there are independent foliated vector fields v, w tangents to the $\mathcal{F}$ closure which commute, as transvers fields, with every global foliated vector field. In particular $[v, e_i] = [w, e_i] = 0$. Writing

$$\begin{aligned} v =& \lambda e_1 + \mu e_2 + \nu e_3 \\ w =& \alpha e_1 + \beta e_2 + \gamma 3_3 \end{aligned}$$

we obtain $v = \lambda e_1$ and $w = \alpha e_1$ which is a contradiction.

– $\mathcal{G}_5$: Let X be the generator of the transversely affine Lie flow on T^3_A. The vector field $(X, \alpha\partial/\partial\theta)$ on $T^3_A \times S^1$, with $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ and θ the coordinate function on S^1, is transversely modeled on $\mathcal{G}_5 = \mathcal{A} + \mathbf{R}$ and $\operatorname{codim} \overline{\mathcal{F}} = 1$.

– $\mathcal{G}_8$ $(h = 0)$: The same construction as before. If $\theta^0, \theta^1, \theta^2, \theta^3$ are the canonical coordinates on $T^3 \times T^1$, the vector field $X = \partial/\partial\theta^0 + \alpha\partial/\partial\theta^1 + \beta\partial/\partial\theta^2$ with α, β rationally independent, admits

$$\left.\begin{aligned} e_1 &= \cos\theta^3 \cdot \partial/\partial\theta^0 + \sin\theta^3 \cdot \partial/\partial\theta^1 \\ e_2 &= -\sin\theta^3 \cdot \partial/\partial\theta^0 + \cos\theta^3 \cdot \partial/\partial\theta^1 \\ e_3 &= -\partial/\partial\theta^3 \end{aligned}\right\}$$

as a transverse parallelism. But e_1, e_2, e_3 is a basis of $\mathcal{G}_8$ with $h = 0$.

– Next we study the remainder algebras $\mathcal{G}_6$, $\mathcal{G}_7$ and $\mathcal{G}_8$ $(h \neq 0)$. As the center of these algebras are trivial, the corresponding connected simply connected groups G_6, G_7, G_8 can be obtained as $e^{t \cdot \operatorname{ad} \alpha}$, $\alpha \in \mathcal{G}_i$ with $i = 1, 2, 3$. We find that these groups can be thougth as $\mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}$ with the product $(p, t) \cdot (p', t') = (p + e^{\Lambda t} \cdot p', t + t')$ and Λ depending on the algebra.

For $\mathcal{G}_6$

$$\Lambda = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad e^{-\Lambda t} = \begin{pmatrix} e^{-t} & -te^{-t} \\ 0 & e^{-t} \end{pmatrix}$$

For $\mathcal{G}_7$

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}, \quad e^{-\Lambda t} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-kt} \end{pmatrix}$$

For $\mathcal{G}_8$

$$\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & h \end{pmatrix}, \quad e^{-\Lambda t} = C(t) \cdot \begin{pmatrix} \cos(\varphi + t) & -\sin t \\ \sin t & \cos(\varphi - t) \end{pmatrix}$$

where $C(t) = \frac{2}{\alpha} e^{\beta t}$ and $\alpha = \sqrt{4 - h^2}$, $\beta = \tan\varphi = h/\alpha$, $(\sin\varphi = h/2, \cos\varphi = \alpha/2)$.

The basis to define the algebras are given by the following left invariant fields.

For $\mathcal{G}_6$

$$\left.\begin{aligned} e_1 &= e^{-t}\frac{\partial}{\partial x} \\ e_2 &= -te^{-t}\frac{\partial}{\partial x} + e^{-t}\frac{\partial}{\partial y} \\ e_3 &= \frac{\partial}{\partial t} \end{aligned}\right\}$$

For $\mathcal{G}_7$

$$\left.\begin{aligned} e_1 &= e^{-t}\frac{\partial}{\partial x} \\ e_2 &= e^{-kt}\frac{\partial}{\partial y} \\ e_3 &= \frac{\partial}{\partial t} \end{aligned}\right\}$$

For $\mathcal{G}_8$

$$\left.\begin{aligned} e_1 &= \frac{2}{\alpha}e^{-\beta t}(\cos(t+\varphi)\frac{\partial}{\partial x} + \sin t\frac{\partial}{\partial y}) \\ e_2 &= \frac{2}{\alpha}e^{-\beta t}(-\sin t\frac{\partial}{\partial x} + \cos(t-\varphi)\frac{\partial}{\partial y}) \\ e_3 &= -\frac{\alpha}{2}\frac{\partial}{\partial t} \end{aligned}\right\}$$

Suppose now that we have a $\operatorname{codim}\overline{\mathcal{F}} = 1$ realization on a compact manifold M of one of this algebras. We shall denote the algebra by $\mathcal{G}$ and the corresponding group by G. The basic fibration is:

$$T^3 \longrightarrow M \longrightarrow T^1$$

and, as $\pi_1(T^3) = \mathbf{Z}^3$, $\pi_1(T^1) = \mathbf{Z}$ and $\pi_2(T^1) = 0$ the corre-ponding homotopy exact sequence is

$$0 \to \mathbf{Z}^3 \longrightarrow \pi_1(M) \longrightarrow \mathbf{Z} \to 0$$

Since this exact sequence has a section, $\pi_1(M)$ is the semidirect product of $\mathbf{Z}^3$ with $\mathbf{Z}$, i.e. $\pi_1(M)$ is the product $\mathbf{Z}^3 \times \mathbf{Z}$ with the operation $(x,t)\cdot(y,s) = (x + t\cdot y, t+s)$ where $t\cdot y$ represents the natural action of $\mathbf{Z}$ on $\mathbf{Z}^3$. To be precise, if $\varphi : T^3 \longrightarrow T^3$ is the diffeomorphism which gives the bundle, then the action is $t\cdot y = \varphi_*^t \cdot y$ where $\varphi_* : \pi_1(T^3) \longrightarrow \pi_1(T^3)$ is the morphism induced by φ. We shall denote this group by $\mathbf{Z}^3 \times_\varphi \mathbf{Z}$.

Since $\mathcal{F}$ is a Lie foliation we have the unfolding diagram:

$$\begin{array}{ccc} \widetilde{M} & \stackrel{D}{\longrightarrow} & G \\ \downarrow & & \\ M & & \end{array}$$

and the holonomy representation $h : \pi_1(M) \longrightarrow h(\pi_1(M)) = \Gamma \subset G$ with $D(\gamma \cdot \tilde{x}) = h(\gamma) \cdot D\tilde{x}$, $\tilde{x} \in \widetilde{M}$, $\gamma \in \pi_1(M)$.

As $M/\overline{\mathcal{F}}$ is diffeomorphic to $G/\overline{\Gamma}$ (cf., for instance, [5]) we have that $\overline{\Gamma}$ is a two dimensional closed subgroup of G.

The Lie algebra $\mathcal{H}$ of $\overline{\Gamma}_e$ (the identity component of $\overline{\Gamma}$) is named the structural Lie algebra of $\mathcal{F}$ and. In the case of flows it is abelian (cf. [1]).

But it is easy to see that the only two dimensional abelian subalgebra of $\mathcal{G}$ is $< e_1, e_2 >$, thus $\mathcal{H} =< e_1, e_2 >$. Looking at the expressions for e_1 and e_2 in $\mathcal{G}_6$, $\mathcal{G}_7$ and $\mathcal{G}_8$ we see that $\mathcal{H} =< \partial/\partial x, \partial/\partial y >$ and hence $\overline{\Gamma} \simeq \mathbf{R}^2 \times \mathbf{Z}\varepsilon$, $\varepsilon > 0$.

Notice that $\overline{\Gamma}_e = \mathbf{R}^2 \times \{0\}$ is abelian.

Lemma. *Let A be an abelian subgroup of Γ. Then A is contained in $\mathbf{R}^2 \times \{0\}$ or there is an element $a = (a_1, a_2, a_3)$ with $a_3 \neq 0$ such that $A = \{a^n, n \in \mathbf{Z}\}$*

Proof. If A is not in $\mathbf{R}^2 \times \{0\}$, then $A \cap (\mathbf{R}^2 \times \{0\}) = 0 \in \mathbf{R}^3$.

Otherwise there is $(p, 0) \in A$, $p \neq 0$, and $(q, t) \in A$, $t \neq 0$. As A is abelian we have that $(p, 0)(q, t) = (q, t)(p, 0)$. Then,

$$q + e^{-\Lambda t} \cdot p = q + p$$

and this implies that $t = 0$, except for $\mathcal{G}_8$ with $h = 0$, but this case is not considered here. Therefore $A \cap (\mathbf{R}^2 \times \{0\}) = 0 \in \mathbf{R}^3$.

In particular A has at most one element in each level $\mathbf{R}^2 \times \{m\varepsilon\}$, $m \in \mathbf{Z}$. In fact, $a_1 \cdot a_2^{-1} \in A \cap (\mathbf{R}^2 \times \{0\}) = 0$ and $a_1 = a_2$.

Let $a = (a_1, a_2, n\varepsilon)$ be the element of A in the lower level. For each $b = (b_1, b_2, m\varepsilon) \in A$, we put $m = nd + r$, then ba^{-d} is an element of A in the $r\varepsilon$ level and hence $r = 0$, i.e. $b = a^d$ and this proves the lemma.

Proposition 1. *Let the notation be as above. Then*

$$(\mathbf{R}^2 \times \{0\}) \cap \Gamma = h(\mathbf{Z}^3)$$

Proof. Applying the lemma we have four possibilities:

(i) $h(\mathbf{Z}^3)$ and $h(\mathbf{Z})$ are both contained in $\mathbf{R}^2 \times \{0\}$. Then Γ, generated by $h(\mathbf{Z}^3)$ and $h(\mathbf{Z})$, is contained in $\mathbf{R}^2 \times \{0\}$ which contradicts $\mathbf{R}^3/\overline{\Gamma} = S^1$.

(ii) $h(\mathbf{Z}^3)$ is contained in $\mathbf{R}^2 \times \{0\}$ and $h(\mathbf{Z}) = \{a^n, n \in \mathbf{Z}\}$ with $a \notin \mathbf{R}^2 \times \{0\}$. As $h(\mathbf{Z}^3)$ is a normal subgroup of Γ, for each $b \in h(\mathbf{Z}^3)$ we have $aba^{-1} = b'$ which is in $h(\mathbf{Z}^3)$. Hence, the elements of $\mathbf{R}^2 \times \{0\} \cap \Gamma$ can be written as

$$\sigma = b_1 a^{r_1} b_2 a^{r_2} b_3 a^{r_3} \cdots b_k a^{r_k}$$

with $\Sigma r_i = 0$ and $b_i \in h(\mathbf{Z}^3)$. That is $\sigma = \tilde{b} \cdot a^{\Sigma r_i} = \tilde{b} \in h(\mathbf{Z}^3)$, i.e. $(\mathbf{R}^2 \times \{0\}) \cap \Gamma = h(\mathbf{Z}^3)$

(iii) $h(\mathbf{Z})$ is contained in $\mathbf{R}^2 \times \{0\}$ and $h(\mathbf{Z}^3) = \{a^n, n \in \mathbf{Z}\}$ with $a \notin \mathbf{R}^2 \times \{0\}$. In this case Γ is abelian because if we let $h(1) = b$ we have $bab^{-1} = a^k$. This implies $k = 1$ and $ab = ba$. As in (ii) this implies that $\Gamma \cap \mathbf{R}^2 \times \{0\} = h(\mathbf{Z})$ which is not dense in $\mathbf{R}^2 \times \{0\}$.

(iv) $h(\mathbf{Z}) = \{a^n, n \in \mathbf{Z}\}$ and $h(\mathbf{Z}^3) = \{b^n, n \in \mathbf{Z}\}$ with $a, b \notin \mathbf{R}^2 \times \{0\}$. As before $aba^{-1} = b^k$ and therefore Γ is abelian. So the elements of $\mathbf{R}^2 \times \{0\}) \cap \Gamma$ can be written as $a^n \cdot b^{-n} = (a \cdot b^{-1})^n$ and this is not dense in $\mathbf{R}^2 \times \{0\}$.

REMARK: Three elements $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{R}^3$ can generate a dense subgroup of $\mathbf{R}^2$. In fact it suffices to take $\mathbf{u} = \lambda \mathbf{v} + \mu \mathbf{w}$ with λ, μ and $\lambda/\mu \in \mathbf{R} \setminus \mathbf{Q}$. So, a priori, it is possible to have $\overline{h(\mathbf{Z})} = \mathbf{R}^2 \times \{0\}$.

Proposition 2. *$\mathcal{G}_6$ is not realizable.*

Proof. If such a realization exists the subgroup $h(\mathbf{Z}^3)$ is normal in Γ. Let $h(\mathbf{Z}^3) =< (p_1, 0), (p_2, 0), (p_3, 0) >$ and $h(\mathbf{Z}) =< (p, t) >$ with $t > 0$ then the normality condition can be written as:

$$e^{-\Lambda t} \cdot p_i = \sum_{j=1}^{3} \lambda_i^j \cdot p_j$$

where $\lambda_i^j \in \mathbf{Z}$.

The matrix $A = (\lambda_i^j)$ corresponds in fact to $\varphi_* : \mathbf{Z}^3 \longrightarrow \mathbf{Z}^3$ then it is invertible, and, as we are assuming orientability, we have $\det A = 1$. Let $v_1 = (a_1, b_1, c_1)$ and $v_2 = (a_2, b_2, c_2)$ where $p_1 = (a_1, a_2)$, $p_2 = (b_1, b_2)$ and $p_3 = (c_1, c_2)$. From the above equations we have that:

$$\left.\begin{aligned} Av_1 =& a \cdot v_1 + a \log a \cdot v_2 \\ Av_2 =& a \cdot v_2 \end{aligned}\right\}$$

where $a = e^{-t}$.

Completing v_1, v_2 to a basis $\{v_1, v_2, v_3\}$ the matrix A can be written:

$$\begin{pmatrix} a & 0 & \alpha \\ a\log a & a & \beta \\ 0 & 0 & a^{-2} \end{pmatrix}$$

and satisfies that:

$$\left.\begin{aligned} 2a + \frac{1}{a^2} =& p \\ a^2 + \frac{2}{a} =& q \end{aligned}\right\}$$

with $p, q \in \mathbf{Z}$, $0 < a < 1$, and $a \in \mathbf{R} \setminus \mathbf{Q}$. But this is impossible, because this equations implie that $pa^2 - 2qa + 3 = 0$ and hence $a = (q \pm \sqrt{q^2 - 3p})/p$. In particular $\sqrt{q^2 - 3p} \in \mathbf{R} \setminus \mathbf{Q}$. Substituting now a in the first of the above equations we conclude, after a short computation, that $p = q = 3$ which is in contradiction with $a \in \mathbf{R} \setminus \mathbf{Q}$. So $\mathcal{G}_6$ is not realizable.

Proposition 3. *The Lie algebras of the $\mathcal{G}_7$ family with $k \in \mathbf{Q}$ are not realizables.*

Proof. Proceeding as in Proposition 2 we obtain that e^{-t} and e^{-kt} are eigenvalues of A.

The characteristic polynomial of A, $x^3 - px^2 + qx - 1$, has three roots, ξ, ξ^k and $\xi^{-(k+1)}$ with $\xi = e^{-t}$. As $t > 0$ we have $0 < \xi < 1$. This implies, from standard arguments in Galois theory (see lemma below), that $k \notin \mathbf{Q}$; i.e. the Lie algebras of $\mathcal{G}_7$ with $k \notin \mathbf{Q}$ are not realizables.

The authors are grateful to P. Ara for his remarks about the following lemma.

Lemma. *Let $f(x) = x^3 - px^2 + qx - 1$ be a polynomial with $p, q \in \mathbf{Z}$. If there are $k \in \mathbf{R}$ and $\xi \in (0, 1)$ such that the roots of $f(x)$ can be written as ξ, ξ^k, $\xi^{-(k+1)}$, then $k \in \mathbf{R} \setminus \mathbf{Q}$.*

Proof. First we observe that any rational root of this polynomial must be 1 or -1, and so it is irreducible over $\mathbf{Q}$. Hence the Galois group of $f(x)$ over $\mathbf{Q}$ is $\mathbf{Z}_3$ or the symmetric group S_3. In both cases there is an automorphism σ of the splitting field $\mathcal{K}$ of order 3 which is the identity over $\mathbf{Q}$. This automorphism permutes the roots, i.e.

$$\sigma(\xi) = \xi^k, \quad \sigma(\xi^k) = \xi^{-k-1}, \quad \sigma(\xi^{-k-1}) = \xi \qquad \text{or}$$
$$\sigma(\xi) = \xi^{-k-1}, \quad \sigma(\xi^k) = \xi, \quad \sigma(\xi^{-k-1}) = \xi^k$$

If $k = p/q$, using that $\sigma(x^{1/q}) = \pm\sigma(x)^{1/q}$, we obtain $\xi^{-k-1} = \sigma(\xi^k) = \sigma(\xi^{p/q}) = \sigma((\xi^p)^{1/q}) = (\sigma(\xi)^p)^{1/q} = \xi^{k^2}$ in the first case and $\xi = \sigma(\xi^x) = \xi^{(-k-1)k}$ in the second. This implies that $k^2 + k + 1 = 0$ which is impossible. Thus $\xi \notin \mathbf{Q}$ and the lemma is proved.

Example: Now we give an example of a Lie flow on a compact manifold M transversely modeled over a Lie algebra $\mathcal{G}$ of the family $\mathcal{G}_7$ with structural Lie algebra of dimemsion 2.

Let

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 3 \end{pmatrix}$$

be an element of $SL(3;\mathbf{R})$.

The eigenvalues are $\lambda_j = 2+2\cos(\frac{6\pi j-4\pi}{9})$ where $j = 1,2,3$. We have $2+2\cos(8\pi/9) < 1 < 2+2\cos(14\pi/9) < 2+2\cos(2\pi/9)$. If we let $\xi = \lambda_2$, there is a $k < 0$ such that $\xi^k = \lambda_3$. In this case $\lambda_1 = \xi^{-k-1}$. Here k is the quotient of logarithms of algebraic numbers. Notice that this is a necessary condition for the corresponding algebra to be realizable.

Thus we have the eigenvectors

$$u_j = (\lambda_j - 3, \lambda_j(\lambda_j - 3) - 1, 1)$$

A computation shows that the components of this vectors are irrational numbers with irrational quotient, i.e. they induce dense linear flows in T^3.

Now we consider the compact manifold $T^4_A = T^3 \times \mathbf{R}/\sim$ where $(x,t) \sim (A\cdot x, t+1)$. As the direction given by u_1 is invariant by A it induces a global flow in T^4_A. This flow is transversely modeled over the Lie algebra of $\mathcal{G}_7$ with $k = \log\lambda_3/\log\lambda_2 < 0$. To verify this we observe that an invariant tranverse parallelism in $T^3 \times \mathbf{R}$ is given by

$$\left.\begin{aligned} e_1 &= \xi^t u_2 \\ e_2 &= \xi^{kt} u_3 \\ e_3 &= -\frac{1}{\log\xi}\frac{\partial}{\partial t} \end{aligned}\right\}$$

and it satisfies that $[e_1, e_2] = 0$, $[e_2, e_3] = e_1$, $[e_1, e_3] = ke_2$.

Remark: We do not know any realization of $\mathcal{G}_8$ with $h \neq 0$ and $\operatorname{codim}\overline{\mathcal{F}} = 1$.

4 Case codim $\overline{\mathcal{F}} = \mathbf{0}$**.**

This is a trivial case because the transverse algebra coincides with the structural algebra and so it is abelian. Only $\mathcal{G}_1$ is realizable (a linear dense flow on T^4).

References

1. Y. Carrière, *Journées sur les structures transverses. Toulouse 1982*, Asterisque **116** (1984).
2. A. ElKacimi and M. Nicolau, *Espaces homogènes moyennables et feuilletages de Lie*, Pub. IRMA, Lille **12** (1988).
3. E. Fedida, *Sur les feuilletages de Lie*, C.R.A.S.P. **277** (1971), 999–1002.
4. C. Godbillon, *Feuilletages*, Institut de la Recherche Mathematique Avancée., Université Louis Pasteur, Strasbourg.
5. M. Llabrés and A. Reventós, *Unimodular Lie foliations*, Annales Fac. Sci. Toulouse IV (5ème série) (1988).
6. P. Molino, *Géométrie globale des feuilletages riemanniens*, Proc. Kon. Nederl. Akad. Ser A1 **85** (1982), 45–76.

Departament de Matemàtiques.
Universitat Autònoma de Barcelona.
08193 Bellaterra, SPAIN.