ON HYPERBOLIC ONCE-PUNCTURED-TORUS BUNDLES III: COMPARING TWO TESSELLATIONS OF THE COMPLEX PLANE

WARREN DICKS AND MAKOTO SAKUMA

Abstract. To each once-punctured-torus bundle, T_φ, over the circle with pseudo-Anosov monodromy φ, there are associated two tessellations of the complex plane: one, $\Delta(\varphi)$, is (the projection from ∞ of) the triangulation of a horosphere at ∞ induced by the canonical decomposition into ideal tetrahedra, and the other, $CW(\varphi)$, is a fractal tessellation given by the Cannon-Thurston map of the fiber group switching back and forth between gray and white each time it passes through ∞. In this paper, we fully describe the relation between $\Delta(\varphi)$ and $CW(\varphi)$.

Dedicated to Prof. Akio Kawauchi on the occasion of his 60th birthday

1. Introduction

Let φ be an (orientation-preserving) pseudo-Anosov homeomorphism of the once-punctured torus $T := (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$ and let

$$T_\varphi := (T \times \mathbb{R})/((x,t) \sim (\varphi(x), t+1))$$

be the bundle over the circle with fiber T and monodromy φ. By Thurston's uniformization theorem for surface bundles ([22, 20]), T_φ admits a complete hyperbolic structure of finite volume. Since T_φ has a single torus cusp, T_φ admits a *canonical decomposition* into ideal tetrahedra which is dual to the Ford domain ([9, 23]). The complete hyperbolic structure and the canonical decomposition of T_φ were constructed by Jørgensen in his famous unfinished work [14], and rigorous treatments of (part of) his results were given in [1, 2, 12, 13, 15, 21].

The canonical decomposition induces a triangulation of any peripheral torus, which in turn lifts to a triangulation, $\Delta(\varphi)$, of the universal covering of the peripheral torus. We may assume that the ideal point ∞ of the upper-half-space model $\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_+$ of hyperbolic space is a parabolic fixed point of the Kleinian group $\Gamma \cong \pi_1(T_\varphi)$ uniformizing T_φ. Then we may regard $\Delta(\varphi)$

The first author is supported jointly by the MEC (Spain) and the EFRD (EU) through Projects MTM2006-13544 and MTM2008-01550.

The second author is supported by JSPS Grants-in-Aid 18340018, and is partially supported by JSPS Core-to-Core Program 18005.

as a triangulation of a horosphere at ∞, and then, by projection from ∞, as a triangulation of the complex plane $\mathbb{C}$ which is invariant by the stabilizer $\Gamma_\infty \cong \mathbb{Z}^2$ of ∞ in Γ.

On the other hand, one can take an external viewpoint and consider the action of the Kleinian group Γ on the Riemann sphere at infinity. In [16], C. T. McMullen constructed the Cannon-Thurston map associated to T_φ, a Riemann-sphere-filling Γ-invariant Peano curve (cf. [3, 4, 7, 18, 19]). Certain natural Peano sub-curves of this Peano curve fill in fractal domains in the Riemann sphere. The first author's work [5], [6] with J. W. Cannon provided some information about such fractal domains which we now mention. In [5], it was found that the boundary of such a domain is the union of two fractal arcs with common endpoints such that each arc is the limit set of a finitely generated Kleinian semigroup. (In the case of the simplest (orientable) hyperbolic punctured-torus bundle, these results, and the existence of the Cannon-Thurston map, were first obtained in the first author's previous joint work with R.C. Alperin and J. Porti [3].) In [6], these results were applied to show that if the Cannon-Thurston map associated to T_φ switches between gray and white each time it passes through the point ∞, then the complex plane becomes painted with a Γ_∞-invariant colored CW-structure, denoted $CW(\varphi)$. In [6], the vertices, the (fractal) 1-cells, the Peano-subcurves giving the (fractal) 2-cells, and the planar symmetry group of $CW(\varphi)$ were fully described.

Since both $\Delta(\varphi)$ and $CW(\varphi)$ are Γ_∞-invariant tessellations of the complex plane which naturally arise from the punctured-torus bundle T_φ, it is reasonable to expect some nice relation between them. The purpose of this paper is to show that this is actually the case (see Theorems 8.1 and 8.9). Figure 1 illustrates the main results, in which the monodromy φ corresponds to an upward translation preserving the tessellations. In fact, we show that $\Delta(\varphi)$ and $CW(\varphi)$ share the same vertex set and that the combinatorial structure of $CW(\varphi)$ can be recovered from that of $\Delta(\varphi)$ and vice versa. To be more precise, $\Delta(\varphi)$ is endowed with a structure of a "layered simplicial complex", which reflects the bundle structure of T_φ (Section 5), whereas we have seen that $CW(\varphi)$ is endowed with a structure of a "colored CW-complex" (Section 7), which reflects the way that the Cannon-Thurston map fills in the Riemann sphere (see Proposition 7.3). For intuitive descriptions of $\Delta(\varphi)$ and $CW(\varphi)$, see Remarks 5.5 and 7.13. We show in Theorem 8.1 that $\Delta(\varphi)$ with the layered structure (combinatorially) determines $CW(\varphi)$ with the colored structure, and vice versa. The theorem is proved by constructing a certain CW-decomposition of the complex plane which serves as a common parent of the two tessellations, in the sense that each of the two tessellations is obtained from the parent CW-decomposition by collapsing certain edges and deleting others (see Definition 8.8 and Theorem 8.9).

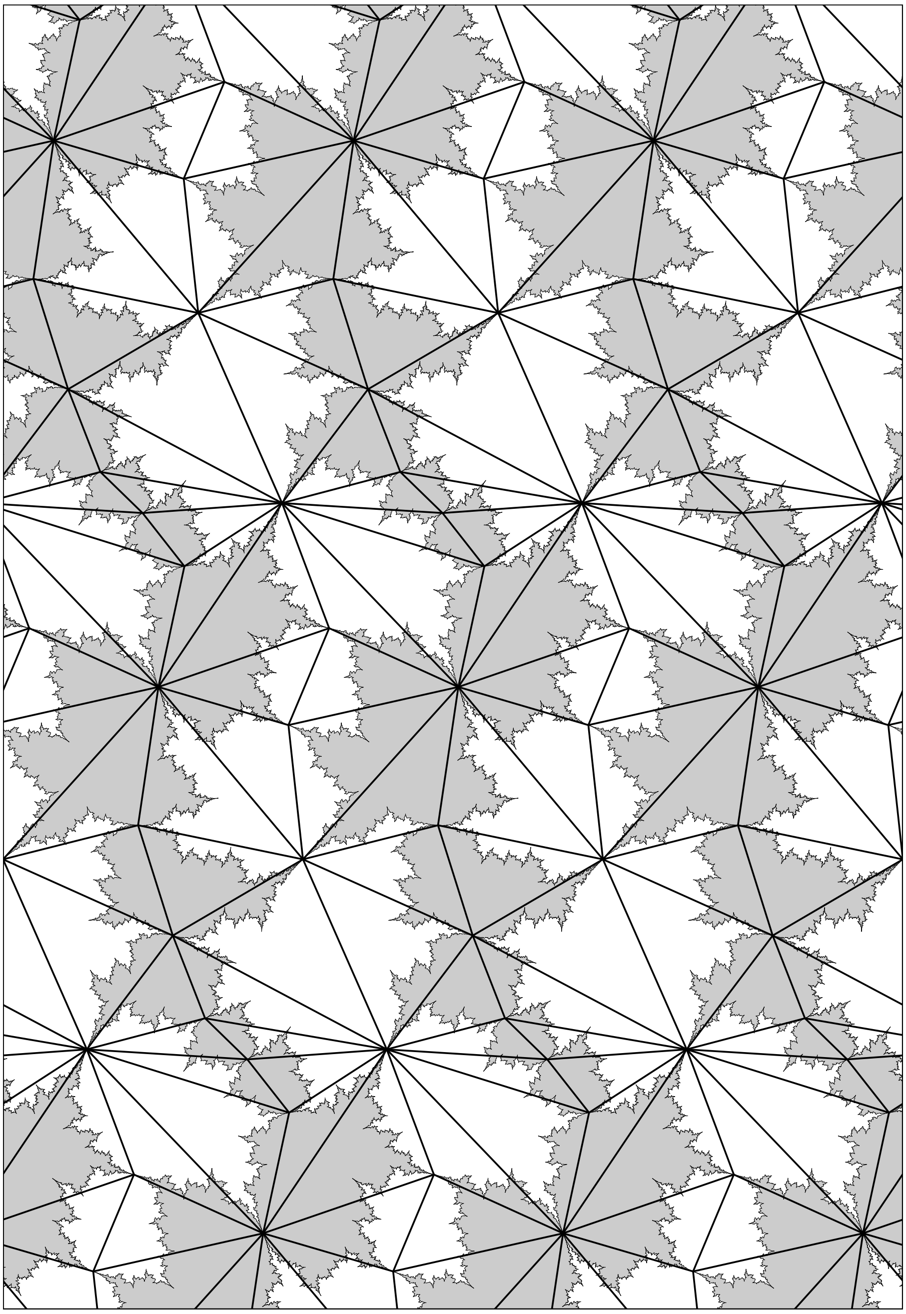

FIGURE 1. Projected-horosphere triangulation $\Delta(\varphi)$ and fractal tessellation $CW(\varphi)$, for $\varphi = RLLRRRLLLL$. The straight line segments etch $\Delta(\varphi)$ while the fractal lines etch $CW(\varphi)$.

This paper is organized as follows. In Section 2, we recall basic facts concerning the orbifold fundamental group $\pi_1(\mathcal{O})$ of the $(2,2,2,\infty)$-orbifold, $\mathcal{O}$, obtained as the quotient of T by the hyper-elliptic involution. The contents in this section give the common language to describe the combinatorial structures of $\Delta(\varphi)$ and $CW(\varphi)$. In Section 3, we describe the normal form of the pseudo-Anosov map φ and fix a convention (Convention 3.1), which we employ throughout the paper. In Section 4, we describe the "type-preserving" $\mathrm{PSL}(2,\mathbb{C})$-representations of $\pi_1(\mathcal{O})$, and fix notation for the punctured-torus bundle T_φ and its natural quotient $\mathcal{O}_\varphi$. In Section 5, we recall the combinatorial description of the canonical decomposition of T_φ, introduce the "layered structure" of $\Delta(\varphi)$ (Definition 5.2), and give a combinatorial description of $\Delta(\varphi)$ in terms of the language prepared in Section 2 (Theorem 5.3 and Proposition 5.4). In Section 6, we recall the combinatorial description of the Cannon-Thurston map associated to T_φ, which was established by Bowditch [4] (Theorem 6.1). In Section 7, we recall the fractal tessellation $CW(\varphi)$ introduced in [6], and, in Theorem 7.10, we give a combinatorial description of $CW(\varphi)$ in terms of the common language developed in Section 2. Finally, in Section 8, we state the main theorem (Theorem 8.1) and give a proof of the theorem.

2. The orbifold $\mathcal{O}$ and its fundamental group

The punctured torus $T = (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$ admits the *hyper-elliptic involution*, induced by the linear automorphism $x \mapsto -x$ of $\mathbb{R}^2 - \mathbb{Z}^2$. The quotient of T by the involution is the $(2,2,2,\infty)$-orbifold $\mathcal{O}$, i.e., the orbifold with underlying space a once-punctured sphere and with three cone points of index 2. The orbifold fundamental group $\pi_1(\mathcal{O})$ is defined to be the covering-transformation group of the universal cover $\tilde{\mathcal{O}}$ of $\mathcal{O}$. Since T is a 2-fold (branched) covering of $\mathcal{O}$, $\tilde{\mathcal{O}}$ is identified with the universal cover $\tilde{T}$ of T, and $\pi_1(T)$ is a subgroup of $\pi_1(\mathcal{O})$ of index 2.

The group $\pi_1(\mathcal{O})$ has the following presentation:

$$\pi_1(\mathcal{O}) = \langle A, B, C \mid A^2 = B^2 = C^2 = 1 \rangle. \tag{2.1}$$

Set $D := CBA$. We may assume that D (resp. D^2) is a peripheral element of $\pi_1(\mathcal{O})$ (resp. $\pi_1(T)$), namely it is represented by a simple loop around the puncture of $\mathcal{O}$ (resp. T). We call D the *distinguished element*.

By picking a complete hyperbolic structure of $\mathcal{O}$ (and hence of T), we identify $\tilde{\mathcal{O}} = \tilde{T}$ with (the upper-half-space model of) the hyperbolic plane $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \Im(z) > 0\}$, and identify $\pi_1(\mathcal{O})$ with a Fuchsian group (see Figure 2).

Then D is identified with the following parabolic transformation having the ideal point ∞ of $\mathbb{H}^2$ as the parabolic fixed point.

$$D(z) = z + 1. \tag{2.2}$$

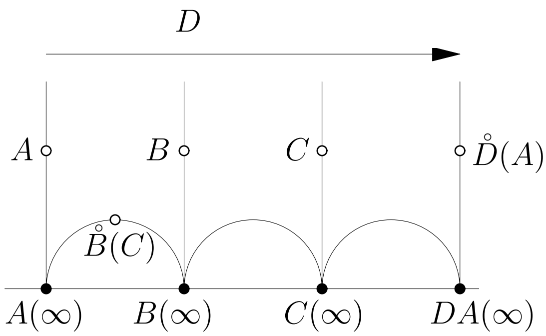

FIGURE 2. Fuchsian group $\langle A, B, C\rangle$. The symbols A, B, C, $\mathring{D}(A)$ and $\mathring{B}(C)$ are situated near the fixed points in $\mathbb{H}^2$ of the involutions they denote.

Then the points $A(\infty)$, $B(\infty)$ and $C(\infty)$ lie on $\mathbb{R}$ from left to right in this order. After a coordinate change, we may assume that the images of the three geodesics joining ∞ with these three points, in the universal abelian cover $\mathbb{R}^2 - \mathbb{Z}^2$ of T, are open arcs of slopes 0, 1 and ∞, joining the puncture $(0, 0)$ with $(1, 0)$, $(1, 1)$ and $(0, 1)$, respectively. Thus the images of these three geodesics in T are mutually disjoint arcs properly embedded in T, which divide T into two ideal triangles, and thus they determine an ideal triangulation of T.

We now recall the well-known correspondence between the ideal triangulations of T and the *Farey triangles*. The *Farey tessellation* is the tessellation of the hyperbolic plane $\mathbb{H}^2$ obtained from the ideal triangle $\langle 0, 1, \infty\rangle$ by successive reflection in its edges. The vertex set of the Farey tessellation is equal to $\hat{\mathbb{Q}} := \mathbb{Q} \cup \{1/0\} \subset \partial\mathbb{H}^2$ and each vertex r determines a properly embedded arc β_r in T of *slope* r, i.e., the arc in T obtained as the image of the straight arc of slope r in $\mathbb{R}^2 - \mathbb{Z}^2$ joining punctures. If $\sigma = \langle r_0, r_1, r_2\rangle$ is a Farey triangle, i.e., a triangle in the Farey tessellation, then the arcs β_{r_0}, β_{r_1} and β_{r_2} are mutually disjoint and they determine an ideal triangulation, $\mathrm{trg}(\sigma)$, of T. In the following we assume that the orientation of $\sigma = \langle r_0, r_1, r_2\rangle$ is coherent with the orientation of the Farey triangle $\langle 0, 1, \infty\rangle$, where the orientation is determined by the order of the vertices. Then the oriented simple loop in T around the puncture representing D^2 meets the edges of $\mathrm{trg}(\sigma)$ of slopes r_0, r_1, r_2 in this cyclic order, for every Farey triangle $\sigma = \langle r_0, r_1, r_2\rangle$.

By using the above notation, the generators A, B and C are described as follows. Consider the ideal triangulation $\mathrm{trg}(\sigma)$ of T determined by the Farey triangle $\sigma = \langle 0, 1, \infty\rangle$. It lifts to a $\pi_1(\mathcal{O})$-invariant tessellation of the universal cover $\tilde{T} = \mathbb{H}^2$. Let $\{e_j\}_{j\in\mathbb{Z}}$ be the edges of the tessellation emanating from the ideal vertex ∞, lying in $\mathbb{H}^2$ from left to right in this order. For each e_j, there is a unique order 2 element, $P_j \in \pi_1(\mathcal{O})$ which inverts e_j. We may assume after a shift of indices that e_{3j}, e_{3j+1} and e_{3j+2} project to the arcs in T of slopes 0, 1 and ∞, respectively, for every $j \in \mathbb{Z}$. Then any triple of consecutive elements $(P_{3j}, P_{3j+1}, P_{3j+2})$ serves as (A, B, C). Throughout this paper, (A, B, C) represents the triple of specific elements of $\pi_1(\mathcal{O})$ obtained in this way. We call $\{P_j\}_{j\in\mathbb{Z}}$ the *sequence of elliptic generators* associated with the Farey triangle σ.

The above construction works for every Farey triangle $\sigma = \langle r_0, r_1, r_2\rangle$, and the sequence of elliptic generators associated with it is defined. (Here we use the assumption that the orientation of $\langle r_0, r_1, r_2\rangle$ is coherent with the orientation of $\langle 0, 1, \infty\rangle$.) Any triple of three consecutive elements in a sequence of elliptic generators is called an *elliptic generator triple*. A member, P, of an elliptic generator triple is called an *elliptic generator*, and its *slope* $s(P) \in \hat{\mathbb{Q}}$ is defined to be the slope of the arc in T obtained as the image of the geodesic $\langle\infty, P(\infty)\rangle$. (Here it should be noted that ∞ is the parabolic fixed point of the distinguished element D.) For example, we have

$$(s(A), s(B), s(C)) = (0, 1, \infty). \tag{2.3}$$

When we say that $\{P_j\}_{j\in\mathbb{Z}}$ is the sequence of elliptic generators associated with a Farey triangle $\sigma = \langle r_0, r_1, r_2\rangle$, we always assume that

$$(s(P_{3m}), s(P_{3m+1}), s(P_{3m+2})) = (r_0, r_1, r_2).$$

Thus the index j is well defined modulo a shift by a multiple of 3. We summarize the properties of elliptic generators (cf. [2, Section 2.1]). We shall use the following non-standard notation.

(2.4) For elements X, Y of a group G, $\mathring{X}(Y)$ denotes XYX^{-1}.

We view $\mathring{X}$ as an element of the automorphism group of G.

Proposition 2.1. (1) *Let $\{P_j\}_{j\in\mathbb{Z}}$ be the sequence of elliptic generators associated with a Farey triangle σ. Then the following hold for every $j \in \mathbb{Z}$.*

(i) $\pi_1(\mathcal{O}) \cong \langle P_j, P_{j+1}, P_{j+2} \mid P_j^2 = P_{j+1}^2 = P_{j+2}^2 = 1\rangle$.
(ii) $P_{j+2}P_{j+1}P_j$ *is equal to the distinguished element D of $\pi_1(\mathcal{O})$.*
(iii) *With the notation of* (2.4), $P_{j+3m} = \mathring{D}^m(P_j)$ *for every $m \in \mathbb{Z}$.*
(iv) $\langle s(P_j), s(P_{j+1}), s(P_{j+2})\rangle$ *is a Farey triangle and its orientation is coherent with $\langle 0, 1, \infty\rangle$.*

(2) *Let P and P' be elliptic generators of the same slope. Then $P' = \mathring{D}^m(P)$ for some $m \in \mathbb{Z}$. Let $\sigma = \langle r_0, r_1, r_2\rangle$ and $\sigma' = \langle r_0', r_1', r_2'\rangle$ be Farey triangles sharing the edge $\langle r_0, r_1\rangle = \langle r_0', r_2'\rangle$, and let $\{P_j\}$ and $\{P_j'\}$, respectively, be the sequences of elliptic generators associated with σ and σ'. Then the following identity holds after a shift of indices by a multiple of* 3 *(see Figure 3).*

$$(P'_{3j}, P'_{3j+1}, P'_{3j+2}) = (P_{3j}, \mathring{P}_{3j+1}(P_{3j+2}), P_{3j+1}). \qquad \square$$

$$
\begin{tikzcd}
& & & P'_2 \arrow[dr] \arrow[d, phantom, "\circlearrowright"] & & & \\
P'_0 \arrow[rr] \arrow[d, equal] & & P'_1 \arrow[ur] \arrow[d, equal] & & P'_3 \arrow[rr] \arrow[d, equal] & & P'_4 \arrow[d, equal] \\
P_0 \arrow[dr] & & P_2 \arrow[rr] & & P_3 \arrow[dr] & & P_5 \\
& P_1 \arrow[ur] \arrow[u, phantom, "\circlearrowleft"] & & & & P_4 \arrow[ur] \arrow[u, phantom, "\circlearrowleft"] &
\end{tikzcd}
$$

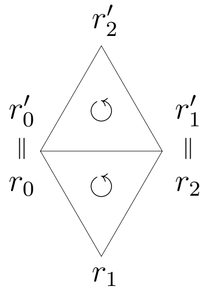

FIGURE 3. Adjacent sequences of elliptic generators. The symbol $\circlearrowleft$, resp. $\circlearrowright$, indicates a triangle in which coherent reading of the vertices is counter-clockwise, resp. clockwise.

The last assertion of the above proposition motivates us to define the *right automorphism* R and the *left automorphism* L of $\pi_1(\mathcal{O})$ by the following rule:

$$
R : (A, B, C) \mapsto (A, \mathring{B}(C), B), \quad L : (A, B, C) \mapsto (B, \mathring{B}(A), C), \tag{2.5}
$$

The proof of the following lemma is straightforward.

Lemma 2.2. *Let $\sigma_0 = \langle 0, \infty, -1\rangle$ and $\sigma_1 = \langle 0, 1, \infty\rangle$. Then the following hold.*

(1) *(A, B, C) is an elliptic generator triple associated with σ_1. Moreover the sequence of elliptic generators associated with σ_1 is as follows:*

$$
\cdots,\ \mathring{D}^{-1}(A),\ \mathring{D}^{-1}(B),\ \mathring{D}^{-1}(C),\ A,\ B,\ C,\ \mathring{D}(A),\ \mathring{D}(B),\ \mathring{D}(C),\ \cdots.
$$

(2) *$(A, C, \mathring{C}(B))$ is an elliptic generator triple associated with σ_0. Moreover the sequence of elliptic generators associated with σ_0 is as follows:*

$$
\cdots,\ \mathring{D}^{-1}(A),\ \mathring{D}^{-1}(C),\ \mathring{D}^{-1}\mathring{C}(B) = \mathring{A}(B),\ A,\ C,\ \mathring{C}(B),\ \mathring{D}(A),\ \cdots.
$$

(3) *Both R and L map the sequence of elliptic generators associated with σ_0 to that associated with σ_1. In fact, we have:*

$$
R(A, C, \mathring{C}(B)) = (A, B, C), \quad L(A, C, \mathring{C}(B)) = (B, C, \mathring{D}(A)).
$$

Moreover, R and L differ only by post composition of a shift of indices of elliptic generators associated with σ_1. To be precise, $LR^{-1}(P_j) = P_{j+1}$, where $\{P_j\}_{j\in\mathbb{Z}}$ is the sequence of elliptic generators associated with σ_1. In particular, $(LR^{-1})^3 = \mathring{D}$. $\square$

Since R and L preserve the distinguished element D, they map elliptic generators to elliptic generators. Moreover, if P and P' are elliptic generators of the same slope, then $R(P)$ and $R(P')$ (resp. $L(P)$ and $L(P')$) have the same slope. Thus R and L act on the set $\hat{\mathbb{Q}}$ of slopes of the elliptic generators.

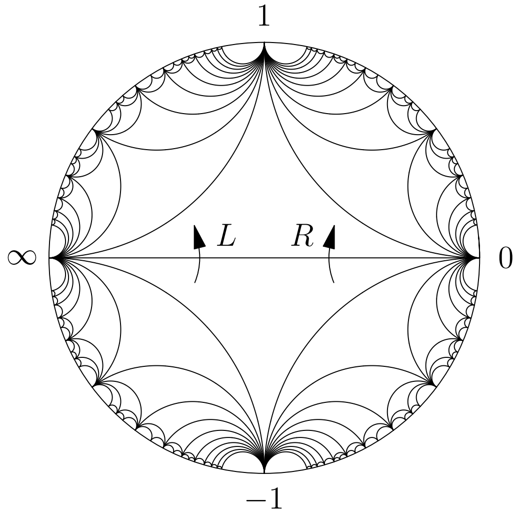

FIGURE 4. The action of R and L on the Farey tessellation

The action R_* (resp. L_*) of R (resp. L) on $\hat{\mathbb{Q}}$ induces an automorphism of the Farey tessellation which acts as a one-unit shift on the bi-infinite sequence of triangles incident on the vertex 0 (resp. ∞), and this shift can be thought of as rotation to the right (resp. left). (See Figure 4.)

3. $SL(2,\mathbb{Z})$ and the Farey tessellation

Recall that the mapping-class group of the once-punctured torus

$$T = (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$$

is identified with $SL(2,\mathbb{Z})$. Thus we may assume the pseudo-Anosov homeomorphism φ is a 'linear' homeomorphism determined by a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

with $|a+d| > 2$.

The homeomorphism φ descends to an automorphism of the orbifold $\mathcal{O}$, denoted by the same symbol, and its action φ_* on the set of slopes of the (elliptic) generators is given by the rule

$$s \mapsto \frac{c+ds}{a+bs}. \tag{3.1}$$

We note that the right and left automorphisms R and L of $\pi_1(\mathcal{O})$ defined by (2.5) are induced by the automorphisms of $\mathcal{O}$ corresponding to the following matrices, which we represent by the same symbols:

$$R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{3.2}$$

The rule (3.1) determines the isometric action φ_* on $\mathbb{H}^2$ preserving the Farey tessellation. Since $|a+d|>2$, $\varphi_*:\mathbb{H}^2\to\mathbb{H}^2$ is a hyperbolic translation, and it has a unique attractive (resp. repulsive) fixed point μ_+ (resp. μ_-) on the ideal boundary $\hat{\mathbb{R}}=\mathbb{R}\cup\{\infty\}$. Since $\mu_\pm$ are irrationals, the oriented geodesic ℓ in $\mathbb{H}^2$ running from μ_- to μ_+ crosses infinitely many Farey triangles $\cdots,\sigma_{-1},\sigma_0,\sigma_1,\sigma_2,\cdots$. This determines a bi-infinite word $\Omega=\prod_{n\in\mathbb{Z}} f_n$ in the letters $\{R,L\}$ by the rule that f_n is R (resp. L) if ℓ exits the Farey triangle σ_n to the right (resp. left) of where it enters. Since φ_* preserves the Farey tessellation, there is a unique (positive) integer p such that

$$\varphi_*(\sigma_n)=\sigma_{n+p}. \tag{3.3}$$

Then $\Omega=(\prod_{n=1}^p f_n)^\infty$, where the product is infinite both to the left and to the right.

After conjugation, we may assume that $\sigma_0=\langle 0,\infty,-1\rangle$ and $\sigma_1=\langle 0,1,\infty\rangle$. Then it follows that φ_* is equal to $\prod_{n=1}^p (f_n)_*$ as isometries of $\mathbb{H}^2$ preserving the Farey tessellation, where $(f_n)_*$ is the isometry R_* or L_* induced by the matrix R or L in (3.2) according as the symbol f_n is R or L. Thus it follows that φ is equal to $\pm\prod_{n=1}^p f_n$ as an element of $SL(2,\mathbb{Z})$. Since the word Ω contains both R and L, we may assume $f_1=R$ and $f_0=f_p=L$ after a shift of indices. This implies the well-known fact that φ is conjugate in $SL(2,\mathbb{Z})$ to

$$\pm R^{a_1}L^{b_1}R^{a_2}L^{b_2}\cdots R^{a_k}L^{b_k} \tag{3.4}$$

for some $k\geqslant 1$ and positive integers a_i and b_i, where $p=\sum_{i=1}^k(a_i+b_i)$. This word, up to cyclic permutation, is uniquely determined by the conjugacy class of φ. We summarize our convention.

Convention 3.1. For the pseudo-Anosov homeomorphism φ of T, $\{\sigma_n\}_{n\in\mathbb{Z}}$ denotes the bi-infinite sequence of Farey triangles and $\{f_n\}_{n\in\mathbb{Z}}$ denotes the bi-infinite sequence of the letters R and L defined in the above. We assume the following conditions are satisfied.

(1) $\sigma_0=\langle 0,\infty,-1\rangle$ and $\sigma_1=\langle 0,1,\infty\rangle$.
(2) $f_0=f_p=L$ and $f_1=R$.
(3) $\varphi=\pm R^{a_1}L^{b_1}R^{a_2}L^{b_2}\cdots R^{a_k}L^{b_k}$, where k, a_i, b_i are positive integers such that $p=\sum_{i=1}^k(a_i+b_i)$.

4. Type-preserving representations of $\pi_1(\mathcal{O})$

A $\mathrm{PSL}(2,\mathbb{C})$-representation of $\pi_1(\mathcal{O})$ (resp. $\pi_1(T)$) is said to be *type-preserving* if it is irreducible (equivalently, it does not have a global fixed point in $\bar{\mathbb{H}}^3$) and sends the distinguished element D (resp. D^2) to a parabolic transformation. It is well known that every type-preserving representation of $\pi_1(T)$ uniquely extends to a type-preserving representation of $\pi_1(\mathcal{O})$ (see e.g. [14, Section 2]). Throughout this paper, we always assume that a type-preserving representation $\rho : \pi_1(\mathcal{O}) \to \mathrm{PSL}(2, \mathbb{C})$ is normalized so that

$$\rho(D) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{4.1}$$

Thus $\rho(D)$ is a parabolic transformation fixing the ideal point ∞.

For an elliptic generator P, we have $\rho(P)(\infty) \neq \infty$ if ρ is faithful, because $\rho(P)$ fixes ∞ if and only if $\rho(DP)$ is an elliptic transformation of order 2 (cf. [2, Proposition 2.4.4]), whereas $DP \in \pi_1(T)$ is a hyperbolic element when we identify $\pi_1(T)$ with a Fuchsian group. We note that DP is represented by a simple loop in T of slope $s(P)$, i.e., a simple loop in $T = (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$ obtained as the image of a line in $\mathbb{R}^2 - \mathbb{Z}^2$ of slope $s(P)$.

Let σ be a Farey triangle and $\{P_j\}_{j\in\mathbb{Z}}$ the sequence of elliptic generators associated with σ. Let $\rho : \pi_1(\mathcal{O}) \to \mathrm{PSL}(2, \mathbb{C})$ be a type-preserving representation and assume that none of the $\rho(P_j)$ fix ∞. Then $\{\rho(P_j)(\infty)\}_{j\in\mathbb{Z}}$ is a sequence of points in $\mathbb{C}$ which is invariant by the Euclidean translation $z \mapsto z + 1$, because

$$\rho(P_{j+3})(\infty) = \rho(DP_jD^{-1})(\infty) = \rho(DP_j)(\infty) = \rho(P_j)(\infty) + 1. \tag{4.2}$$

We denote by $\mathcal{L}(\rho, \sigma)$ the periodic (possibly self-intersecting) piecewise-straight line in $\mathbb{C}$ obtained by joining the points $\{\rho(P_j)(\infty)\}_{j\in\mathbb{Z}}$ successively. If ρ is a faithful discrete $\mathrm{PSL}(2, \mathbb{R})$-representation of $\pi_1(\mathcal{O})$, then $\mathcal{L}(\rho, \sigma)$ gives a triangulation of the real line, and the hyperbolic plane lying above the real line is identified with the universal cover $\tilde{\mathcal{O}}$. Moreover, the vertical geodesic joining ∞ and the vertex $\rho(P_j)(\infty)$ corresponds to the edge e_j introduced in Section 2. In general, (the conjugacy class of) the representation ρ can be recovered from $\mathcal{L}(\rho, \sigma)$ (cf. [2, Section 2.4]), and it plays a key role in the construction of the triangulation $\Delta(\varphi)$ induced by the canonical decomposition of the punctured-torus bundle T_φ.

Since the hyper-elliptic involution of T generates the center of the mapping-class group of T, it extends to a fiber-preserving involution of T_φ. Moreover, it is realized by an isometric involution of the hyperbolic manifold T_φ. The quotient orbifold, $\mathcal{O}_\varphi$, is a bundle over S^1 with fiber $\mathcal{O}$ and admits a complete hyperbolic structure of finite volume. Let $\hat{\Gamma} \cong \pi_1(\mathcal{O}_\varphi)$ be the Kleinian group uniformizing the hyperbolic orbifold, and let $\rho_{\hat{\mathbb{C}}} : \pi_1(\mathcal{O}_\varphi) \to \hat{\Gamma} \subset \mathrm{PSL}(2, \mathbb{C})$ be the holonomy representation. (As in [6], the notation $\rho_{\hat{\mathbb{C}}}$ records the fact that the limit set of $\hat{\Gamma}$ is the whole Riemann sphere $\hat{\mathbb{C}}$.) The bundle structure gives an exact sequence

$$1 \longrightarrow \pi_1(\mathcal{O}) \longrightarrow \pi_1(\mathcal{O}_\varphi) \longrightarrow \mathbb{Z} \longrightarrow 1, \tag{4.3}$$

and the restriction of $\rho_{\hat{\mathbb{C}}}$ to $\pi_1(\mathcal{O})$ is type-preserving.

The orbifold $\mathcal{O}_\varphi$ can be compactified with a single cusp with associated group $\mathbb{Z}^2$. Deleting a small open neighborhood of the $\mathbb{Z}^2$-cusp leaves a compact orbifold with the same orbifold fundamental group as $\mathcal{O}_\varphi$ and with one boundary component; this boundary is a torus which we consider fixed and we call it the *peripheral torus*, and by abuse of notation we denote it by $\partial\mathcal{O}_\varphi$. This terminology lifts from $\mathcal{O}_\varphi$ to T_φ.

With respect to lifting the $\mathbb{Z}^2$-cusp to ∞, the fundamental group of the peripheral torus $\partial\mathcal{O}_\varphi$ is generated by the distinguished element D and an element, $D^\dagger$, where $D^\dagger$ projects to the generator 1 of $\mathbb{Z}$ in the exact sequence (4.3). Since $\rho_{\hat{\mathbb{C}}}(D) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ by the normalization, we have $\rho_{\hat{\mathbb{C}}}(D^\dagger) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ for some non-real complex number λ. Under a suitable orientation convention, we may assume $\Im(\lambda) > 0$. Thus the stabilizers, Γ_∞ and $\hat{\Gamma}_\infty$, of the ideal point ∞ with respect to the actions of Γ and $\hat{\Gamma}$, respectively, are given as follows.

$$\Gamma_\infty = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \right\rangle \cong \pi_1(\partial T_\varphi) \tag{4.4}$$

$$\hat{\Gamma}_\infty = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \right\rangle \cong \pi_1(\partial\mathcal{O}_\varphi). \tag{4.5}$$

We may choose our peripheral torus ∂T_φ so that its preimage in hyperbolic three-space is a family of horospheres. The horosphere at ∞ is acted on by Γ_∞, and can be identified with $\mathbb{C}$ by projection from ∞, and thus ∂T_φ is identified with the quotient space $\mathbb{C}/\Gamma_\infty$. Similar identifications hold for $\partial\mathcal{O}_\varphi$.

5. The canonical decomposition of T_φ

Recall that each Farey triangle σ determines a (topological) ideal triangulation $\mathrm{trg}(\sigma)$ of the punctured torus T. Moreover, if σ and σ' are adjacent Farey triangles, then $\mathrm{trg}(\sigma')$ is obtained from $\mathrm{trg}(\sigma)$ by a "diagonal exchange", i.e., by deleting any one of the three edges and then inserting a new edge in the unique possible way. As illustrated in Figure 5, $\mathrm{trg}(\sigma)$ and $\mathrm{trg}(\sigma')$ can be regarded as the bottom and top faces of an immersed topological ideal tetrahedron (with two pairs of edges identified) in $T \times \mathbb{R}$. We denote this immersed topological ideal tetrahedron in $T \times \mathbb{R}$ by $\mathrm{trg}(\sigma, \sigma')$.

The immersed topological ideal tetrahedra $\{\mathrm{trg}(\sigma_n, \sigma_{n+1})\}_{n\in\mathbb{Z}}$ can be stacked up to form a topological ideal triangulation of $T \times \mathbb{R}$. Since $\varphi_*(\sigma_n) = \sigma_{n+p}$ for every integer n, we may assume that the covering transformation $(x, t) \mapsto (\varphi(x), t+1)$, of the infinite-cyclic covering $T \times \mathbb{R}$ of T_φ, sends $\mathrm{trg}(\sigma_n, \sigma_{n+1})$ to $\mathrm{trg}(\sigma_{n+p}, \sigma_{n+p+1})$ and hence it preserves the topological ideal triangulation of $T \times \mathbb{R}$. Thus there is an induced topological ideal triangulation of T_φ consisting of p ideal tetrahedra and $2p$ ideal triangles and p ideal edges. The following

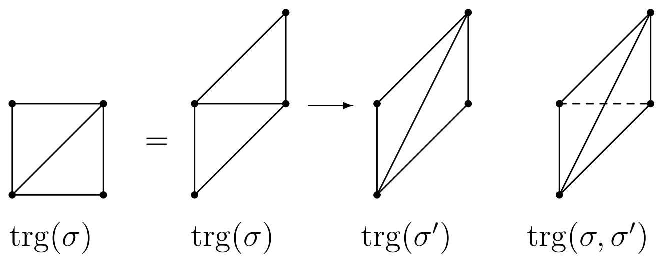

FIGURE 5. trg(σ) and trg(σ') form the immersed topological ideal tetrahedron trg(σ, σ')

theorem was found by Jørgensen [14] (cf. [11]) and rigorous treatments were given in [1, 15, 12, 13, 14, 21].

Theorem 5.1. *The topological ideal triangulation of T_φ described in the above is homeomorphic to the canonical decomposition, defined in the Introduction, of the complete hyperbolic manifold T_φ.* □

In particular, the triangulation of the chosen peripheral torus ∂T_φ induced by the canonical tetrahedral decomposition of T_φ is combinatorially isomorphic to the triangulation induced by the above combinatorial tetrahedral decomposition. We now describe this combinatorial triangulation following [12]. Since $\mathrm{trg}(\sigma_n)$ is an ideal triangulation of a fiber surface T consisting of two ideal triangles, it induces a triangulation, $C(\sigma_n)$, of some chosen *peripheral circle* (to be precise, the circular *link* of the ideal point) ∂T in T. The triangulation $C(\sigma_n)$ consists of 6 edges, which correspond to the 6 ideal vertex neighborhoods of the two ideal triangles. The region in $\partial T \times \mathbb{R}$ bounded by $C(\sigma_n)$ and $C(\sigma_{n+1})$ consists of 4 triangles, which correspond to the 4 ideal vertex neighborhoods of the tetrahedron $\mathrm{trg}(\sigma_n, \sigma_{n+1})$ (see Figure 6).

Since the family $\{\mathrm{trg}(\sigma_n)\}_{n\in\mathbb{Z}}$ forms the 2-skeleton of the ideal triangulation of $T \times \mathbb{R}$, invariant by the covering transformation $(x, t) \mapsto (\varphi(x), t + 1)$, the family $\{C(\sigma_n)\}_{n\in\mathbb{Z}}$ forms the 1-skeleton of a triangulation of $\partial T \times \mathbb{R}$ invariant by the covering transformation. It descends to a triangulation of the peripheral torus ∂T_φ. This is by definition the triangulation induced by the topological ideal triangulation of T_φ. Let $\Delta^*(\varphi)$ be the lift of this triangulation of ∂T_φ to its universal cover $\widetilde{\partial T_\varphi}$. Since $\widetilde{\partial T_\varphi}$ is identified with the universal cover $\widetilde{\partial \mathcal{O}_\varphi}$ of $\partial \mathcal{O}_\varphi$ and since the above triangulation of ∂T_φ is invariant by the hyperelliptic involution, $\Delta^*(\varphi)$ gives a triangulation of $\widetilde{\partial \mathcal{O}_\varphi}$ invariant by $\pi_1(\partial \mathcal{O}_\varphi)$. We identify $\widetilde{\partial \mathcal{O}_\varphi}$ with the complex plane $\mathbb{C}$, where the action of the generators D and $D^\dagger$ of $\pi_1(\partial \mathcal{O}_\varphi)$ are given by the following formulas.

$$D(z) = z + 1, \quad D^\dagger(z) = z + \lambda. \tag{5.1}$$

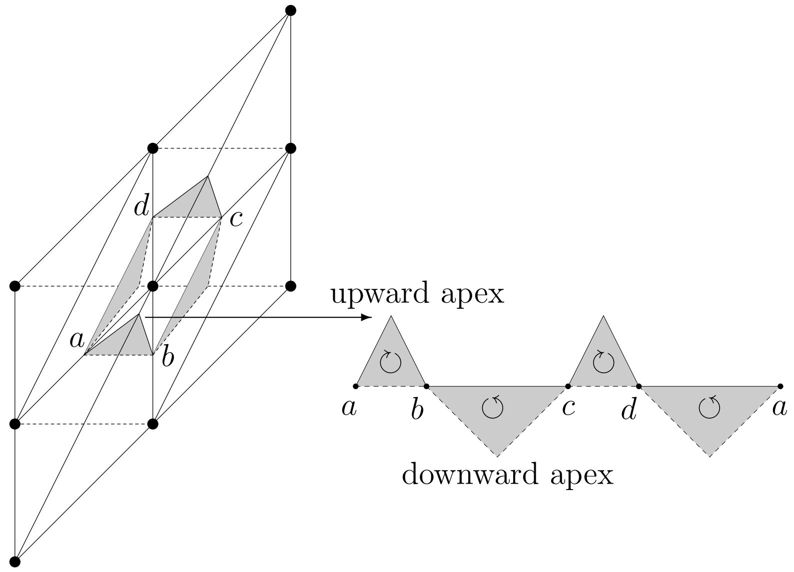

FIGURE 6. The developed image of the triangles corresponding to the 4 ideal vertices of the ideal tetrahedron

Here λ is the complex number in (4.5). Then the inverse image of $C(\sigma_n)$ in $\mathbb{C}$ is a bi-infinite, horizontal, piecewise-straight line, $\mathcal{L}(\sigma_n)$, which is invariant by D. The translation D shifts each vertex and edge of $\mathcal{L}(\sigma_n)$ to the right by 3 combinatorial units. The piecewise-straight line $\mathcal{L}(\sigma_{n+1})$ lies above $\mathcal{L}(\sigma_n)$, and the 1-skeleton of $\Delta^*(\varphi)$ is obtained by stacking up these bi-infinite piecewise-straight lines. Thus the set $\{\mathcal{L}(\sigma_n)\}_{n\in\mathbb{Z}}$ gives a layered structure of $\Delta^*(\varphi)$ in the sense of Definition 5.2 (1) below. Moreover, $(\Delta^*(\varphi), \{\mathcal{L}(\sigma_n)\}_{n\in\mathbb{Z}})$ can be regarded as a layered $\pi_1(\partial\mathcal{O}_\varphi)$-simplicial complex in the sense of Definition 5.2 (3) below. See Figure 7, where the vertices are "opened up" in order to emphasize the layered structure.

Definition 5.2. (1) By a *layered structure* of a 2-dimensional simplicial complex K with underlying space $\mathbb{C}$, we mean a family of 1-dimensional subcomplexes $\{\mathcal{L}_n\}_{n\in\mathbb{Z}}$ indexed by $\mathbb{Z}$, satisfying the following conditions.

(i) Each $\mathcal{L}_n$ gives a triangulation of a subspace homeomorphic to the real line $\mathbb{R}$, and $\cup_{n\in\mathbb{Z}}\mathcal{L}_n$ forms the 1-skeleton of K.

(ii) For each pair $\mathcal{L}_m$ and $\mathcal{L}_n$ with $m \neq n$, the union of $\mathcal{L}_m$ and $\mathcal{L}_n$ cuts off a region in $\mathbb{C}$ homomorphic to a (possibly-pinched) infinite strip.

(iii) $\{\mathcal{L}_n\}_{n\in\mathbb{Z}}$ lie in $\mathbb{C}$ in the order of the index from bottom to top.

We call the pair $(K, \{\mathcal{L}_n\}_{n\in\mathbb{Z}})$ a *layered simplicial complex*. It is often abbreviated to $(K, \{\mathcal{L}_n\})$.

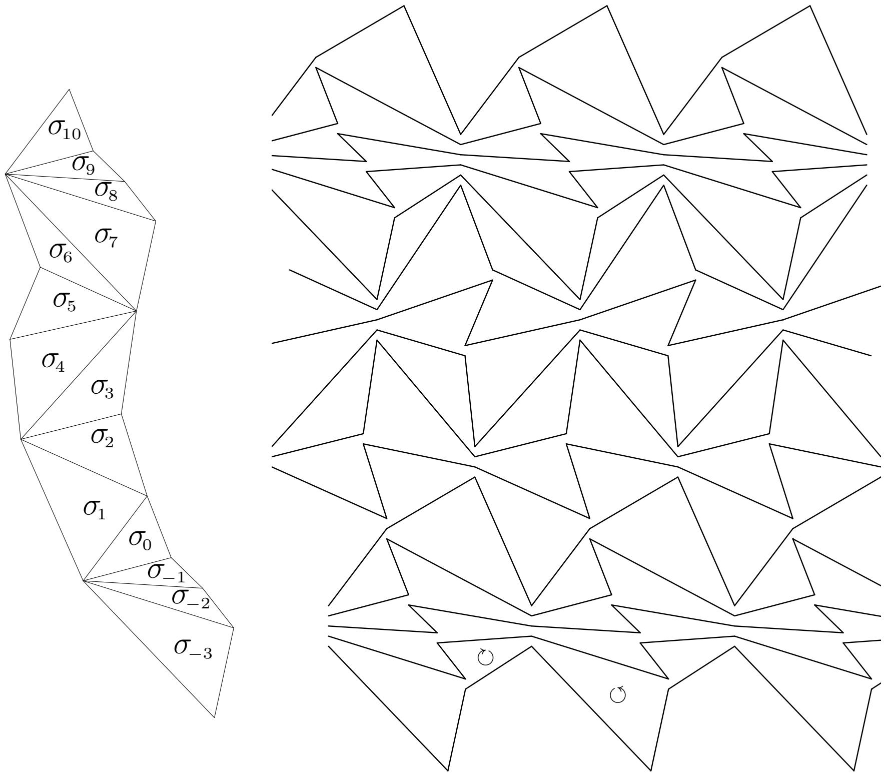

FIGURE 7. The layered simplicial complex $(\Delta^*(\varphi), \{\mathcal{L}(\sigma_n)\})$ for $\varphi = RLLRRRLLLL$

(2) By an *isomorphism* between two layered simplicial complexes $(K, \{\mathcal{L}_n\})$ and $(K', \{\mathcal{L}'_n\})$ we mean a simplicial isomorphism from K to K' which maps $\mathcal{L}_n$ to $\mathcal{L}_{n+d}$, where d is an integer independent of n.

(3) For a group G, a *layered G-simplicial complex* is a layered simplicial complex $(K, \{\mathcal{L}_n\})$ together with an action by G where each element of G acts as an automorphism of $(K, \{\mathcal{L}_n\})$. An *isomorphism* between layered G-simplicial complexes is defined to be a G-equivariant isomorphism between the layered simplicial complexes.

By Theorem 5.1, $\Delta^*(\varphi)$ gives a combinatorial model of $\Delta(\varphi)$. To describe the correspondence, consider the $\pi_1(\mathcal{O})$-invariant ideal triangulation of $\mathbb{H}^3$, obtained as the lift of the canonical decomposition of the hyperbolic manifold T_φ. Then $\Delta(\varphi)$ is obtained as its intersection with the horosphere $\mathbb{C}\times\{t_0\}$, where t_0 is a sufficiently large positive real number. Thus a vertex of $\Delta(\varphi)$ corresponds to a vertical edge (i.e., an edge emanating from ∞) of the lifted canonical decomposition, and vice versa. Similarly, an edge of $\Delta(\varphi)$ corresponds to a vertical face (i.e., a face having ∞ as a vertex) of the lifted canonical decomposition, and vice versa. Recall that we identify $\Delta(\varphi)$ with its projection to the complex plane $\mathbb{C} \subset \partial\mathbb{H}^3$. Then a vertex of $\Delta(\varphi)$ is equal to the endpoint of the corresponding vertical edge of the lifted canonical decomposition, and an edge of $\Delta(\varphi)$ is equal to the line segment joining two vertices obtained as the projection of the corresponding face of the lifted canonical decomposition.

Since the layer $\mathcal{L}(\sigma_n)$ of $\Delta^*(\varphi)$ arises from the ideal triangulation $\mathrm{trg}(\sigma_n)$, it follows that the vertex set of its image in $\Delta(\varphi)$ is equal to $\{\rho_{\hat{\mathbb{C}}}(P_j^{(n)})(\infty)\}_{j\in\mathbb{Z}}$, where $\{P_j^{(n)}\}_{j\in\mathbb{Z}}$ is the sequence of elliptic generators associated with σ_n, and $\rho_{\hat{\mathbb{C}}} : \pi_1(\mathcal{O}) \to \mathrm{PSL}(2,\mathbb{C})$ is the type-preserving representation obtained from the holonomy representation of the complete hyperbolic orbifold $\mathcal{O}_\varphi$. Thus the layer $\mathcal{L}(\sigma_n)$ of $\Delta^*(\varphi)$ corresponds to the periodic piecewise-straight line $\mathcal{L}(\rho_{\hat{\mathbb{C}}}, \sigma_n)$. Hence we obtain the following theorem.

Theorem 5.3. *The family $\{\mathcal{L}(\rho_{\hat{\mathbb{C}}}, \sigma_n)\}_{n\in\mathbb{Z}}$ forms a layered structure of $\Delta(\varphi)$ invariant by the action of $\pi_1(\partial\mathcal{O}_\varphi)$, and the layered $\pi_1(\partial\mathcal{O}_\varphi)$-simplicial complexes $(\Delta(\varphi), \{\mathcal{L}(\rho_{\hat{\mathbb{C}}}, \sigma_n)\})$ and $(\Delta^*(\varphi), \{\mathcal{L}(\sigma_n)\})$ are isomorphic. In particular, $\Delta(\varphi)$ is described as follows.*

(1) *The vertex set of $\Delta(\varphi)$ consists of the points*

$$\rho_{\hat{\mathbb{C}}}(P_j^{(n)})(\infty).$$

(2) *The edge set of $\Delta(\varphi)$ consists of*

$$\langle \rho_{\hat{\mathbb{C}}}(P_j^{(n)})(\infty), \rho_{\hat{\mathbb{C}}}(P_{j+1}^{(n)})(\infty)\rangle.$$

(3) *The face set of $\Delta(\varphi)$ consists of the convex hulls of*

$$\{\rho_{\hat{\mathbb{C}}}(P_j^{(n)})(\infty), \rho_{\hat{\mathbb{C}}}(P_{j+1}^{(n)})(\infty), \rho_{\hat{\mathbb{C}}}(P_{j+2}^{(n)})(\infty)\},$$

such that $(P_j^{(n)}, P_{j+2}^{(n)})$ is a pair of successive elements in the sequence of elliptic generators associated with σ_{n-1} or σ_{n+1}.

Here $\{P_j^{(n)}\}_{j\in\mathbb{Z}}$ is the sequence of elliptic generators associated with σ_n. □

The above argument also implies the following combinatorial description of the topological model $\Delta^*(\varphi)$ of $\Delta(\varphi)$.

Proposition 5.4. (1) *The vertex set of $\Delta^*(\varphi)$ is identified with the set of the elliptic generators P such that $s(P)$ is a vertex of σ_n for some $n \in \mathbb{Z}$.*

(2) *The edge set of $\Delta^*(\varphi)$ is identified with the set of pairs (P, Q) of elliptic generators, such that P and Q are two successive elements in the sequence of elliptic generators associated with σ_n for some $n \in \mathbb{Z}$.*

(3) *The face set of $\Delta^*(\varphi)$ is identified with the set of triples (P, Q, R) of elliptic generators, such that the following hold for some $n \in \mathbb{Z}$.*

(i) *(P, Q, R) is an elliptic generator triple associated with σ_n.*

(ii) *P and R are two successive elements of the sequence of elliptic generators associated with σ_{n-1} or σ_{n+1}.*

(4) *The layer $\mathcal{L}(\sigma_n)$ corresponds to the union of the bi-infinite family of edges $\{\langle P_j^{(n)}, P_{j+1}^{(n)}\rangle\}_{j\in\mathbb{Z}}$, where $\{P_j^{(n)}\}_{j\in\mathbb{Z}}$ is the sequence of elliptic generators associated with σ_n.* □

The following remark, pointed out by the referee, gives heuristic relationships between the word φ and $\Delta(\varphi)$.

Remark 5.5. Quotienting out by all covering transformations and the hyperelliptic involution, we have the following: Each maximal subword of φ of the form R^n or L^n gives rise to one vertex of degree $2n+4$ and $(n-1)$ vertices of degree 4 of $\Delta(\varphi)$. (The average degree is of course 6.) Conversely, Figure 7 provides a way of reading the infinite word Ω generated by φ from $\Delta(\varphi)$ (cf. Remark 7.13).

6. The Cannon-Thurston map

In this section, we recall the combinatorial description of the Cannon-Thurston map following [5, 6]. Recall that we have identified $\pi_1(\mathcal{O})$ with a Fuchsian group by choosing a complete hyperbolic metric of $\mathcal{O}$ of finite area. Let $\rho_{\hat{\mathbb{R}}} : \pi_1(\mathcal{O}) \to \mathrm{PSL}(2,\mathbb{R}) \subset \mathrm{PSL}(2,\mathbb{C})$ be the holonomy representation inducing the identification. Then the *limit set*, $\Lambda(\rho_{\hat{\mathbb{R}}})$, of the group $\rho_{\hat{\mathbb{R}}}(\pi_1(\mathcal{O}))$ is equal to the round circle $\hat{\mathbb{R}}$.

We recall a combinatorial description of the $\pi_1(\mathcal{O})$-space $\Lambda(\rho_{\hat{\mathbb{R}}})$ following [3] (cf. [10]). Let $\mathcal{E}$ be the space of the ends of the group $\pi_1(\mathcal{O})$. Here an *end* of $\pi_1(\mathcal{O})$ is an infinite reduced word in A, B, C, i.e., an infinite sequence $X_1X_2\cdots X_n\cdots$ such that $X_n \in \{A,B,C\}$ and $X_n \neq X_{n+1}$ for every $n \geqslant 1$. For a finite reduced word F in A, B, C, let $[F]$ be the subset of $\mathcal{E}$ consisting of those infinite words which have F as an initial segment. The space $\mathcal{E}$ is endowed with the topology such that the family $\{[F]\}$, where F runs over all finite reduced words, forms a base of the topology. Then $\mathcal{E}$ is a Cantor set, i.e., a compact, Hausdorff, totally disconnected, topological space. Note that the left multiplication of $\pi_1(\mathcal{O})$ induces a left action of $\pi_1(\mathcal{O})$ on $\mathcal{E}$.

For an infinite reduced word W, let W_n be the n-th initial segment of W. Then $\lim \rho_{\hat{\mathbb{R}}}(W_n)(\infty)$ exists in $\hat{\mathbb{R}}$ for every infinite reduced word $W \in \mathcal{E}$, and the map $\mathcal{E} \to \hat{\mathbb{R}} = \Lambda(\rho_{\hat{\mathbb{R}}})$ defined by $W \mapsto \lim \rho_{\hat{\mathbb{R}}}(W_n)(\infty)$ is a continuous $\pi_1(\mathcal{O})$-equivariant surjective map. Moreover, if $\sim$ denotes the equivalence relation on $\mathcal{E}$ generated by the relation

$$WD^{\infty} \sim WD^{-\infty} \tag{6.1}$$

where W runs over elements of $\pi_1(\mathcal{O})$, then the above continuous map induces a $\pi_1(\mathcal{O})$-equivariant homeomorphism from $\mathcal{E}/\sim$ to $\Lambda(\rho_{\hat{\mathbb{R}}}) = \hat{\mathbb{R}}$. It should be noted that the equivalence relation $\sim$ is independent of the choice of a complete hyperbolic structure of $\mathcal{O}$.

Recall the type-preserving representation $\rho_{\hat{\mathbb{C}}} : \pi_1(\mathcal{O}) \to \mathrm{PSL}(2, \mathbb{C})$ associated with the complete hyperbolic orbifold $\mathcal{O}_\varphi$. Then the limit set $\Lambda(\rho_{\hat{\mathbb{C}}})$ of the Kleinian group $\rho_{\hat{\mathbb{C}}}(\pi_1(\mathcal{O}))$ is equal to the whole Riemann sphere $\hat{\mathbb{C}}$. It was proved by McMullen [16] that for every infinite word $W \in \mathcal{E}$ the sequence $\rho_{\hat{\mathbb{C}}}(W_n)(\infty)$ converges in $\hat{\mathbb{C}}$ and that the map $\mathcal{E} \to \hat{\mathbb{C}}$ defined by $W \mapsto \lim \rho_{\hat{\mathbb{C}}}(W_n)(\infty)$ is a continuous $\pi_1(\mathcal{O})$-equivariant surjective map, which in turn induces a continuous $\pi_1(\mathcal{O})$-equivariant surjective map

$$\kappa : \hat{\mathbb{R}} = \Lambda(\rho_{\hat{\mathbb{R}}}) \to \Lambda(\rho_{\hat{\mathbb{C}}}) = \hat{\mathbb{C}}.$$

This is called the *Cannon-Thurston map* associated to the punctured-torus bundle T_φ.

Following [5, 6], we now recall the combinatorial description of the Cannon-Thurston map κ established by Bowditch [4]. Since we have fixed a complete hyperbolic structure on $\mathcal{O}$, we can identify the universal cover of $\mathcal{O}$ with the upper half-space $\mathbb{C}_+ := \{z \in \mathbb{C} \mid \Im z > 0\}$. Thus we obtain the following tower of (topological) coverings:

$$\mathbb{C}_+ \to (\mathbb{R}^2 - \mathbb{Z}^2) \to T \to \mathcal{O}. \tag{6.2}$$

Recall the attractive fixed point, μ_+, of the linear fractional action of φ_* on $\bar{\mathbb{C}}_+$ defined by (3.1), i.e., the slope of the expanding eigenspace of the linear map $\varphi \in SL(2, \mathbb{Z})$. Consider the foliation of T determined by the foliation of $\mathbb{R}^2 - \mathbb{Z}^2$ by lines of slope μ_+. This is the stable foliation of the pseudo-Anosov homeomorphism φ, and it lifts to a foliation of $\mathbb{C}_+$ whose leaves are homeomorphic to the open interval. Each end of each leaf has a well-defined endpoint on $\hat{\mathbb{R}}$, and the closure of each leaf, which we call a *closed-up leaf*, is homeomorphic to the closed interval. Now let $\mathbb{P}$ be the set of the parabolic fixed points of the Fuchsian group $\rho_{\hat{\mathbb{R}}}(\pi_1(\mathcal{O}))$. Then two closed-up leaves have a common point if and only if they share a point, say p, in $\mathbb{P}$. Moreover, either of these closed-up leaves is translated to the other by a parabolic element of $\rho_{\hat{\mathbb{R}}}(\pi_1(\mathcal{O}))$ with parabolic fixed point p. For each $p \in \mathbb{P}$, the union of the closed-up leaves with an endpoint p is called the *spider* with *head* p (cf. [6, Section 5]). Each of the closed-up leaves in the spider is called a *leg*, and the endpoint of a leg different from the head is called a *foot*. It should be noted that the image of a spider in $\mathbb{R}^2 - \mathbb{Z}^2$ consists of a puncture and the pair of rays of slope μ_+ emanating from the puncture. We thus obtain a partition, $\mathcal{P}_+$, of $\bar{\mathbb{C}}_+ = \mathbb{C}_+ \cup \hat{\mathbb{R}}$ into (i) spiders, (ii) closed-up leaves disjoint from $\mathbb{P}$, and (iii) singletons in $\mathbb{R}$ disjoint from the endpoints of the closed-up leaves.

By applying the complex conjugate to (6.2), we obtain yet another tower of (topological) coverings:

$$\mathbb{C}_- \to (\mathbb{R}^2 - \mathbb{Z}^2) \to T \to \mathcal{O},$$

where $\mathbb{C}_- := \{z \in \mathbb{C} \mid \Im z < 0\}$ is the lower half-space. Starting from the foliation of $\mathbb{R}^2 - \mathbb{Z}^2$ by lines of slope μ_-, the repulsive fixed point of φ_*, we obtain a similar partition, $\mathcal{P}_-$, of $\bar{\mathbb{C}}_- = \mathbb{C}_- \cup \hat{\mathbb{R}}$. A piece in $\mathcal{P}_+$ and a piece of $\mathcal{P}_-$, which are not singletons, intersect if and only if they are spiders with a common head, say $p \in \mathbb{P}$, and their intersection is reduced to $\{p\}$. Their union is called the *double spider* with head p. It should be noted that the image of a double spider in $\mathbb{R}^2 - \mathbb{Z}^2$ consists of a puncture and the two pairs of rays of slopes μ_+ and μ_- emanating from the puncture. So the feet of a double spider, contained in $\mathcal{P}_+$ and $\mathcal{P}_-$ are arranged in $\mathbb{R}$ alternately. The 'union' of $\mathcal{P}_+$ and $\mathcal{P}_-$ determines a partition, $\mathcal{P}$, of $\hat{\mathbb{C}}$ into (i) double spiders, (ii) closed-up leaves disjoint from $\mathbb{P}$, and (iii) singletons in $\mathbb{R}$ disjoint from the endpoints of the closed-up leaves.

The continuous map $\hat{\mathbb{R}}/\mathcal{P} \to \hat{\mathbb{C}}/\mathcal{P}$ between the quotient spaces induced by the inclusion $\hat{\mathbb{R}} \to \hat{\mathbb{C}}$ is surjective, because every piece in $\mathcal{P}$ intersects $\hat{\mathbb{R}}$. Since the actions of the Fuchsian group $\rho_{\hat{\mathbb{R}}}(\pi_1(\mathcal{O}))$ on $\hat{\mathbb{R}}$ and $\hat{\mathbb{C}}$ respect the partition $\mathcal{P}$, both quotient spaces inherit $\pi_1(\mathcal{O})$-actions, and the above surjective continuous map is $\pi_1(\mathcal{O})$-equivariant.

As is shown in [5, Appendix], the Moore decomposition theorem implies that the quotient space $\hat{\mathbb{C}}/\mathcal{P}$ is homeomorphic to the 2-sphere. We denote this topological 2-sphere with the $\pi_1(\mathcal{O})$-action by $\mathbb{S}^2_{\mathcal{P}}$. Then we have the following tower of continuous $\pi_1(\mathcal{O})$-equivariant surjective maps:

$$\hat{\mathbb{R}} = \Lambda(\rho_{\hat{\mathbb{R}}}) \to \hat{\mathbb{R}}/\mathcal{P} \to \hat{\mathbb{C}}/\mathcal{P} = \mathbb{S}^2_{\mathcal{P}}.$$

Let $\kappa_{\mathcal{P}} : \hat{\mathbb{R}} = \Lambda(\rho_{\hat{\mathbb{R}}}) \to \hat{\mathbb{C}}/\mathcal{P} = \mathbb{S}^2_{\mathcal{P}}$ be the composition of these surjective maps and call it the *model Cannon-Thurston map*. The following theorem was proved by Bowditch [4].

Theorem 6.1 (Bowditch)**.** *The model Cannon-Thurston map $\kappa_{\mathcal{P}}$ gives a combinatorial model of the Cannon-Thurston map $\kappa : \hat{\mathbb{R}} \to \hat{\mathbb{C}}$. Namely, there is a $\pi_1(\mathcal{O})$-equivariant homeomorphism*

$$\Phi : \mathbb{S}^2_{\mathcal{P}} \to \Lambda(\rho_{\hat{\mathbb{C}}}) = \hat{\mathbb{C}}$$

such that $\kappa = \Phi \circ \kappa_{\mathcal{P}}$. □

By the above theorem and the description of the model Cannon-Thurston map, it follows that the inverse image $\kappa^{-1}(x)$ of a point $x \in \hat{\mathbb{C}}$ consists of one, two, or a countably infinite number of points. If $|\kappa^{-1}(x)| = 2$, then $\kappa^{-1}(x)$ consists of the endpoints of a closed-up leaf of one of the two foliations. If $|\kappa^{-1}(x)| = \infty$, then x is a parabolic fixed point of $\rho_{\hat{\mathbb{C}}}(\pi_1(\mathcal{O}))$, i.e., there is an element D' of $\pi_1(\mathcal{O})$ conjugate to D such that x is the fixed point of the parabolic transformation $\rho_{\hat{\mathbb{C}}}(D')$. In this case $\kappa^{-1}(x)$ consists of the fixed point of $\rho_{\hat{\mathbb{R}}}(D')$ and the feet of the double spider having the point as the head.

7. The fractal tessellation $CW(\varphi)$

In this section, we review the construction of the fractal tessellation $CW(\varphi)$ introduced in [6]. Recall that the Cannon-Thurston map is obtained by shrinking each lifted closed-up leaf of the stable and unstable foliations to a point (in particular, each double spider is shrunk to a point). The fractal tessellation reflects the way the Cannon-Thurston map fills in $\hat{\mathbb{C}}$.

Let $\mathfrak{s}$ be the double spider with head ∞, the parabolic fixed point of $\rho_{\hat{\mathbb{R}}}(D)$, and let $\{\ell_m\}_{m\in\mathbb{Z}}$ be the legs of $\mathfrak{s}$, and let w_m be the foot of ℓ_m. We assume that the elements of $\{w_m\}$ are located in $\mathbb{R}$ in increasing order and that ℓ_m is contained in $\bar{\mathbb{C}}_+$ or $\bar{\mathbb{C}}_-$ according as m is even or odd. It should be noted that $\rho_{\hat{\mathbb{R}}}(D)$ maps ℓ_m to ℓ_{m+2} and that $\kappa^{-1}(\infty) = \{w_m\}_{m\in\mathbb{Z}} \cup \{\infty\}$. Thus $\hat{\mathbb{R}} - \kappa^{-1}(\infty)$ is the disjoint union of open intervals $\sqcup_{m\in\mathbb{Z}}(w_m, w_{m+1})$.

Let $\bar{\ell}_m \subset \hat{\mathbb{C}}$ be the complex conjugate of ℓ_m, and let $\hat{\partial}_m$ be the image of $\bar{\ell}_m$ under the quotient map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}/\mathcal{P} = \mathbb{S}^2_{\mathcal{P}}$. The surjective map $\bar{\ell}_m \to \hat{\partial}_m$ is the quotient map that identifies the two endpoints ∞ and w_m of $\bar{\ell}_m$, because $\bar{\ell}_m$ intersects any member of $\mathcal{P}$, except $\mathfrak{s}$, at most in one point ([6, Proposition 7.8]). Thus $\hat{\partial}_m$ is a simple closed curve in $\mathbb{S}^2_{\mathcal{P}}$ passing through $\infty_{\mathcal{P}}$, where $\infty_{\mathcal{P}} = \Phi^{-1}(\infty) \in \mathbb{S}^2_{\mathcal{P}}$. Hence $\partial_m := \hat{\partial}_m - \{\infty_{\mathcal{P}}\}$ is homeomorphic to the real line and is properly embedded in the model complex plane, $\mathbb{C}_{\mathcal{P}} := \mathbb{S}^2_{\mathcal{P}} - \{\infty_{\mathcal{P}}\}$.

We orient ∂_m as follows. We first orient the loop $\ell_m \cup \bar{\ell}_m$ so that it proceeds from ∞ through $\mathbb{C}_-$ to w_m and through $\mathbb{C}_+$ back to ∞. Since $\hat{\partial}_m$ is equal to the image of the loop $\ell_m \cup \bar{\ell}_m$ in $\mathbb{S}^2_{\mathcal{P}}$, it inherits an orientation from $\ell_m \cup \bar{\ell}_m$ and so does ∂_m. The following proposition is proved in [6, Section 7].

Proposition 7.1. (1) $\partial_m \cap \partial_{m'} = \emptyset$ *whenever* $|m - m'| > 1$.

(2) $\partial_m \cap \partial_{m+1}$ *is a discrete subset of* ∂_m *which accumulates at* $\infty_{\mathcal{P}}$ *from both directions.*

(3) *The closure of each component of* $\mathbb{C}_{\mathcal{P}} - \cup_{m\in\mathbb{Z}}\partial_m$ *is homeomorphic to the disk.* □

By the first and the second assertions of Proposition 7.1, $\partial_{m-1} \cap \partial_m$ and $\partial_m \cap \partial_{m+1}$ are disjoint discrete subsets of the line ∂_m, and their union forms the vertex set of a CW-decomposition of ∂_m. By the last assertion of Proposition 7.1, the union of the 1-dimensional cell complexes $\{\partial_m\}_{m\in\mathbb{Z}}$ etches a $\langle D\rangle$-invariant CW-decomposition of $\mathbb{C}_{\mathcal{P}}$ satisfying the following conditions.

(i) The vertex set is $\cup_{m\in\mathbb{Z}}(\partial_m \cap \partial_{m+1})$.
(ii) The edge set is the set of the closures of components of $\partial_m - (\partial_{m-1} \cup \partial_{m+1})$ where m runs over $\mathbb{Z}$.

(iii) The face set is the set of the closures of the components of $\mathbb{C}_{\mathcal{P}} - \cup_{m \in \mathbb{Z}} \partial_m$. We denote this CW-decomposition of $\mathbb{C}_{\mathcal{P}}$ by $CW(\varphi)$. Note that the $\pi_1(\mathcal{O})$-equivariant homeomorphism $\Phi : \mathbb{S}^2_{\mathcal{P}} \to \Lambda(\rho_{\hat{\mathbb{C}}}) = \hat{\mathbb{C}}$ restricts to a $\langle D \rangle$-equivariant homeomorphism $\mathbb{C}_{\mathcal{P}} \to \mathbb{C}$. We identify the $\langle D \rangle$-space $\mathbb{C}$ with $\mathbb{C}_{\mathcal{P}}$ through this homeomorphism. Then the $\langle D \rangle$-invariant CW-decomposition $CW(\varphi)$ of $\mathbb{C}_{\mathcal{P}}$ determines a $\langle D \rangle$-invariant CW-decomposition of $\mathbb{C}$, which we continue to denote by the same symbol $CW(\varphi)$. We also continue to denote the image of ∂_m in $\mathbb{C}$ by the same symbol.

We give a structure of a colored CW-complex to $CW(\varphi)$, in the sense of Definition 7.2 (1) below, by declaring that the vertices and the faces which lie between ∂_m and ∂_{m+1} are white or gray according as m is even or odd. Thus $CW(\varphi)$ admits a structure of colored $\langle D \rangle$-CW-complex in the sense of Definition 7.2 (3) below.

Definition 7.2. (1) By a *colored CW-complex*, we mean a 2-dimensional CW-complex in which each vertex and each open 2-cell is assigned a value in the set {white, gray}.

(2) A *color-preserving CW-isomorphism* is a homeomorphism between colored CW-complexes which carries cells homeomorphically to cells, and respects the colorings. Two colored CW-complexes are said to be *isomorphic* if there is a color-preserving CW-isomorphism between them.

(3) For a group G, a *colored G-CW-complex* is a colored CW-complex W together with an action by G such that each element of G acts as a color-preserving CW-automorphism of W. An *isomorphism* between two colored G-CW-complexes is defined to be a G-equivariant color-preserving CW-isomorphism between them.

For the actual picture of $CW(\varphi)$, see Figure 1, where the color of a vertex is the color of the two regions occupying most of a small neighborhood of that vertex. The following result shows that this colored CW-complex $CW(\varphi)$ reflects the way the Cannon-Thurston map κ fills in $\hat{\mathbb{C}}$ (see [6, Introduction and Proposition 7.7]).

Proposition 7.3. *The image of the open interval (w_m, w_{m+1}) by the Cannon-Thurston map κ is equal to the closed region bounded by ∂_m and ∂_{m+1}. Moreover, κ fills in the region (cell by cell) from top to bottom or from bottom to top, according as the region is white or gray.* $\square$

Next, we give an explicit description of the vertex set of $CW(\varphi)$. Recall the bi-infinite word $\Omega = \prod_{n \in \mathbb{Z}} f_n$ in the letters $\{R, L\}$ in Section 3.

Definition 7.4. Let $\mathcal{F}_n$ $(n \in \mathbb{Z})$ be the bi-infinite sequence of automorphisms of $\pi_1(\mathcal{O})$ defined by the following rules:

$$\mathcal{F}_0 = 1, \quad \mathcal{F}_n = \mathcal{F}_{n-1} f_n.$$

Then the following proposition holds (see [6, Definition 7.13]).

Proposition 7.5. (1) *The vertex set of* ∂_0 *is equal to*

$$\{\rho_{\hat{\mathbb{C}}}(\mathcal{F}_n(B))(\infty)\}_{n\in\mathbb{Z}},$$

in increasing order. More generally, the vertex set of ∂_{2m} *is equal to*

$$\{\rho_{\hat{\mathbb{C}}}(\mathring{D}^m\mathcal{F}_n(B))(\infty)\}_{n\in\mathbb{Z}} = \{m + \rho_{\hat{\mathbb{C}}}(\mathcal{F}_n(B))(\infty)\}_{n\in\mathbb{Z}}$$

in increasing order.

(2) *The vertex set of* ∂_1 *is equal to*

$$\{\rho_{\hat{\mathbb{C}}}(\mathcal{F}_n\mathring{C}(B))(\infty)\}_{n\in\mathbb{Z}},$$

in increasing order. More generally, the vertex set of ∂_{2m+1} *is equal to*

$$\{\rho_{\hat{\mathbb{C}}}(\mathring{D}^m\mathcal{F}_n\mathring{C}(B))(\infty)\}_{n\in\mathbb{Z}} = \{m + \rho_{\hat{\mathbb{C}}}(\mathcal{F}_n\mathring{C}(B))(\infty)\}_{n\in\mathbb{Z}}$$

in increasing order. □

Remark 7.6. (1) In the above proposition, B and C are elliptic generators as in Lemma 2.2, and the identities among the sets follow from the identity (4.2).

(2) In [6, Definition 7.13], the vertex

$$\rho_{\hat{\mathbb{C}}}(\mathring{D}^m\mathcal{F}_n\mathring{C}(B))(\infty) = \rho_{\hat{\mathbb{C}}}(\mathcal{F}_n\mathring{C}(B))(\infty) + m$$

is described as

$$\rho_{\hat{\mathbb{C}}}(\mathring{D}^{m+1}\mathcal{F}_n(AC))(\infty) = \rho_{\hat{\mathbb{C}}}(\mathring{D}\mathcal{F}_n(AC))(\infty) + m.$$

These points coincide, because

$$\begin{aligned}\rho_{\hat{\mathbb{C}}}(\mathring{D}\mathcal{F}_n(AC))(\infty) &= \rho_{\hat{\mathbb{C}}}(\mathcal{F}_n\mathring{D}(AC))(\infty)\\ &= \rho_{\hat{\mathbb{C}}}(\mathcal{F}_n(CBCD^{-1}))(\infty)\\ &= \rho_{\hat{\mathbb{C}}}(\mathcal{F}_n\mathring{C}(B))(\infty).\end{aligned}$$

In order to describe how the adjacent vertical lines ∂_m and ∂_{m+1} intersect each other, we prepare the following notation.

Definition 7.7. (1) For each $(m,n)\in\mathbb{Z}^2$, let $P_{m,n}$ be the elliptic generator defined by the following formulas:

$$P_{2m,n} := \mathring{D}^m\mathcal{F}_{n-1}(B), \quad P_{2m+1,n} := \mathring{D}^m\mathcal{F}_n\mathring{C}(B).$$

(2) For each $(m,n)\in\mathbb{Z}^2$, let $p_{m,n}$ be the point in $\mathbb{C}$ defined by the following formula:

$$p_{m,n} := \rho_{\hat{\mathbb{C}}}(P_{m,n})(\infty).$$

(3) For each $n \in \mathbb{Z}$, set:

$$\begin{aligned} n_+ &:= \min\{k \in \mathbb{Z} \mid f_k = f_n \text{ and } k > n\}, \\ n_- &:= \max\{k \in \mathbb{Z} \mid f_k = f_n \text{ and } k < n\}. \end{aligned}$$

It should be noted that $(n_+)_- = n = (n_-)_+$ and $f_{n_+} = f_n = f_{n_-}$.

Then we have the following (see [6, Definition 7.13 (iii)]).

Proposition 7.8. (1) *For each $m \in \mathbb{Z}$, the set $\{p_{m,n}\}_{n\in\mathbb{Z}}$ forms the vertex set of ∂_m.*

(2) *The intersection of two adjacent elements of $\{\partial_m\}$ is as follows.*

$$\begin{aligned} \partial_{2m-1} \cap \partial_{2m} &= \{p_{2m,n} \mid f_n = L\} = \{p_{2m-1,n} \mid f_n = L\}, \\ \partial_{2m} \cap \partial_{2m+1} &= \{p_{2m,n} \mid f_n = R\} = \{p_{2m+1,n} \mid f_n = R\}. \end{aligned}$$

Moreover, we have the following identities among the vertices:

$$p_{2m,n} = \begin{cases} p_{2m+1,n_+} & \text{if } f_n = R, \\ p_{2m-1,n_+} & \text{if } f_n = L. \end{cases}$$

Equivalently, we have the following:

$$p_{2m+1,n} = \begin{cases} p_{2m,n_-} & \text{if } f_n = R, \\ p_{2m+2,n_-} & \text{if } f_n = L. \end{cases}$$ □

In order to give an explicit combinatorial model of $CW(\varphi)$, we introduce the following definition (see [6, Definition 4.2] and Figure 8).

Definition 7.9. Let $CW'(\varphi)$ be the CW-decomposition of $\mathbb{R}^2$ defined as follows: The vertex set is $\mathbb{Z}^2$, where each vertex (m, n) is endowed with the label $f_n \in \{L, R\}$. The edge set consists of the *vertical* edges and the *slanted* edges, which are defined as follows:

(E1) The vertical edges are

$$\langle (m, n), (m, n + 1)\rangle.$$

(E2) The slanted edges are

$$\begin{aligned} &\langle (2m, n), (2m + 1, n_+)\rangle \quad \text{if } f_n = R, \\ &\langle (2m, n), (2m - 1, n_+)\rangle \quad \text{if } f_n = L. \end{aligned}$$

The face set of $CW'(\varphi)$ is $\{c'_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$, where $c'_{m,n}$ is described as follows.

(F1) If $f_n = R$ (and hence $f_{n_\pm} = R$), then $c'_{m,n}$ is the convex hull of

$$\{(2m, n_-), (2m, n), (2m + 1, n), (2m + 1, n_+)\},$$

and we assign the color 'white' to its interior (see Figure 8).

(F2) If $f_n = L$ (and hence $f_{n_\pm} = L$), then $c'_{m,n}$ is the convex hull of

$$\{(2m-1, n), (2m-1, n_+), (2m, n_-), (2m, n)\},$$

and we assign the color 'gray' to its interior.

Thus the interior of each 2-cell of $CW'(\varphi)$ has the color white or gray according as it lies in the vertical strip $[m, m+1] \times \mathbb{R}$ with m even or odd. It should be noted that the colored complex $CW'(\varphi)$ admits the action of the infinite cyclic group $\langle D \rangle$ by setting $D(x, y) = (x + 2, y)$.

In the above definition, the vertex (m, n) of $CW'(\varphi)$ corresponds to the vertex $p_{m,n}$ of $CW(\varphi)$, and collapsing the slanted edges of $CW'(\varphi)$ corresponds to the identities in Proposition 7.8(2). This motivates us to define yet another CW-complex, $CW^*(\varphi)$, as the colored CW-complex obtained from $CW'(\varphi)$ by collapsing each closed slanted edge in $CW'(\varphi)$ to a point. The image of the 2-cell $c'_{m,n}$ is a 2-cell of $CW^*(\varphi)$, denoted by $c^*_{m,n}$, and the set $\{c^*_{m,n}\}_{(m,n)\in\mathbb{Z}}$ is the set of the 2-cells of $CW^*(\varphi)$. The open 2-cells of $CW^*(\varphi)$ inherit colors from those of $CW'(\varphi)$. Each vertex of $CW^*(\varphi)$ is the image of a slanted edge of $CW'(\varphi)$, and it is endowed with the color white or gray according as the slanted edge is contained in the white or gray strip. This determines the colored structure on $CW^*(\varphi)$. The action of $\langle D \rangle$ on $CW'(\varphi)$ descends to an action of $\langle D \rangle$ on $CW^*(\varphi)$. Thus $CW^*(\varphi)$ is a colored $\langle D \rangle$-CW-complex, and it gives a combinatorial model of the colored $\langle D \rangle$-CW-complex $CW(\varphi)$. To be precise, we have the following theorem (see [6, Theorem 7.12]).

Theorem 7.10. *There is a color-preserving $\langle D \rangle$-isomorphism from $CW^*(\varphi)$ to $CW(\varphi)$, sending the vertex $[(m, n)]$ to the vertex $p_{m,n}$. The isomorphism maps the image of the vertical line $\{m\} \times \mathbb{R}$ in $CW'(\varphi)$ to the vertical line ∂_m in $CW(\varphi)$. Here $[(m, n)]$ denotes the vertex of $CW^*(\varphi)$ represented by the vertex (m, n) of $CW'(\varphi)$.* □

Remark 7.11. Since $\rho_{\hat{\mathbb{C}}}$ is a faithful discrete representation, two elliptic generators $P_{m,n}$ and $P_{m',n'}$ coincide if and only if the two points $p_{m,n}$ and $p_{m',n'}$ coincide. Thus the slanted edges of $CW'(\varphi)$ represent identities among the elliptic generators, too (see Lemma 8.6(3)). Thus we may identify the vertex set of $CW^*(\varphi)$ with the set $\{P_{m,n}\}$ of elliptic generators, by the correspondence $[(m, n)] \mapsto P_{m,n}$.

Let $c_{m,n}$ be the 2-cell of $CW(\varphi)$ obtained as the homeomorphic image of $c^*_{m,n}$. Then $\{c_{m,n}\}_{(m,n)\in\mathbb{Z}}$ is the set of the 2-cells of $CW(\varphi)$. In the remainder of this section, we give a description of $c_{m,n}$. To this end, let

$$\mathcal{F}'_n = \begin{cases} \mathcal{F}_{n-1}L & \text{if } f_n = R, \\ \mathcal{F}_{n-1}R & \text{if } f_n = L. \end{cases}$$

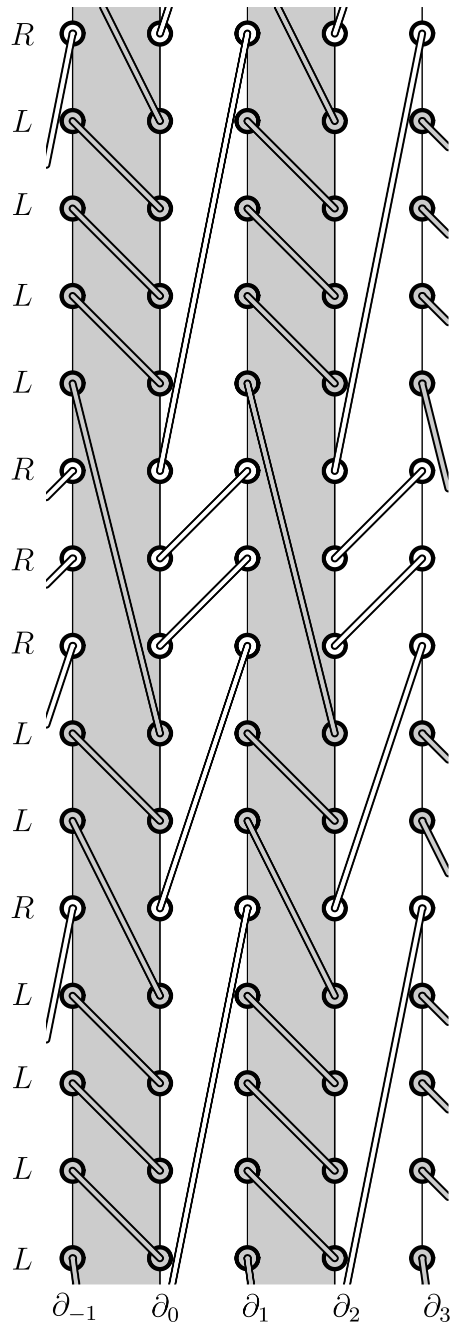

FIGURE 8. The CW-complex $CW'(\varphi)$ for $\varphi = RLLRRRLLLL$. The dots represents the points $P_{m,n}$. The vertical line segments represent vertical edges and the double line segments represent slanted edges.

For example, if $n > 0$, then $\mathcal{F}'_n$ and $\mathcal{F}_n$, viewed as words in R and L, are the same except that their last letters are different. In any event, $\mathcal{F}'_n$ is an automorphism of $\pi_1(\mathcal{O})$ preserving the distinguished element D. Thus it induces a $\langle D\rangle$-equivariant homeomorphism of the space $\mathcal{E}$ of ends of $\pi_1(\mathcal{O})$, preserving the equivalence relation $\sim$ defined by (6.1). Since the limit set $\Lambda(\rho_{\hat{\mathbb{R}}}) = \hat{\mathbb{R}}$ is identified with $\mathcal{E}/\sim$, $\mathcal{F}'_n$ induces a $\langle D\rangle$-equivariant self-homeomorphism of $\Lambda(\rho_{\hat{\mathbb{R}}}) = \hat{\mathbb{R}}$, which we continue to denote by the same symbol. Then we have the following description of the 2-cells of $CW(\varphi)$ ([6, Theorem 7.12]).

Proposition 7.12. *The 2-cell $c_{m,n}$ of $CW(\varphi)$ is equal to $\kappa(\mathring{D}^m\mathcal{F}'_n([B]))$.* $\square$

In the above proposition, B is the elliptic generator of Lemma 2.2, and $[B]$ denotes the image, in $\hat{\mathbb{R}} = \mathcal{E}/\sim$, of the subset $[B]$ of the space $\mathcal{E}$, consisting of those infinite words which have B as an initial segment.

The following remark, pointed out by the referee, gives heuristic relationships between the word φ and $CW(\varphi)$.

Remark 7.13. Definition 7.9 and Theorem 7.10 imply the following. Each letter R (resp. L) of the infinite word Ω generated by φ produces one white (resp. gray) region, modulo the action of $\langle D\rangle$; more precisely, each subword $R(L^a)\boxed{R}(L^b)R$ (with a, b possibly zero) gives rise to a white region corresponding to the boxed R with a spikes (gray vertices) on its left boundary and b spikes on its right boundary. The gray region corresponding to $L(R^a)\boxed{L}(R^b)L$ has a spikes to the right and b spikes to the left. Conversely, we can read Ω off $CW(\varphi)$ as follows. Count the numbers of spikes on the left sides of successive gray regions (going up): $(0, 0, 0, 1, 0, 3)$ in Figure 1; then insert these numbers as exponents to obtain $L(R^0)L(R^0)L(R^0)L(R^1)L(R^0)L(R^3) = LLLLRLLRRR = \varphi$ up to conjugacy. (This can be also applied, with an offset, to the right side.) Using the white regions instead, one finds $(2, 0, 0, 4)$ spikes yielding the word $R(L^2)R(L^0)R(L^0)R(L^4) = RLLRRRLLLL = \varphi$. See Remark 4.3 and the proof of Proposition 4.4 in [6].

8. Statement and the proof of the main result

Theorem 8.1. *The vertex set of the fractal tessellation $CW(\varphi)$ is identical with the vertex set of the projected horosphere triangulation $\Delta(\varphi)$. Moreover, the combinatorial structure of the colored $\langle D\rangle$-CW-complex $CW(\varphi)$ can be recovered from that of the layered $\langle D\rangle$-simplicial complex*

$$(\Delta(\varphi), \{\mathcal{L}(\rho_{\hat{\mathbb{C}}}, \sigma_n)\}),$$

and vice versa.

An explicit description of the second statement of the theorem is presented in italics in mid-course of the proof.

We now begin the proof of the main theorem.

The following concerns the bi-infinite sequence of Farey triangles $\{\sigma_n\}_{n\in\mathbb{Z}}$ occurring in Convention 3.1, and the slope $s(P_{m,n})$ of the elliptic generator $P_{m,n}$ occurring in Definition 7.7.

Lemma 8.2. (1) *$\sigma_n = (\mathcal{F}_n)_*(\sigma_0)$ and $\sigma_{n+1} = (\mathcal{F}_n)_*(\sigma_1)$.*

(2) *$s(P_{2m,n}) = (\mathcal{F}_{n-1})_*(1)$, and it is equal to the vertex of σ_n which is not contained in σ_{n-1}.*

(3) *$s(P_{2m+1,n}) = (\mathcal{F}_n)_*(-1)$, and it is equal to the vertex of σ_n which is not contained in σ_{n+1}.*

Proof. (1) If $n = 0$, then $\mathcal{F}_0 = id$ and the assertion obviously holds. Assume that $n > 0$ and the assertion holds for $n-1$. Since $\mathcal{F}_n = (\mathcal{F}_{n-1} f_n \mathcal{F}_{n-1}^{-1})\mathcal{F}_{n-1}$, the inductive hypothesis implies

$$((\mathcal{F}_n)_*(\sigma_0), (\mathcal{F}_n)_*(\sigma_1)) = ((\mathcal{F}_{n-1} f_n \mathcal{F}_{n-1}^{-1})_*(\sigma_{n-1}), (\mathcal{F}_{n-1} f_n \mathcal{F}_{n-1}^{-1})_*(\sigma_n)).$$

Since $(\mathcal{F}_{n-1})_*$ maps the pair (σ_0, σ_1) to the pair (σ_{n-1}, σ_n), the conjugate $(\mathcal{F}_{n-1} f_n \mathcal{F}_{n-1}^{-1})_*$ of $(f_n)_*$ by $(\mathcal{F}_{n-1})_*$ rotates the pair (σ_{n-1}, σ_n) to the right or left, according as f_n is R or L. By the definition of f_n, this implies that $(\mathcal{F}_{n-1} f_n \mathcal{F}_{n-1}^{-1})_*$ maps the pair (σ_{n-1}, σ_n) to (σ_n, σ_{n+1}). Thus the assertion is valid for n. This proves the assertion when n is non-negative. A parallel argument works for the case when n is negative.

(2) and (3). These assertions follow from (1) and the fact that 1 (resp. -1) is the vertex of σ_1 (resp. σ_0) which is not contained in σ_0 (resp. σ_1). □

Definition 8.3. Let $\{Q_{m,1}\}_{m\in\mathbb{Z}}$ denote the sequence of elliptic generators associated with σ_1 such that

$$(Q_{0,1}, Q_{1,1}, Q_{2,1}) = (A, B, C).$$

Let $Q_{m,n}$ be the elliptic generator defined by

$$Q_{m,n} = \mathcal{F}_{n-1}(Q_{m,1}).$$

(The second identity is consistent with the definition of $Q_{m,1}$ even when $n = 1$, because $\mathcal{F}_0 = id$.)

Lemma 8.4. (1) *For each $n \in \mathbb{Z}$, $\{Q_{m,n}\}_{m\in\mathbb{Z}}$ is the sequence of elliptic generators associated with σ_n. Thus $\{Q_{m,n}\}_{m\in\mathbb{Z}}$ is equal to $\{P_m^{(n)}\}_{m\in\mathbb{Z}}$ as in Theorem 5.3, after a shift of indices m.*

(2) *For each $n \in \mathbb{Z}$, the sequences $\{Q_{m,n}\}_{m\in\mathbb{Z}}$ and $\{Q_{m,n+1}\}_{m\in\mathbb{Z}}$ are related as follows.*

(i) *If $f_n = R$, then*

$$Q_{3m,n} = Q_{3m,n+1}, \quad Q_{3m+1,n} = Q_{3m+2,n+1}.$$

(ii) *If* $f_n = L$, *then*

$$Q_{3m+1,n} = Q_{3m,n+1}, \quad Q_{3m+2,n} = Q_{3m+2,n+1}.$$

Moreover, these are the only identities among the members of the two sequences.

(3) *Two elements* $Q_{m,n}$ *and* $Q_{m',n'}$ *are identical if and only if they are related by a finite sequence of the above identities. To be precise,* $Q_{m,n} = Q_{m',n'}$ *if and only if* (m,n) *and* (m',n') *are equivalent with respect to the equivalence relation generated by the following elementary relations:*

(i) *If* $f_n = R$, *then*

$$(3m,n) \sim (3m,n+1), \quad (3m+1,n) \sim (3m+2,n+1).$$

(ii) *If* $f_n = L$, *then*

$$(3m+1,n) \sim (3m,n+1), \quad (3m+2,n) \sim (3m+2,n+1).$$

Proof. (1) Since $\{Q_{m,1}\}_{m\in\mathbb{Z}}$ is the sequence of elliptic generators associated with σ_1, $\{\mathcal{F}_{n-1}(Q_{m,1})\}_{m\in\mathbb{Z}}$ is the sequence of elliptic generators associated with $(\mathcal{F}_{n-1})_*(\sigma_1)$, which is equal to σ_n by Lemma 8.2(1). Hence $\{Q_{m,n}\}_{m\in\mathbb{Z}}$ is the sequence of elliptic generators associated with σ_n.

(2) Since $\mathcal{F}_n = (\mathcal{F}_{n-1} f_n \mathcal{F}_{n-1}^{-1})\mathcal{F}_{n-1}$, we have

$$\begin{aligned} Q_{k,n+1} &= \mathcal{F}_n(Q_{k,1}) \\ &= (\mathcal{F}_{n-1} f_n \mathcal{F}_{n-1}^{-1})\mathcal{F}_{n-1}(Q_{k,1}) \\ &= \mathcal{F}_{n-1} f_n \mathcal{F}_{n-1}^{-1}(Q_{k,n}). \end{aligned}$$

Since $\mathcal{F}_{n-1}$ maps $Q_{k,1}$ to $Q_{k,n}$, this implies that the triple

$$(Q_{3m,n+1}, Q_{3m+1,n+1}, Q_{3m+2,n+1})$$

is obtained from the triple $(Q_{3m,n}, Q_{3m+1,n}, Q_{3m+2,n})$ by applying the first or second rule in (2.5) according as $f_n = R$ or L, where (A,B,C) is replaced with the triple $(Q_{3m,n}, Q_{3m+1,n}, Q_{3m+2,n})$. Namely,

$$\begin{aligned} &(Q_{3m,n+1}, Q_{3m+1,n+1}, Q_{3m+2,n+1}) \\ &= \begin{cases} (Q_{3m,n}, \mathring{Q}_{3m+1,n}(Q_{3m+2,n}), Q_{3m+1,n}) & \text{if } f_n = R, \\ (Q_{3m+1,n}, \mathring{Q}_{3m+1,n}(Q_{3m,n}), Q_{3m+2,n}) & \text{if } f_n = L. \end{cases} \end{aligned}$$

Hence we obtain (2).

(3) Since the 'if' part is obvious, we prove the 'only if' part. Suppose $Q_{m,n} = Q_{m',n'}$, and denote this elliptic generator by Q. We may assume $m' = m + r$ for some non-negative integer r. Then, for every i $(0 \leqslant i \leqslant r)$, the slope $s(Q)$ is a vertex of the Farey triangle σ_{n+i} and hence Q belongs to

the sequence of elliptic generators associated with σ_{n+i}. Thus we can find by (2) a sequence $\{m_i\}_{0\leqslant i\leqslant r}$ of integers such that $m_0 = m$ and

$$(m,n) = (m_0,n) \sim (m_1,n+1) \sim \cdots \sim (m_r,n+r).$$

Since $Q_{m',n'} = Q_{m,n} = Q_{m_r,n+r} = Q_{m_r,n'}$, we have $m' = m_r$, because the sequence of elliptic generators associated with any Farey triangle consists of mutually distinct elements. Hence $(m,n) \sim (m_r,n+r) = (m',n')$. Thus we obtain the desired result. □

The following lemma describes the relation between the elliptic generators $\{P_{m,n}\}$ in Definition 7.7 and $\{Q_{m,n}\}$ in Definition 8.3.

Lemma 8.5. *The sets $\{P_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ and $\{Q_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ are identical. To be precise, the following hold.*

(1) $P_{2m,n} = Q_{3m+1,n}$ *and* $P_{2m+1,n} = Q_{3m+d(n),n}$, *where*

$$d(n) = \begin{cases} 2 & \text{if } f_n = R, \\ 3 & \text{if } f_n = L. \end{cases}$$

(2) *Conversely, the following hold.*

$$\begin{aligned} Q_{3m+1,n} &= P_{2m,n}, \\ Q_{3m+2,n} &= \begin{cases} P_{2m+1,n} & \text{if } f_n = R, \\ P_{2m+1,n+r} & \text{if } f_n = L, \end{cases} \\ Q_{3m+3,n} &= \begin{cases} P_{2m+1,n+r} & \text{if } f_n = R, \\ P_{2m+1,n+1} & \text{if } f_n = L, \end{cases} \end{aligned}$$

where r is the smallest positive integer such that $f_{n+r} \neq f_n$.

Proof. (1) Since

$$P_{2m+k,n} = \mathring{D}^m(P_{k,n}), \quad Q_{3m+k,n} = \mathring{D}^m(Q_{k,n}),$$

we have only to prove the identities when $m = 0$.

The first identity is proved as follows.

$$P_{0,n} = \mathcal{F}_{n-1}(B) = \mathcal{F}_{n-1}(Q_{1,1}) = Q_{1,n}.$$

We prove the second identity for $n \geqslant 0$ by induction on n. The identity for $n = 0$ is proved as follows.

$$\begin{aligned} P_{1,0} &= \mathcal{F}_0\mathring{C}(B) = \mathring{C}(B) = L^{-1}\mathring{D}(A) = L^{-1}(Q_{3,1}) \\ &= \mathcal{F}_{-1}(Q_{3,1}) = Q_{3,0} = Q_{d(0),0}. \end{aligned}$$

In the above, the third identity follows from Lemma 2.2(3), the fifth identity follows from the fact that $\mathcal{F}_{-1} = f_0^{-1} = L^{-1}$ (see Convention 3.1), and the last identity follows from the convention $f_0 = L$. Now suppose the desired identity holds for some non-negative integer n, namely $P_{1,n} = Q_{d(n),n}$. Then

$$\begin{aligned}
(8.1)\qquad P_{1,n+1} &= \mathfrak{F}_{n+1}\mathring{C}(B) = (\mathfrak{F}_n f_{n+1}\mathfrak{F}_n^{-1})\mathfrak{F}_n\mathring{C}(B) \\
&= \mathfrak{F}_n f_{n+1}\mathfrak{F}_n^{-1}(P_{1,n}) = \mathfrak{F}_n f_{n+1}\mathfrak{F}_n^{-1}(Q_{d(n),n}) \\
&= \mathfrak{F}_n f_{n+1} f_n^{-1}\mathfrak{F}_{n-1}^{-1}(Q_{d(n),n}) = \mathfrak{F}_n f_{n+1} f_n^{-1}(Q_{d(n),1}).
\end{aligned}$$

If $f_{n+1} = f_n$, then $d(n) = d(n+1)$ and hence the last element in the above identity is equal to

$$\mathfrak{F}_n(Q_{d(n),1}) = Q_{d(n),n+1} = Q_{d(n+1),n+1},$$

and therefore the desired identity holds for $n+1$. Suppose $(f_n, f_{n+1}) = (R, L)$. By Lemma 2.2(3), we have

$$f_{n+1}f_n^{-1}(Q_{m,1}) = LR^{-1}(Q_{m,1}) = Q_{m+1,1}.$$

Hence the last term of (8.1) is equal to

$$\mathfrak{F}_n(Q_{d(n)+1,1}) = Q_{d(n)+1,n+1} = Q_{d(n+1),n+1}.$$

Here the last identity follows from the fact that $(d(n), d(n+1)) = (2, 3)$. Thus the desired identity holds for $n+1$. The case $(f_n, f_{n+1}) = (L, R)$ is treated similarly. Thus we have proved, by induction, the desired identity for every non-negative integer n. By a parallel argument, we can also show that the identity holds for every integer n.

(2) Suppose $f_n = R$. Then, by Lemma 8.5(1), we have

$$Q_{3m+1,n} = P_{2m,n}, \quad Q_{3m+2,n} = P_{2m+1,n}.$$

By Lemma 8.4(2), we have $Q_{3m+3,n} = Q_{3m+3,n+1}$. By applying the same lemma repeatedly, we have

$$Q_{3m+3,n} = Q_{3m+3,n+1} = \cdots = Q_{3m+3,n+r},$$

because $f_n = \cdots = f_{n+r-1} = R$. Since $f_{n+r} = L$, Lemma 8.5(1) implies $Q_{3m+3,n+r} = P_{2m+1,n+r}$. Hence we obtain the desired identity. A similar argument also works for the case $f_n = L$. $\square$

The following lemma completely describes the identities among the elliptic generators $\{P_{m,n}\}$ and $\{Q_{m,n}\}$.

Lemma 8.6. (1) *If $f_n = R$, then $P_{2m,n} = P_{2m+1,n_+}$. Moreover $Q_{m',n'}$ is equal to this element if and only if (m', n') belongs to the following set.*

$$\{(3m+1, n)\} \cup \{(3m+2, k) \mid n+1 \leqslant k \leqslant n_+\}.$$

(2) *If $f_n = L$, then $P_{2m,n} = P_{2m-1,n_+}$. Moreover $Q_{m',n'}$ is equal to this element if and only if (m', n') belongs to the following set.*

$$\{(3m+1, n)\} \cup \{(3m, k) \mid n+1 \leqslant k \leqslant n_+\}.$$

(3) *The following are the only identities among the* $P_{m,n}$.

$$P_{2m,n} = \begin{cases} P_{2m+1,n_+} & \text{if } f_n = R, \\ P_{2m-1,n_+} & \text{if } f_n = L. \end{cases}$$

Proof. (1) Suppose $f_n = R$. Then by Lemmas 8.5(1) and 8.4(2), we see

$$P_{2m,n} = Q_{3m+1,n} = Q_{3m+2,n+1}.$$

Since $f_k = L$ for every k $(n+1 \leqslant k \leqslant n_+ - 1)$, Lemma 8.4(2) implies that the above element is equal to $Q_{3m+2,k}$ for every k $(n+1 \leqslant k \leqslant n_+)$. Moreover, by Lemma 8.4(3), we see that $Q_{m',n'}$ is equal to $Q_{3m+1,n}$ if and only if (m', n') appears in the above. (Check that there are no elementary relations relating $(3m+1, n)$ with $(*, n-1)$ and those relating $(3m, n_+)$ with $(*, n_+ + 1)$.) Since $f_{n_+} = f_n = R$, we have $Q_{3m+2,n_+} = P_{2m+1,n_+}$ by Lemma 8.5(2). This completes the proof of (1).

(2) is proved by an argument parallel to that of (1).

(3) The desired identity is already proved by (1) and (2). Let (r, s) and (r', s') be elements of $\mathbb{Z}^2$ such that $P_{r,s} = P_{r',s'}$. Suppose first that r and r' have the same parity. Then we have $s = s'$ by Lemma 8.2(2),(3). So, by Definition 7.7, $P_{r',s'} = P_{r',s}$ is the conjugate of $P_{r,s}$ by $D^{(r'-r)/2}$. Since $P_{r,s} = P_{r',s'}$, this implies $r = r'$ and hence $(r, s) = (r', s')$. Suppose r and r' have different parity, say $r = 2m$ and $r' = 2m' + 1$ for some $m, m' \in \mathbb{Z}$. Then, by Lemma 8.2(2),(3) again, s' is uniquely determined by s. On the other hand, we have $P_{r,s} = P_{2m,s}$ is equal to P_{2m+1,s_+} or P_{2m-1,s_+} according as $f_s = R$ or L. Thus we have $s' = s_+$. By the argument for the same-parity case, we see that s' is equal to $2m+1$ or $2m-1$ according as $f_s = R$ or L. Thus we obtain the conclusion. □

Remark 8.7. As observed in Remark 7.11, the assertion (3) in the above lemma is essentially equivalent to the identities in Proposition 7.8(2).

Lemmas 8.4 and 8.5 motivate us to introduce the following CW-decomposition of $\mathbb{R}^2$ (see Figures 9 and 10).

Definition 8.8. $CW\Delta(\varphi)$ denotes the CW-decomposition of $\mathbb{R}^2$ defined as follows. The vertex set is equal to $\mathbb{Z}^2$, where (m, n) is labeled with $Q_{m,n}$, and the edge set consists of the *horizontal edges*, the *vertical edges* and the *slanted edges*, which are described as follows.

(1) The *horizontal edges* are:

$$\langle (m, n), (m+1, n) \rangle.$$

(2) The *vertical edges* are:

$$\langle (3m+1, n), (3m+1, n+1) \rangle,$$
$$\langle (3m + d(n), n), (3m + d(n+1), n+1) \rangle.$$

(3) For each $n \in \mathbb{Z}$, the *slanted edges* lying between $\mathbb{R}\times\{n\}$ and $\mathbb{R}\times\{n+1\}$ are:
  if $f_n = R$,

$$\langle(3m,n),(3m,n+1)\rangle,$$
$$\langle(3m+1,n),(3m+2,n+1)\rangle,$$

and, if $f_n = L$,

$$\langle(3m+1,n),(3m,n+1)\rangle,$$
$$\langle(3m+2,n),(3m+2,n+1)\rangle.$$

For each $n \in \mathbb{Z}$, the horizontal edges $\{\langle(m,n),(m+1,n)\rangle\}_{m\in\mathbb{Z}}$ of $CW\Delta(\varphi)$ correspond to the edges of the layer $\mathcal{L}(\sigma_n)$ of $\Delta^*(\varphi)$ (cf. Proposition 5.4(4) and Lemma 8.4(1)). For each $m \in \mathbb{Z}$, the vertical edges

$$\{\langle(3m+1,n),(3m+1,n+1)\rangle\}_{n\in\mathbb{Z}}$$

correspond to the edges in the vertical line ∂_{2m} in $CW(\varphi)$, and the vertical edges

$$\{\langle(3m+d(n),n),(3m+d(n+1),n+1)\rangle\}_{n\in\mathbb{Z}}$$

correspond to the edges in the vertical line ∂_{2m+1} in $CW(\varphi)$ (cf. Proposition 7.8(2) and Lemma 8.5(1)). The slanted edges correspond to the elementary relations in Lemma 8.4(3). Thus $CW\Delta(\varphi)$ can be regarded as a common parent of the two tessellations $\Delta(\varphi)$ and $CW(\varphi)$. To be precise, we have the following theorem, for which all terms have been defined.

Theorem 8.9. (1) *Let $\Delta^{**}(\varphi)$ be the CW-complex obtained from $CW\Delta(\varphi)$ by removing (the interior of) each vertical edge and collapsing each slanted edge to a point. Then $\Delta^{**}(\varphi)$ is combinatorially isomorphic to $\Delta^*(\varphi)$ (and hence to $\Delta(\varphi)$), where the vertex of $\Delta^{**}(\varphi)$ represented by (m,n) corresponds to the vertex $Q_{m,n}$ of $\Delta^*(\varphi)$. Here, we employ the identification described by Proposition 5.4.*

(2) *Let $CW^{**}(\varphi)$ be the CW-complex obtained from $CW\Delta(\varphi)$ by removing (the interior of) each horizontal edge and collapsing each slanted edge to a point. Then $CW^{**}(\varphi)$ is combinatorially isomorphic to $CW^*(\varphi)$ (and hence to $CW(\varphi)$), where the vertex of $CW^{**}(\varphi)$ represented by $(3m+1,n)$ (resp. $(3m+d(n),n)$) corresponds to the vertex $P_{2m,n}$ (resp. $P_{2m+1,n}$) of $CW^*(\varphi)$. Here, we employ the identification described by Remark 7.11.*

Proof. (1) By Proposition 5.4 and Lemma 8.4(1), the vertex set of $\Delta^*(\varphi)$ is identified with the set $\{Q_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$. On the other hand, since the slanted edges of $CW\Delta(\varphi)$ correspond to the elementary relations in Lemma 8.4(3), the vertex set of $\Delta^{**}(\varphi)$ is identified with the set $\{Q_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$. Hence there is a natural bijection between the vertex sets of $\Delta^*(\varphi)$ and $\Delta^{**}(\varphi)$. By virtue of the characterization of the edge set and the face set of $\Delta^*(\varphi)$ by Proposition

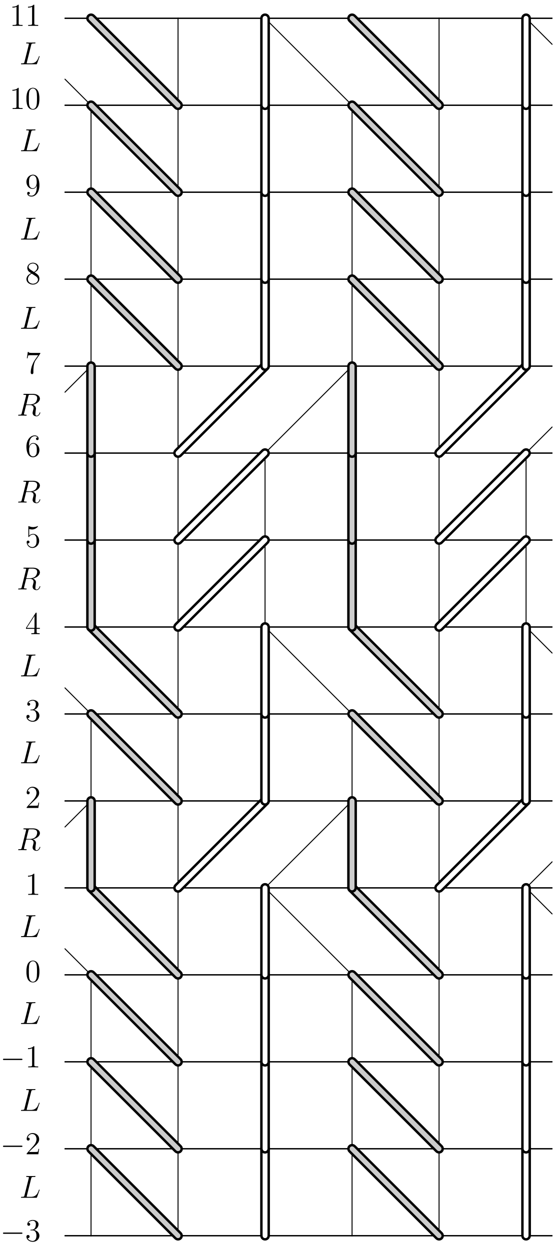

FIGURE 9. The CW-complex $CW\Delta(\varphi)$ for $\varphi = RLLRRRLLLL$. The horizontal line segments represent horizontal edges, the thin non-horizontal line segments represent vertical edges, and the thick line segments represent slanted edges.

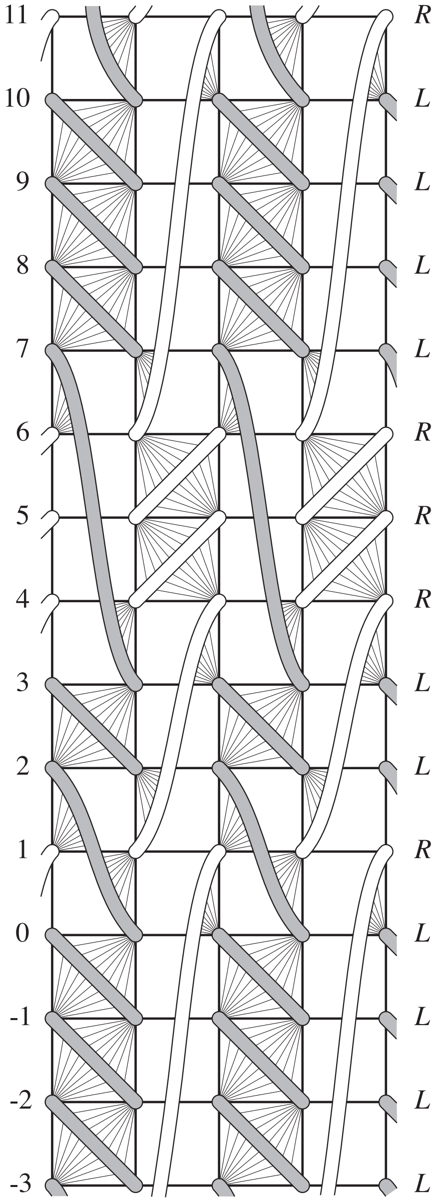

Figure 10. Another view of $CW\Delta(\varphi)$ for $\varphi = RLLRRRLLLL$. In contrast to Figure 9, here the "vertical", "horizontal" and "slanted" lines have the indicated properties. The slanted edges are represented by thick, white or gray connections between the vertices $Q_{m,n}$ (there is a vertex wherever a slanted edge intersects a horizontal line). The isotopies that exist between vertical and horizontal edges (after collapsing the slanted ones) sweep out the small, dashed area.

5.4, the bijection between the vertex sets of $\Delta^*(\varphi)$ and $\Delta^{**}(\varphi)$ extends to an isomorphism between the two CW-complexes.

(2) By Remark 7.11, the vertex set of $CW^*(\varphi)$ is identified with the set $\{P_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$, which in turn is equal to the set

$$\{Q_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$$

by Lemma 8.5(1). On the other hand, as in the proof of Theorem 8.9(1), the vertex set of $CW^{**}(\varphi)$ is also identified with the set $\{Q_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$. Thus there is a natural bijection between the vertex sets of $CW^*(\varphi)$ and $CW^{**}(\varphi)$. By the correspondences mentioned in the paragraph preceding the proposition, this bijection extends to an isomorphism between the two CW-complexes. □

Remark 8.10. (1) The subcomplex of $CW\Delta(\varphi)$ obtained by deleting the vertical edges is a tessellation by four types of pentagons. These pentagons, which become triangles after slanted-edge collapsing, can be thought of as being equal to the pentagons that appeared in Figure 3.

(2) Each vertical edge becomes identified (up to isotopy with fixed endpoints) with exactly one horizontal edge.

We now give a proof of the first assertion of Theorem 8.1. By Theorem 5.3 and Lemma 8.4(1), the vertex set of $\Delta(\varphi)$ is equal to $\{\rho_{\hat{\mathbb{C}}}(Q_{m,n})(\infty)\}_{(m,n)\in\mathbb{Z}^2}$. By the definition of $CW(\varphi)$, the vertex set of $CW(\varphi)$ is equal to

$$\{\rho_{\hat{\mathbb{C}}}(P_{m,n})(\infty)\}_{(m,n)\in\mathbb{Z}^2}.$$

Since these two sets are identical by Lemma 8.5, we obtain the first assertion of Theorem 8.1.

Next, we show that the combinatorial structure of the colored $\langle D\rangle$-CW-complex $CW(\varphi)$ can be recovered from that of the layered $\langle D\rangle$-simplicial complex $(\Delta(\varphi), \{\mathcal{L}(\rho_{\hat{\mathbb{C}}}, \sigma_n)\})$. In the following, we identify $\Delta(\varphi)$ with $\Delta^{**}(\varphi)$ and $CW(\varphi)$ with $CW^{**}(\varphi)$. We also employ the identification described by Proposition 5.4 and Remark 7.11. Thus the vertices of $\Delta(\varphi)$ and $CW(\varphi)$ are represented by elliptic generators, and in particular, the layer $\mathcal{L}(\rho_{\hat{\mathbb{C}}}, \sigma_n)$ of $\Delta(\varphi)$ is identified with the following bi-infinite family of edges of $\Delta^{**}(\varphi)$.

$$\mathcal{L}_n := \{\langle Q_{m,n}, Q_{m+1,n}\rangle\}_{m\in\mathbb{Z}}$$

Definition 8.11. A vertex of $\mathcal{L}_n$ is called an *upward apex* (resp. a *downward apex*) if it is not a vertex of $\mathcal{L}_{n-1}$ (resp. $\mathcal{L}_{n+1}$) (see Figure 6).

Lemma 8.12. (1) *For each $n \in \mathbb{Z}$, a vertex of $\mathcal{L}_n$ is an upward apex of $\mathcal{L}_n$ if, and only if, it is equal to $P_{2m,n}$ for some $m \in \mathbb{Z}$. Moreover, $P_{2m,n+1}$ is a vertex of $\mathcal{L}_{n+1}$ adjacent to $P_{2m,n}$ in $\mathcal{L}_{n+1}$. Furthermore, $P_{2m,n+1}$ is the unique upward apex of $\mathcal{L}_{n+1}$ which is adjacent to $P_{2m,n}$ in $\Delta(\varphi)$.*

(2) *For each $n \in \mathbb{Z}$, a vertex of $\mathcal{L}_n$ is a downward apex of $\mathcal{L}_n$ if, and only if, it is equal to $P_{2m+1,n}$ for some $m \in \mathbb{Z}$. Moreover, $P_{2m+1,n-1}$ is a vertex of $\mathcal{L}_{n-1}$ adjacent to $P_{2m+1,n}$ in $\mathcal{L}_{n-1}$. Furthermore, $P_{2m+1,n-1}$ is the unique downward apex of $\mathcal{L}_{n-1}$ which is adjacent to $P_{2m+1,n}$ in $\Delta(\varphi)$.*

Proof. Since the proofs of (1) and (2) are parallel, we prove only (1). By Lemma 8.2, the slope of $P_{m,n}$ is the vertex of σ_n which is not contained in σ_{n-1}. Hence $P_{m,n}$ is an upward apex of $\mathcal{L}_n$. By the periodicity of $\mathcal{L}_n$ and $\mathcal{L}_{n-1}$, every upward apex of $\mathcal{L}_n$ is obtained from a single upward apex, say $P_{0,n}$, by taking the conjugate by D^m for some $m \in \mathbb{Z}$. Thus it is equal to the following element by Lemma 8.5(1).

$$\mathring{D}^m(P_{0,n}) = \mathring{D}^m(Q_{1,n}) = Q_{3m+1,n} = P_{2m,n}$$

Hence $\{P_{2m,n}\}_{m\in\mathbb{Z}}$ are the only upward apexes of $\mathcal{L}_n$.

By Lemma 8.6, we have

$$P_{2m,n} = Q_{3m+1,n} = \begin{cases} Q_{3m+2,n+1} & \text{if } f_n = R, \\ Q_{3m,n+1} & \text{if } f_n = L. \end{cases}$$

Hence $P_{2m,n}$ is a vertex of $\mathcal{L}_{n+1}$ adjacent to $Q_{3m+1,n+1} = P_{2m,n+1}$. The other vertex of $\mathcal{L}_{n+1}$ adjacent to $Q_{3m+1,n+1}$ in $\Delta(\varphi)$ is equal to $Q_{3m,n+1}$ or $Q_{3m+2,n+1}$ according as $f_n = R$ or L. By Lemma 8.4(2), it is equal to $Q_{3m,n}$ or $Q_{3m+2,n}$ accordingly. Thus it is contained in $\mathcal{L}_n$ and therefore it is not an upward apex of $\mathcal{L}_{n+1}$. Hence we obtain the last assertion of (1). □

Lemma 8.12 implies that, for each upward apex of $\mathcal{L}_n$, there is a unique upward apex of $\mathcal{L}_{n+1}$ adjacent to it in $\Delta(\varphi)$. Thus *by successively joining adjacent upward apexes, we obtain a bi-infinite 'vertical' edge path in $\Delta(\varphi)$. We call it an* upward line. *Similarly, we define a* downward line *to be a vertical edge path in $\Delta(\varphi)$ obtained by successively joining adjacent downward apexes.* By Lemmas 8.12 and 8.5(1), the vertical line in $CW\Delta(\varphi)$ with vertex set $\{(3m+1,n)\}_{n\in\mathbb{Z}}$ projects homeomorphically onto an upward line in $\Delta(\varphi)$, for each $m \in \mathbb{Z}$. Similarly, the vertical line in $CW\Delta(\varphi)$ with vertex set $\{(3m+1,d(n))\}_{n\in\mathbb{Z}}$ projects homeomorphically onto a downward line in $\Delta(\varphi)$, for each $m \in \mathbb{Z}$. Since these two kinds of vertical lines in $CW\Delta(\varphi)$ are mutually disjoint, and since all upward lines and downward lines are obtained in this way, an upward line and a downward line never cross each other though they may share some common edges. Moreover, upward lines and downward lines are located in $\mathbb{C}$ alternately. Hence we may number them so that $\{\partial^\Delta_m\}_{m\in\mathbb{Z}}$ *is the set of upward/downward lines located in $\Delta(\varphi)$ from left to right, where ∂^Δ_m is an upward line or a downward line according as m is even or odd.* It should be noted that D maps ∂^Δ_m to ∂^Δ_{m+2} (see Figure 11).

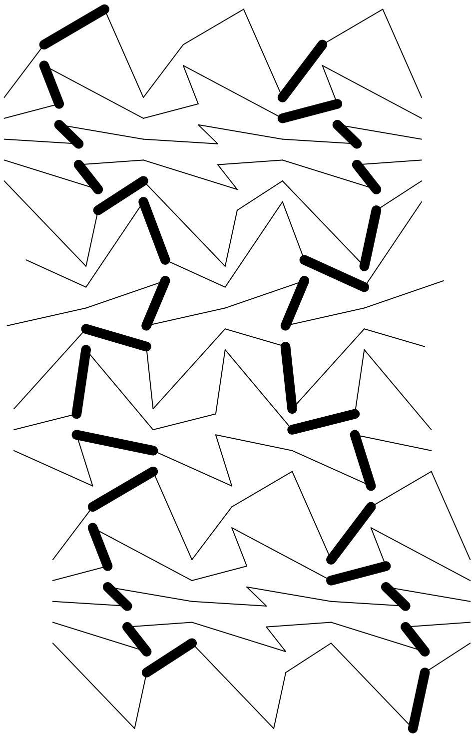

FIGURE 11. The upward/downward lines $\{\partial_m^\Delta\}_{m\in\mathbb{Z}}$ in $\Delta(\varphi)$

The union of $\{\partial_m^\Delta\}_{m\in\mathbb{Z}}$ *forms a* $\langle D\rangle$*-CW-decomposition of* $\mathbb{C}$*. We blow up each common edge of adjacent upward/downward lines into a bigon to produce a new* $\langle D\rangle$*-CW-decomposition,* $CW^{\#}(\varphi)$*, and let* $\partial_m^{\#}$ *be the image of* ∂_m^Δ *in* $CW^{\#}(\varphi)$*. We color the interior of the 2-cells in the new cell-complex bounded by* $\partial_m^{\#}$ *and* $\partial_{m+1}^{\#}$ *white or gray according as* m *is even or odd. We also color the vertices shared by* $\partial_m^{\#}$ *and* $\partial_{m+1}^{\#}$ *white or gray by the same rule. Then, by Theorem 8.9(2), we see that the resulting colored* $\langle D\rangle$*-CW-complex* $CW^{\#}(\varphi)$ *is isomorphic to the colored* $\langle D\rangle$*-CW-complex* $CW(\varphi)$*.* Thus we have shown that the combinatorial structure of the colored $\langle D\rangle$-CW-complex $CW(\varphi)$ can be recovered from that of the layered $\langle D\rangle$-simplicial complex $(\Delta(\varphi),\{\mathcal{L}_n\})$.

Next, we show conversely that the combinatorial structure of the layered $\langle D\rangle$-simplicial complex $(\Delta(\varphi),\{\mathcal{L}_n\})$ can be recovered from that of the colored $\langle D\rangle$-CW-complex $CW(\varphi)$.

Lemma 8.13. (1) *Suppose* $f_n = R$*. Then the open* 2*-cell* $c_{m,n}$ *of* $CW(\varphi)$ *has the color white and its boundary is the union of an edge path,* α*, of* ∂_{2m} *and an edge path,* β*, of* ∂_{2m+1} *which satisfy the following conditions.*

(i) *$\alpha \cap \beta = \partial\alpha = \partial\beta$ and it consists of the following two vertices:*

$$v_+ := P_{2m,n} = P_{2m+1,n_+},$$
$$v_- := P_{2m,n_-} = P_{2m+1,n}.$$

Moreover, the inverse image of v_+ in $CW\Delta(\varphi)$ is the edge path, $\tilde{v}_+$, spanned by the following $n_+ - n + 1$ vertices:

$$\{(3m+1, n)\} \cup \{(3m+2, k) \mid n+1 \leqslant k \leqslant n_+\}.$$

Similarly, the inverse image of v_- in $CW\Delta(\varphi)$ is the edge path, $\tilde{v}_-$, spanned by the following $n - n_- + 1$ vertices:

$$\{(3m+1, n_-)\} \cup \{(3m+2, k) \mid n_- + 1 \leqslant k \leqslant n\}.$$

Furthermore, the vertices v_+ and v_- are contained in the layer $\mathfrak{L}_n$ of $\Delta(\varphi)$ in this order.

(ii) *α has $n - n_- + 1$ vertices $\{P_{2m,k} \mid n_- \leqslant k \leqslant n\}$, and it is the homeomorphic image of the edge path, $\tilde{\alpha}$, in $CW\Delta(\varphi)$ with vertex set:*

$$\{(3m+1, k) \mid n_- \leqslant k \leqslant n\}.$$

(iii) *β has $n_+ - n + 1$ vertices $\{P_{2m+1,k} \mid n \leqslant k \leqslant n_+\}$, and it is the homeomorphic image of the edge path, $\tilde{\beta}$, in $CW\Delta(\varphi)$ with vertex set:*

$$\{(3m+2, n)\} \cup \{(3m+3, k) \mid n+1 \leqslant k \leqslant n_+ - 1\}$$
$$\cup \{(3m+2, n_+)\}.$$

(2) *Suppose $f_n = L$. Then the open 2-cell $c_{m,n}$ of $CW(\varphi)$ has the color gray and its boundary is the union of an edge path, α, of ∂_{2m} and an edge path, β, of ∂_{2m-1} which satisfy the following conditions.*

(i) *$\alpha \cap \beta = \partial\alpha = \partial\beta$ and it consists of the following two vertices:*

$$v_+ := P_{2m,n} = P_{2m-1,n_+},$$
$$v_- := P_{2m,n_-} = P_{2m-1,n_-}.$$

Moreover the inverse image of v_+ in $CW\Delta(\varphi)$ is the edge path, $\tilde{v}_+$, spanned by the following $n_+ - n + 1$ vertices:

$$\{(3m+1, n)\} \cup \{(3m, k) \mid n+1 \leqslant k \leqslant n_+\}.$$

Similarly, the inverse image of v_- in $CW\Delta(\varphi)$ is the edge path, $\tilde{v}_-$, spanned by the following $n - n_- + 1$ vertices:

$$\{(3m+1, n_-)\} \cup \{(3m, k) \mid n_- + 1 \leqslant k \leqslant n\}.$$

Furthermore, the vertices v_- and v_+ are contained in the layer $\mathfrak{L}_n$ of $\Delta(\varphi)$ in this order.

(ii) *α has $n-n_-+1$ vertices $\{P_{2m,k} \mid n_- \leqslant k \leqslant n\}$, and it is the homeomorphic image of the edge path, $\tilde{\alpha}$, in $CW\Delta(\varphi)$ with vertex set:*

$$\{(3m+1,k) \mid n_- \leqslant k \leqslant n\}.$$

(iii) *β has n_+-n+1 vertices $\{P_{2m-1,k} \mid n \leqslant k \leqslant n_+\}$, and it is the homeomorphic image of the edge path, $\tilde{\beta}$, in $CW\Delta(\varphi)$ with vertex set:*

$$\{(3m,n)\} \cup \{(3m-1,k) \mid n+1 \leqslant k \leqslant n_+-1\} \cup \{(3m,n_+)\}.$$

Proof. By the definition of $CW^*(\varphi)$ and by Remark 7.11, the boundary of $c_{m,n}$ is the union of an edge path α in ∂_{2m} and β in $\partial_{2m\pm1}$ such that $\alpha\cap\beta = \partial\alpha = \partial\beta$, the vertex set of α is $\{P_{2m,k} \mid n_- \leqslant k \leqslant n\}$, and the vertex set of β is $\{P_{2m\pm1,k} \mid n \leqslant k \leqslant n_+\}$. Here the signs $\pm$ stand for $+$ or $-$ according as $f_n = R$ or L. By Lemma 8.6, we see that the condition (i) is satisfied, except the last statement that v_+ and v_- are contained in the layer $\mathcal{L}_n$ of $\Delta(\varphi)$ in this order. The last statement in (i), for the case $f_n = R$, follows from the fact that v_+ and v_-, respectively, are the images of the vertices $(3m+1,n)$ and $(3m+2,n)$ of $CW\Delta(\varphi)$ and the fact that the layer $\mathcal{L}_n$ is the image of the horizontal line of height n in $CW\Delta(\varphi)$. The statement for the case $f_n = L$ is proved similarly. The remaining conditions (ii) and (iii) follow from Lemma 8.5(1). □

In the above lemma, the union of the edge path $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{v}_+$ and $\tilde{v}_-$ forms a simple closed edge path in $CW\Delta(\varphi)$. Let $\tilde{c}_{m,n}$ be the 2-cell bounded by it (see Figure 12). Consider first the case where the 2-cell $c_{m,n}$ is colored white (i.e., $f_n = R$). Then the horizontal edges of $CW\Delta(\varphi)$ contained in $\tilde{c}_{m,n}$ are as follows:

$$\begin{aligned} &\langle(3m+1,k),(3m+2,k)\rangle \quad (n_-+1 \leqslant k \leqslant n),\\ &\langle(3m+2,k),(3m+3,k)\rangle \quad (n \leqslant k \leqslant n_+-1). \end{aligned}$$

Here, one or both families can be empty, if $f_n = f_{n-1}$ and/or $f_n = f_{n+1}$. By using Lemma 8.13, we see that the images of these edges in $c_{m,n}$ constitute the following three families of arcs, which have mutually disjoint interiors:

- an arc joining v_- and v_+,
- arcs joining v_- with the vertices contained in the interior of α,
- arcs joining v_+ with the vertices contained in the interior of β.

The same result holds when the color of $c_{m,n}$ is gray. This observation leads us to the following recipe for recovering $\Delta(\varphi)$ from the colored $\langle D\rangle$-CW-complex $CW(\varphi)$.

Let c be a 2-cell of $CW(\varphi)$. Then it contains exactly two vertices, v_+ and v_-, which share the same color with c. We assume v_+ lies 'above' v_-. These two vertices divide ∂c into two arcs, α and β. We assume α lies on the left or right hand side of c according as the color of c is white or gray. In the 2-cell

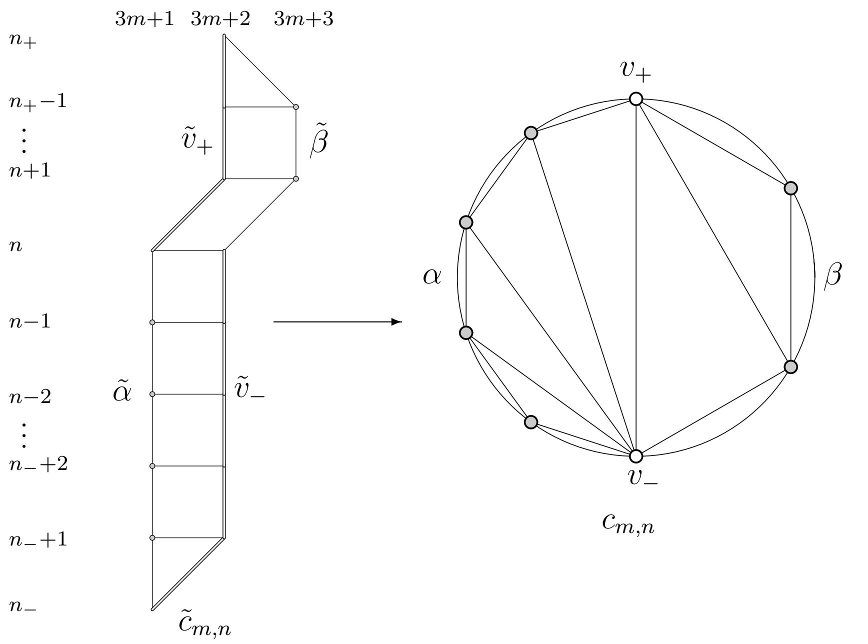

FIGURE 12. The 2-cell $c_{m,n}$ in $CW(\varphi)$ and the chosen region $\tilde{c}_{m,n}$ in $CW\Delta(\varphi)$, with $\varphi = RLLRRRLLLL$. Here, $n = 1$ and $f_n = f_1 = R$. Compare with Figure 9.

c, we draw the following families of arcs with mutually disjoint interiors (see Figure 12)*:*

- *an arc joining v_- and v_+,*
- *arcs joining v_- with the vertices contained in the interior of α,*
- *arcs joining v_+ with the vertices contained in the interior of β,*
- *arcs joining consecutive vertices of α not already joined by the foregoing,*
- *arcs joining consecutive vertices of β not already joined by the foregoing.*

Next, shrink each bigon into a straight edge until there are no bigons.

The number of bigons equals the number of vertices and is 2 *more than the number of triangles (see Figure* 12)*. The arc joining v_- and v_+ is called the* central edge *associated with the* 2*-cell c.*

Perform the above operation at every 2*-cell of $CW(\varphi)$. Then it follows from the preceding observation that the resulting CW-decomposition of $\mathbb{C}$ is combinatorially isomorphic to $\Delta(\varphi)$.*

In order to reconstruct the layered structure $\mathcal{L}_n$ of $\Delta(\varphi)$, we need the following observation. Let v be a vertex of $\Delta(\varphi)$, and let

$$\mathcal{L}_p, \mathcal{L}_{p+1}, \cdots, \mathcal{L}_q$$

be the layers of $\Delta(\varphi)$ containing v. For each n ($p \leqslant n \leqslant q$), let $e_n^{(\ell)}$, resp. $e_n^{(r)}$, be the edge of $\mathcal{L}_n$ which is incident on v and lies on the left-hand, resp. right-hand, side of v. Then the edges incident on v are located around v in the following clockwise cyclic order:

$$e_p^{(\ell)}, e_{p+1}^{(\ell)}, \cdots, e_q^{(\ell)}, e_q^{(r)}, e_{q-1}^{(r)}, \cdots, e_p^{(r)}. \tag{8.2}$$

We call $e_p^{(\ell)}$, $e_p^{(r)}$, $e_q^{(\ell)}$ and $e_q^{(r)}$, respectively, the *lower-left edge*, the *lower-right edge*, the *upper-left edge*, and the *upper-right edge* of v. If we could identify one of the above four 'characteristic' edges of v, then we would know the 'local structure' of the layered structure around the vertex v, because the information on the characteristic edge together with the cyclic order (8.2) tells us how the edges incident on v are paired by the layered structure. In fact, if we could identify, say the lower-left edge $e_p^{(\ell)}$, of v, then the cyclic order (8.2) extends to an order, by regarding $e_p^{(\ell)}$ as the first edge, such that the i-th edge and $(2d+1-i)$-th edge belong to the same layer for each i ($1 \leqslant i \leqslant d$), where $d = q - p + 1$ (and hence $2d$ is the degree of v). The following lemma enables us to identify the four 'characteristic' edges of each of the vertices of $CW(\varphi)$ (see Figure 13).

Lemma 8.14. *Under the setting of Lemma 8.13, the following holds. If $c_{m,n}$ is a white 2-cell, then the central edge $\langle v_+, v_- \rangle$ is the upper-left edge of v_- and is the lower-right edge of v_+. If $c_{m,n}$ is a gray 2-cell, then the central edge $\langle v_-, v_+ \rangle$ is the upper-right edge of v_- and is the lower-left edge of v_+.*

Proof. We prove the lemma when $c_{m,n}$ is a white 2-cell. The other case is proved similarly. So, assume $c_{m,n}$ is a white 2-cell. Then v_+ and v_-, respectively, are the images of the vertices $(3m+1, n)$ and $(3m+2, n)$ of $CW\Delta(\varphi)$, and the layer $\mathcal{L}_n$ of $\Delta(\varphi)$ contains the edge $\langle v_+, v_- \rangle$ by Lemma 8.13. Thus the image, w_+, of $(3m, n)$ is the predecessor of v_+ in $\mathcal{L}_n$, and the image, w_-, of $(3m+3, n)$ is the successor of v_- in $\mathcal{L}_n$.

We first show that $\langle v_+, v_- \rangle$ is the lower-right edge of v_+. Observe that there is no slanted edge of $CW\Delta(\varphi)$ which has endpoint $(3m+1, n)$ and which is contained in $\mathbb{R} \times [n-1, n]$ (cf. Definition 8.8 and Figure 14). Consider a small half circle $\tilde{c}$ with center $(3m+1, n)$ contained in $\mathbb{R} \times [n-1, n]$ with endpoints $(3m+1 \pm \epsilon, n)$ for some small positive real number ϵ. Then the interior of c is contained in an open 2-cell of $CW\Delta(\varphi)$ which has a horizontal edge of height $n-1$ and has the following horizontal edges:

$$\langle (3m, n), (3m+1, n) \rangle, \langle (3m+1, n), (3m+2, n) \rangle.$$

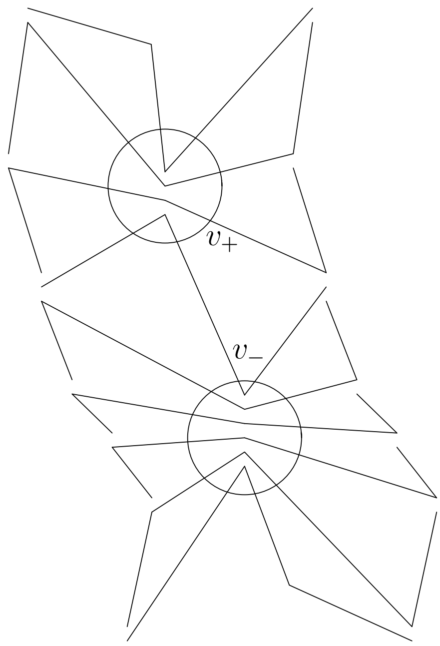

FIGURE 13. The local layered structure around the vertices v_+ and v_-

Thus $\tilde{c}$ projects to an arc c around the vertex v_+ such that (i) c is contained in a triangle of $\Delta(\varphi)$, (ii) $\langle w_+, v_+\rangle$ and $\langle v_+, v_-\rangle$ are edges of the triangle and each of them contains an endpoint of c, and that (iii) the remaining edge of the triangle belongs to the layer $\mathcal{L}_{n-1}$. Then it follows that $\langle w_+, v_+, v_-\rangle$ is the triangle of $\Delta(\varphi)$ containing c and that the edge $\langle w_+, v_-\rangle$ belongs to the layer $\mathcal{L}_{n-1}$. Hence $\langle v_+, v_-\rangle$ is the lower-right edge of v_+.

Next, we show that $\langle v_+, v_-\rangle$ is the upper-left edge of v_-. To this end, note that the assumption $f_n = R$ implies that $CW\Delta(\varphi)$ has the following two slanted edges (see Figure 14):

$$\langle (3m+1, n), (3m+2, n+1)\rangle, \langle (3m+3, n), (3m+3, n+1)\rangle.$$

Observe that these two slanted edges and the three horizontal edges

$$\langle (3m+1, n), (3m+2, n)\rangle, \langle (3m+2, n), (3m+3, n)\rangle,$$
$$\langle (3m+2, n+1), (3m+3, n+1)\rangle$$

bound a 2-cell of $CW\Delta(\varphi)$. This implies that $\langle v_+, v_-, w_-\rangle$ is a 2-simplex of $\Delta(\varphi)$ and that $\langle v_+, w_-\rangle$ is an edge of the layer $\mathcal{L}_{n+1}$. Hence $\langle v_+, v_-\rangle$ is the upper-left edge of v_-. $\square$

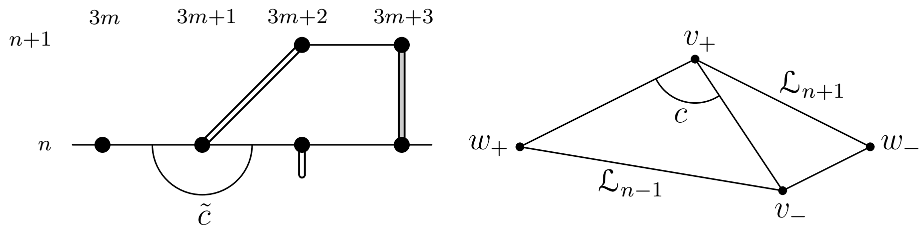

FIGURE 14. The layer $\mathcal{L}_n$ passing through v_+ and v_-

The above lemma leads us to the following recipe for recovering the layered structure $\{\mathcal{L}_n\}_{n\in\mathbb{Z}}$ of $\Delta(\varphi)$ from the colored $\langle D\rangle$-CW-complex $CW(\varphi)$.

Let v be a vertex of $\Delta(\varphi)$. Since it is also a vertex of $CW(\varphi)$, it has a color white or gray. Let $e_1, e_2, \cdots, e_{2d}$ be the edges of $\Delta(\varphi)$ incident on v in counter-clockwise or clockwise order according as v is white or gray. By the construction of $CW(\varphi)$, there are precisely two 2-cells of $CW(\varphi)$ which have v as a vertex and share the same color with v. Let c be such a 2-cell which lies below v, and let $\langle v_+, v_-\rangle$ be the central edge associated with c, where $v = v_+$. We may assume, after cyclic permutation, that $e_1 = \langle v_+, v_-\rangle$. Then the couplings $\{(e_i, e_{2d-i+1})\}_{1\leqslant i\leqslant d}$ gives the desired "local layered structure around v". (In fact, e_1 is the lower-right or upper-right vertex of v according as v is white or gray by Lemma 8.14. So the above local layered structure at each vertex v is consistent with the layered structure $\{\mathcal{L}_n\}$ of $\Delta(\varphi)$.) By combining this local information at the vertices of $\Delta(\varphi)$, we obtain the layered structure. Thus we have shown that the combinatorial structure of the layered $\langle D\rangle$-simplicial complex $(\Delta(\varphi), \{\mathcal{L}_n\})$ can be recovered from that of the colored $\langle D\rangle$-CW-complex $CW(\varphi)$. This completes the proof of Theorem 8.1.

Remark 8.15. The main theorem, Theorem 8.1, is actually valid for all doubly degenerate punctured-torus groups with "bounded geometry". (See the celebrated paper [17] by Minsky, for the classification of punctured-torus groups.) In fact, the description of the Cannon-Thurston maps given by Bowditch [4] is valid for every such group and thus a fractal tessellation of the complex plane is naturally associated with the group, for which an analogue of Theorem 7.10 holds. On the other hand, the canonical decompositions of the quotient hyperbolic manifolds associated with the punctured-torus groups are determined by Akiyoshi [1] and Gueritaud [13]. In particular, an analogue of Theorem 5.3 holds for all punctured-torus groups. These two results guarantee that the proof of Theorem 8.1 works for all doubly degenerate punctured-torus groups with bounded geometry, and hence we have an analogue of Theorem 8.1 for such groups.

Added February, 2010: It has been established by McMullen [16] (see also [18, 19]) that the Cannon-Thurston map exists for all punctured-torus groups.

Moreover, Mj and his student Das ([8]) have recently proved that the description of the Cannon-Thurston map in Theorem 6.1 is valid for all punctured-torus groups without accidental parabolics, possibly with unbounded geometry. This shows that Theorem 8.1 can be extended to arbitrary doubly degenerate punctured-torus groups.

Remark 8.16. The referee has kindly provided a proof of Conjectures 14.3 of [6] which proposed that, at each vertex of $CW(\varphi)$, the four cyclically ordered incident edges form well-defined angles, two of 180° and two of 0°, and that the two incident open two-cells that fill in the angles of 180° have the same color as the vertex. The referee's argument, which shows slightly more, is as follows.

Let us view $CW(\varphi)$ within $\hat{\mathbb{C}}$, where each of the multi-pinched columns now has as closure a multi-pinched annulus with two 'spikes' terminating in the accumulation point ∞. Every spike is considered to be the 'opposite' of the other spike in the annulus, and two opposite spikes form an angle of 180°. The boundary of an annulus gives four cyclically ordered, oriented (fractal) curves, ending at ∞, forming four well-defined angles with cyclic sequence $(180°, 0°, 180°, 0°)$.

Let us consider an arbitrary vertex expressed as $\rho_{\hat{\mathbb{C}}}(P)(\infty)$ for some elliptic generator $P \in \pi_1(\mathcal{O})$. Observe that $\rho_{\hat{\mathbb{C}}}(P)(\infty)$ lies in exactly three columns of $CW(\varphi)$ and that we are now extending these columns to annuli. Moreover, $\rho_{\hat{\mathbb{C}}}(P)(\infty)$ is incident to four of the 2-cells of $CW(\varphi)$ which we think of as being to the left, top, right, and bottom of $\rho_{\hat{\mathbb{C}}}(P)(\infty)$. By Proposition 7.3 (see [6, Introduction and Proposition 7.7]), we know the following about the cyclic sequence of visits of the Peano curve κ to the two points ∞ and $\rho_{\hat{\mathbb{C}}}(P)(\infty)$: κ makes one or more visits to $\rho_{\hat{\mathbb{C}}}(P)(\infty)$ on the left, and then κ visits ∞ filling in two adjacent spikes in succession, and then κ makes one or more visits to $\rho_{\hat{\mathbb{C}}}(P)(\infty)$ on the top and the bottom, and then κ visits ∞ filling in the opposite of the spike last visited and then an adjacent spike, and then κ makes one or more visits to $\rho_{\hat{\mathbb{C}}}(P)(\infty)$ on the right, and then κ makes infinitely many visits to ∞ filling in the complement of the two pairs of adjacent spikes, and then κ makes one or more visits to $\rho_{\hat{\mathbb{C}}}(P)(\infty)$ on the left, restarting the cycle. (The interested reader might like to study the behavior of the model Cannon-Thurston map given by Theorem 6.1 by first analyzing how a double spider can intersect with the complex conjugate of another double spider.) There are four cyclically ordered, oriented germs of curves, ending at ∞, forming four well-defined angles with cyclic sequence $(180°, 0°, 180°, 0°)$.

If we now apply $\rho_{\hat{\mathbb{C}}}(P^{-1})$, or, equivalently, $\rho_{\hat{\mathbb{C}}}(P)$, we find that κ makes one or more visits to ∞, and then κ visits $\rho_{\hat{\mathbb{C}}}(P)(\infty)$ filling in a pair of adjacent spikes, and then κ makes one or more visits to ∞, and then κ visits $\rho_{\hat{\mathbb{C}}}(P)(\infty)$ filling in a pair of adjacent spikes that includes the opposite of the spike last visited, and then κ makes one or more visits to ∞, and then κ makes infinitely many visits to $\rho_{\hat{\mathbb{C}}}(P)(\infty)$ filling in the complement of the two pairs of adjacent spikes, and then κ makes one or more visits to ∞, restarting the cycle. Also, there are four cyclically ordered, oriented curves, ending at $\rho_{\hat{\mathbb{C}}}(P)$, forming four well-defined angles with cyclic sequence $(180°, 0°, 180°, 0°)$. Since the germs of these curves are those determined by the visits κ makes to ∞, these germs are precisely the germs of the edges of $CW(\varphi)$ that are incident to $\rho_{\hat{\mathbb{C}}}(P)(\infty)$.

It remains to verify that the two 180°-angles at $\rho_{\hat{\mathbb{C}}}(P)(\infty)$ correspond to Jordan domains that have the same color as the vertex. By symmetry, we may suppose that $\rho_{\hat{\mathbb{C}}}(P)(\infty)$ is a white vertex. Then the unique white column containing $\rho_{\hat{\mathbb{C}}}(P)(\infty)$ has exactly two Jordan domains which contain $\rho_{\hat{\mathbb{C}}}(P)(\infty)$, and their intersection is $\{\rho_{\hat{\mathbb{C}}}(P)(\infty)\}$. Now κ fills in the top Jordan domain and passes through $\rho_{\hat{\mathbb{C}}}(P)(\infty)$ and fills in the bottom Jordan domain. This cannot be done with a visit that fills in a single spike while arriving and then a single adjacent spike while retreating; hence it is necessarily in this visit that we form the two 180°-angles, as desired.

This proves [6, Conjectures 14.3].

It now follows that the planar symmetry group of $CW(\varphi)$ respects the columns. Thus, the column-respecting planar symmetry group of $CW(\varphi)$ computed in [6, Section 14] is the full planar symmetry group of $CW(\varphi)$.

Acknowledgment. The authors thank the referee for reading the first draft extremely carefully, and for valuable comments, particularly, the comments included as Remarks 5.5, 7.13, and 8.16, and for the beautiful Figure 10.

References

[1] H. Akiyoshi, *On the Ford domains of once-punctured torus groups*, in *Hyperbolic spaces and related topics* RIMS, Kyoto, Kokyuroku 1104 (1999), 109-121.

[2] H. Akiyoshi, M. Sakuma, M. Wada and Y. Yamashita, *Punctured torus groups and 2-bridge knot groups I*, Lecture Notes in Mathematics, 1909. Springer, Berlin, 2007.

[3] R. C. Alperin, W. Dicks and J. Porti, *The boundary of the Gieseking tree in hyperbolic three-space*, Topology Appl. 93 (1999), 219-259.

[4] B. H. Bowditch, *The Cannon-Thurston map for punctured-surface groups*, Math. Z. 255 (2007), 35-76.

[5] J. W. Cannon and W. Dicks, *On hyperbolic once-punctured-torus bundles*, Geom. Dedicata 94 (2002), 141-183.
Errata and addenda: http://mat.uab.cat/∼dicks/Cannon.html

[6] J. W. Cannon and W. Dicks, *On hyperbolic once-punctured-torus bundles II*, Geom. Dedicata 126 (2006), 11-63.
Errata and addenda: http://mat.uab.cat/∼dicks/spiders.html

[7] J. W. Cannon and W. P. Thurston, *Group invariant Peano curves*, Geom. Topol. 11 (2007), 1315-1355.

[8] S. Das and M. Mj, *Addendum to Ending laminations and Cannon-Thurston maps: parabolics*, preprint.

[9] D. B. A. Epstein and R. C. Penner, *Euclidean decompositions of noncompact hyperbolic manifolds*, J. Diff. Geom. 27 (1988) 67-80.

[10] W. Floyd, *Group completions and limit sets of Kleinian groups*, Invent. Math. 57 (1980), 205–218.

[11] W. Floyd and A. Hatcher, *Incompressible surfaces in punctured torus bundles*, Topology Appl. 13 (1982), 263-282.

[12] F. Gueritaud, *On canonical triangulations of once-punctured torus bundles and two-bridge link complements. With an appendix by David Futer*, Geom. Topol. 10 (2006), 1239–1284.

[13] F. Gueritaud, *Triangulated cores of punctured-torus groups*, J. Differential Geom. 81 (2009), 91-142.

[14] T. Jørgensen, *On pairs of punctured tori*, unfinished manuscript, available in Proceedings of the workshop "Kleinian groups and hyperbolic 3-manifolds" (edited by Y. Komori, V. Markovic and C. Series), London Math. Soc., Lect. Notes 299 (2003), 183–207.

[15] M. Lackenby, *The canonical decomposition of once-punctured torus bundles*, Comment. Math. Helv. 78 (2003) 363-384.

[16] C. T. McMullen, *Local connectivity, Kleinian groups and geodesics on the blowup of the torus*, Inv. Math. 146 (2001), 35–91.

[17] Y. N. Minsky, *The classification of punctured-torus groups*, Ann. of Math. 149 (1999), 559–626.

[18] M. Mj, *Cannon-Thurston maps, i-bounded geometry and a theorem of McMullen*, arXiv:math/0511104v2 [mathGT].

[19] M. Mj, *Ending laminations and Cannon-Thurston maps*, arXiv:math/0702162 [mathGT].

[20] J. P. Otal, *The hyperbolization theorem for fibered 3-manifolds*, Translated from the 1996 French original by Leslie D. Kay. SMF/AMS Texts and Monographs, vol 7, American Mathematical Society, 2001.

[21] J. R. Parker, *Tetrahedral decomposition of punctured torus bundles*, Kleinian groups and hyperbolic 3-manifolds (Warwick, 2001), 275–291, London Math. Soc. Lecture Note Ser., 299, Cambridge Univ. Press, Cambridge, 2003.

[22] W. P. Thurston, *Hyperbolic Structures on 3-manifolds, II: Surface groups and 3-manifolds which fiber over the circle*, 32 pages. http://arxiv.org/abs/math/9801045

[23] J. Weeks, *Convex hulls and isometries of cusped hyperbolic manifolds*, Topology Appl. 52 (1993), 127-149.

Warren Dicks, Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Spain.
E-mail address: `dicks@mat.uab.cat`

Makoto Sakuma, Department of Mathematics, Graduate School of Science, Hiroshima University, Kagamiyama 1-3-1, Higashi-Hiroshima, Hiroshima 739-8526, Japan
E-mail address: `sakuma@math.sci.hiroshima-u.ac.jp`