A torsion-free non-left-orderable group

Example (Passman). *The group* $G = \langle x, y \mid (x^2)^y = (x^2)^{-1},\ (y^2)^x = (y^2)^{-1}\rangle$ *is torsion free and is not left orderable.*

Proof (Passman, with later shortcuts by Dicks). Suppose that $<$ is a left order on G. On abelianizing G, we see that $x \neq 1 \neq y$; hence, there exist (odd) $i, j \in \{-1, 1\}$ such that $1 < x^i$ and $1 < y^j$. Here, $(x^{2i})^{y^j} = \big((x^2)^{y^j}\big)^i = \big((x^2)^{(-1)^j}\big)^i = x^{-2i}$; hence, $x^{2i}y^j = y^j x^{-2i}$. Similarly, $y^{2j}x^i = x^i y^{-2j}$. Now $1 < x^{2i}y^j \cdot x^i \cdot y^{2j}x^i = y^j x^{-2i} \cdot x^i \cdot x^i y^{-2j} = y^{-j} < 1$, a contradiction. Thus, G is not left orderable.

Let $t \in \operatorname{tor}(G)$, the torsion subset of G. We shall show that $t = 1$. Set $N := \langle x^2, y^2\rangle$. From the presentation of G, we see that $N \lhd G$ and that $G/N = \langle \overline{x}, \overline{y} \mid \overline{x}^2 = \overline{y}^2 = 1\rangle$ where $\overline{x} := xN$ and $\overline{y} := Ny$. As $\left(\begin{smallmatrix} -1 & r \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -1 & s \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & r-s \\ 0 & 1 \end{smallmatrix}\right)$, there exists a homomorphism $\gamma : G/N \to \mathrm{GL}_2(\mathbb{Z})$ such that $\gamma(\overline{x}) = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\gamma(\overline{y}) = \left(\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Then $\gamma(\overline{x}\,\overline{y}) = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$, which has infinite order; thus, $\overline{x}\,\overline{y}$ has infinite order. It can be seen that $G/N = \langle \overline{x}\,\overline{y}\rangle \cup \overline{x}\langle \overline{x}\,\overline{y}\rangle = \langle \overline{x}\,\overline{y}\rangle \cup \overline{x}^{\langle \overline{x}\,\overline{y}\rangle} \cup \overline{y}^{\langle \overline{x}\,\overline{y}\rangle}$ and $tN \in \operatorname{tor}(G/N) = \{1\} \cup \overline{x}^{\langle \overline{x}\,\overline{y}\rangle} \cup \overline{y}^{\langle \overline{x}\,\overline{y}\rangle}$. By conjugating t by a power of xy, we may assume that $tN \in \{1, \overline{x}, \overline{y}\}$. Then $t \in N \cup xN \cup Ny$. Since $(x^2)^{y^2} = (x^2)^{(-1)^2} = x^2$, N is abelian. Hence, there exist $i, j \in \mathbb{Z}$ such that $t = x^i y^j$ and i or j is even. As $\big((\overline{x}\,\overline{y})^2\big)^{\overline{x}} = \big((\overline{x}\,\overline{y})^2\big)^{-1}$, there exists a homomorphism $\alpha : G \to G/N \times G/N$ such that $\alpha(x) = (\overline{x}\,\overline{y},\ \overline{x})$ and $\alpha(y) = (\overline{x},\ \overline{x}\,\overline{y})$. Then $\big((\overline{x}\,\overline{y})^i\overline{x}^j,\ \overline{x}^i(\overline{x}\,\overline{y})^j\big) = \alpha(x^i y^j) \in \operatorname{tor}(G/N \times G/N)$. Since i or j is even, we then see that $i = j = 0$. Hence, $t = 1$. Thus, G is torsion free. $\square$