

AN IMPROVED PROOF OF THE ALMOST STABILITY THEOREM

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ABSTRACT. In 1989, Dicks and Dunwoody proved the Almost Stability Theorem, which has among its corollaries the Stallings-Swan theorem that groups of cohomological dimension one are free. In this article, we use a nestedness result of Bergman, Bowditch, and Dunwoody to simplify somewhat the proof of the finitely generable case of the Almost Stability Theorem. We also simplify the proof of the non finitely generable case.

The proof we give here of the Almost Stability Theorem is essentially self contained, except that in the non finitely generable case we refer the reader to the original argument for the proofs of two technical lemmas about groups acting on trees.

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To Martin Dunwoody, on the occasion of his 80th birthday.

1. INTRODUCTION

Throughout, G will denote a discrete, multiplicative group.

Unexplained terminology and notation used in the first two sections will be defined in Section 3.

1.1. Definitions. For any sets E and Z , we write $\text{Maps}(E, Z)$ to denote the set of all maps of sets from E to Z , with each $v \in \text{Maps}(E, Z)$ written as $v: E \rightarrow Z$, $e \mapsto \langle v, e \rangle$. For any $v, w \in \text{Maps}(E, Z)$, we write

$$v \nabla w := \{e \in E \mid \langle v, e \rangle \neq \langle w, e \rangle\};$$

if this set is finite, then we say that v and w are *almost equal*, and write $v =_a w$. Almost equality is an equivalence relation on $\text{Maps}(E, Z)$; its equivalence classes are called *almost equality classes*.

If E and Z are (left) G -sets, then $\text{Maps}(E, Z)$ is a G -set, with the conjugation G -action, that is, if $v \in \text{Maps}(E, Z)$, $g \in G$, and $e \in E$, then $\langle gv, e \rangle := g\langle v, g^{-1}e \rangle$, and, hence, $g\langle v, e \rangle = \langle gv, ge \rangle$.

A G -set is said to be G -free if each element's G -stabilizer is trivial, and is said to be G -quasifree if each element's G -stabilizer is finite. \square

The following is one form of [5, III.8.5]; see Remarks 1.3(ii) below.

1.2. The Almost Stability Theorem. *If E and Z are any G -sets such that E is G -quasifree and each element's G -stabilizer stabilizes some element of Z , then each G -stable almost equality class in $\text{Maps}(E, Z)$ is the vertex G -set of some G -tree. Any such G -tree automatically has G -quasifree edge G -set.*

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The purpose of this article is to give a revised proof of Theorem 1.2 that incorporates various simplifications which have become available since the original proof was published.

Let $\mathcal{B}(G)$ denote the Boolean algebra of almost right G -stable subsets of G . For G finitely generated, Bergman [1] defined a well-ordered measure on $\mathcal{B}(G)$, and Bowditch and Dunwoody [2, 8.1] used the well-orderedness of Bergman's measure to show that each Boolean G -subalgebra of $\mathcal{B}(G)$ is generated by some nested G -subset. We shall recall their proofs, and then use their results to simplify the proof of the case of Theorem 1.2 where G is finitely generable. Although it closely follows part of the proof in [5], this argument had not been recorded before now; Bowditch and Dunwoody [2, 14.2] had noted the weaker conclusion that each G -stable almost equality class *embeds* in the vertex G -set of some G -tree, in the case where G is finitely generable and $Z = \mathbb{Z}/2\mathbb{Z}$.

In the complementary case, where G is not finitely generable, we shall describe some further simplifications in the proof of Theorem 1.2.

The proof we give of the Almost Stability Theorem 1.2 is essentially self contained, except that in the non finitely generable case we refer the reader to the original argument in [5] for the proofs of two technical lemmas about groups acting on trees.

The article has the following structure.

In Section 2, to provide motivation, we digress to show that Theorem 1.2 yields one form of the result of Stallings [17, 6.8] and Swan [19] that groups of cohomological dimension one are free.

In Section 3, we record rather a large number of definitions, which will provide much of the basic terminology that we shall be using.

In Section 4, we recall from [6] and [5] Dunwoody's construction of trees from nested sets, here with a simplification by Roller [14].

In Section 5, we recall from [1] Bergman's well-ordered measure, and we recall from [2] the Bowditch-Dunwoody construction of nested generating sets.

In Section 6, we use the results of Section 4 and the nested sets of Section 5 to construct a tree, and we deduce a result from [5] which strengthened a result of Dunwoody [6].

In Section 7, to provide motivation, we digress to deduce one form of Stallings' Ends Theorem [18, 4.1].

In Section 8, we recall from [5] the deduction of the finitely generable case of Theorem 1.2 from the results of Section 6.

We then consider the non finitely generable case, closely following [5] but with an improved transfinite induction procedure.

In Section 9, we record, without proofs, two lemmas about trees proved in [5].

In Section 10, we fix notation that applies for the remainder of the proof.

In Section 11, we give results and proofs about finitely generable extensions.

In Section 12, we give results and proofs about countably generable extensions.

In Section 13, we give the proof of the general case.

In Section 14, we give the proof of the analogue for extensions.

In this article, we shall work with trees, and not discuss Bass-Serre theory. We shall mention in each of the two digressions that certain information about trees may be translated by Bass-Serre theory into information about groups.

1.3. Remarks. Let E and Z be any G -sets, and V be any G -stable almost equality class in the G -set $\text{Maps}(E, Z)$.

(i). We denote by $\text{Complete}(V)$ the G -graph with vertex G -set V and edge G -set $\{(v, w) \in V \times V \mid v \neq w\}$, where each edge (v, w) has initial vertex v and terminal vertex w ; here, the G -stabilizer of (v, w) is a subgroup of the G -stabilizer of $v \nabla w$,

and $v \nabla w$ is a finite, nonempty subset of E . Thus, if E is G -quasifree, then the edge G -set of $\text{Complete}(V)$ is also G -quasifree, and, in particular, any G -tree with vertex G -set V has G -quasifree edge G -set.

(ii). Consider the following conditions.

- (a) G stabilizes each element of Z .
- (b) E is such that each element's G -stabilizer stabilizes some element of Z ; equivalently, there exists some G -map from E to Z ; equivalently, G stabilizes some element of $\text{Maps}(E, Z)$.
- (c) Each finite subgroup of G stabilizes some element of V .
- (z) E is G -quasifree.

Notice that (b) and (z) are the hypotheses in Theorem 1.2. Since equivalence classes are nonempty by definition, $V \neq \emptyset$, and $\text{Maps}(E, Z) \neq \emptyset$; hence, if $E \neq \emptyset$, then $Z \neq \emptyset$. It is easy to see that (a) \Rightarrow (b) and that (c)+(z) \Rightarrow (b). It is not difficult to use properties of almost equality to prove that (b) implies (c). Thus, if (z) holds, then (b) \Leftrightarrow (c).

In 1989, Dicks and Dunwoody [5, III.8.5] proved the case of Theorem 1.2 where (a) holds. In this article, we shall see that (b), as opposed to (a), is the condition that was used in that proof.

Since (c) is a *necessary* condition for the G -set V to be the vertex G -set of a G -tree, we now see that Theorem 1.2 says that if (z) holds, then the G -set V is the vertex G -set of some G -tree if and only if (b) holds.

(iii). In Theorem 1.2, each hypothesis on E determines a corresponding condition on $\text{Complete}(V)$, and we have the following formulation: If E is G -quasifree, then, first, the edge G -set of $\text{Complete}(V)$ is G -quasifree, and, secondly, the G -set $\text{Maps}(E, Z)$ has some G -stable element if and only if the G -set consisting of the maximal subtrees of $\text{Complete}(V)$ has some G -stable element.

In the simplest case, where V is the almost equality class of a G -stable element v of $\text{Maps}(E, Z)$, there exists a G -stable maximal subtree of $\text{Complete}(V)$ with edge set $\{v\} \times (V - \{v\})$. \square

2. DIGRESSION 1: THE STALLINGS-SWAN THEOREM

In this section, to motivate interest in the Almost Stability Theorem, we recall how it implies the Stallings-Swan result that groups of cohomological dimension one are free. This and many other applications may be found in [5, Chapter IV].

Let $\mathbb{Z}G$ denote the integral group ring, and $\omega\mathbb{Z}G$ denote its augmentation ideal.

In 1953, Fox [9, (2.3)] proved, but did not state, that if the group G is free, then the left $\mathbb{Z}G$ -module $\omega\mathbb{Z}G$ is free. In 1956, this implication was made explicit by Cartan and Eilenberg [3, X.5], who further observed that if G is a nontrivial free group, then the projective dimension of the left $\mathbb{Z}G$ -module \mathbb{Z} is equal to 1. In 1957, Eilenberg and Ganea [8] defined 'the dimension of a group G ', now called the *cohomological dimension of G* , to be the projective dimension of the left $\mathbb{Z}G$ -module \mathbb{Z} . Thus, by definition, G has cohomological dimension at most one if and only if the left $\mathbb{Z}G$ -module $\omega\mathbb{Z}G$ is projective. Hence, by Fox's result, all free groups have cohomological dimension at most one. Eilenberg and Ganea remarked that they did not know whether or not all groups of cohomological dimension one are free. In 1968, Stallings [17, 6.8] proved that all finitely generable groups of cohomological dimension one are free; in 1969, Swan [19] proved that *all* groups of cohomological dimension one are free. In the academic year within this same period, 1968–9, Serre gave a course on what is now called *Bass-Serre theory*, and one of the many new results presented was the fact that the group G is free if and only if G acts freely on some tree [15, I.3.2.15 and I.3.3.4]. To my knowledge, neither direction

had previously been stated in the literature in exactly this form. It seems plausible that Dehn knew the ‘only if’ direction in 1910, in the context of his work on Cayley graphs in [4]. Reidemeister came close to knowing the ‘if’ direction in 1932 in the context of his tree-based proof in [13, 4.20] of the Nielsen-Schreier theorem that all subgroups of free groups are free.

In summary then, the left $\mathbb{Z}G$ -module $\omega\mathbb{Z}G$ is projective if and only if the group G is free if and only if G acts freely on some tree.

The currently known proofs of the Almost Stability Theorem 1.2 use many of the arguments of Stallings and Swan. We shall now recall that one form of their theorem is in turn a consequence of Theorem 1.2.

2.1. The Stallings-Swan Theorem. *If the left $\mathbb{Z}G$ -module $\omega\mathbb{Z}G$ is projective, then G acts freely on some tree.*

Proof. By hypothesis, there exists some left $\mathbb{Z}G$ -module Q such that the left $\mathbb{Z}G$ -module $\omega\mathbb{Z}G \oplus Q$ is free. There then exists some free left \mathbb{Z} -module A such that the (free) left $\mathbb{Z}G$ -modules $\omega\mathbb{Z}G \oplus Q$ and $AG := \mathbb{Z}G \otimes_{\mathbb{Z}} A$ are isomorphic, and may be identified. In a natural way, $\text{Maps}(G, A)$ is a left $\mathbb{Z}G$ -module, and we may identify AG with the (G -stable) almost equality class of 0 in $\text{Maps}(G, A)$; here, it is to be understood that G stabilizes each element of A . Each element r of AG has a unique expression as $p + q$ with $p \in \omega\mathbb{Z}G$ and $q \in Q$, and here we shall write $r = p \oplus q$.

Let g and x represent variable elements of G ranging over all of G .

We set $\widehat{g-x} := (g-x) \oplus 0 \in \omega\mathbb{Z}G \oplus Q = AG \subseteq \text{Maps}(G, A)$.

Notice that $\widehat{g-x} =_a 0$ in $\text{Maps}(G, A)$ and $\langle \widehat{g-x}, x \rangle \in A$. Essentially following Specker [16], we consider the element v of $\text{Maps}(G, A)$ defined by $\langle v, x \rangle := \langle \widehat{1-x}, x \rangle$, and show that $gv = \widehat{g-1} + v =_a 0 + v$ in $\text{Maps}(G, A)$ as follows:

$$\begin{aligned} \langle gv, x \rangle &= g\langle v, g^{-1}x \rangle \stackrel{\text{def } v}{=} g\langle \widehat{1-g^{-1}x}, g^{-1}x \rangle \\ &= \langle \widehat{g-x}, x \rangle = \langle \widehat{g-1} + \widehat{1-x}, x \rangle \stackrel{\text{def } v}{=} \langle \widehat{g-1} + v, x \rangle. \end{aligned}$$

Hence, the almost equality class $v + AG$ in $\text{Maps}(G, A)$ is G -stable. By the Almost Stability Theorem 1.2, the G -set $v + AG$ is the vertex G -set of some G -tree.

It remains to show that the vertex G -set $v + AG$ is G -free. Suppose then that we have some $g \in G$ and some $r \in AG$ such that $g(v+r) = v+r$ in $\text{Maps}(G, A)$, that is, $(1-g)r = (g-1)v = \widehat{g-1}$. Write $r = p \oplus q$ with $p \in \omega\mathbb{Z}G$ and $q \in Q$. Then $(1-g)p \oplus (1-g)q = (g-1) \oplus 0$ in $\omega\mathbb{Z}G \oplus Q$. Thus, $g(p+1) = p+1$ in $\mathbb{Z}G \subseteq \text{Maps}(G, \mathbb{Z})$. Hence, $p+1$ is constant on each $\langle g \rangle$ -orbit $\langle g \rangle x$ in G . Since $p \in \omega\mathbb{Z}G$, $0 \neq p+1 =_a 0$ in $\text{Maps}(G, \mathbb{Z})$. Hence, $\langle g \rangle$ is finite. Now the augmentation map carries $p+1$ to a \mathbb{Z} -multiple of $|\langle g \rangle|$ and also to 1. Thus, $|\langle g \rangle| = 1$. Hence, $g = 1$, as desired. \square

2.2. Remark. The foregoing argument applies to give Dunwoody’s characterization of the groups G such that the left RG -module ωRG is projective, where R is any nonzero associative ring with 1 and ωRG denotes the augmentation ideal of the group ring RG ; see [6, 1.1], [5, IV.3.13]. \square

3. TERMINOLOGY

In this section, we collect together definitions of many of the concepts that we shall be using.

3.1. Notation. We write $f|_D$ to indicate the map obtained from a map f by restricting the domain of f to a subdomain D .

We shall find it useful to have notation for intervals in \mathbb{Z} that is different from the notation for intervals in \mathbb{R} . Let $i, j \in \mathbb{Z}$. We define the sequence

$$\llbracket i \uparrow j \rrbracket := \begin{cases} (i, i+1, \dots, j-1, j) \in \mathbb{Z}^{j-i+1} & \text{if } i \leq j, \\ () \in \mathbb{Z}^0 & \text{if } i > j. \end{cases}$$

The subset of \mathbb{Z} underlying $\llbracket i \uparrow j \rrbracket$ is denoted $[i \uparrow j] := \{i, i+1, \dots, j-1, j\}$.

We set $\llbracket i \uparrow \infty \rrbracket := (i, i+1, i+2, \dots)$ and $[i \uparrow \infty[:= \{i, i+1, i+2, \dots\}$.

Suppose we have a set V and a map $[i \uparrow j] \rightarrow V$, $\ell \mapsto v_\ell$. We define the corresponding sequence in V by

$$v_{\llbracket i \uparrow j \rrbracket} := \begin{cases} (v_i, v_{i+1}, \dots, v_{j-1}, v_j) \in V^{i-j+1} & \text{if } i \leq j, \\ () \in V^0 & \text{if } i > j. \end{cases}$$

By abuse of notation, we shall also express this sequence as $(v_\ell \mid \ell \in \llbracket i \uparrow j \rrbracket)$, although “ $\ell \in \llbracket i \uparrow j \rrbracket$ ” on its own will not be assigned a meaning. The set of terms of $v_{\llbracket i \uparrow j \rrbracket}$ is denoted $v_{[i \uparrow j]}$.

We set $v_{\llbracket i \uparrow \infty \rrbracket} := (v_i, v_{i+1}, v_{i+2}, \dots)$ and $v_{[i \uparrow \infty[} := \{v_i, v_{i+1}, v_{i+2}, \dots\}$. \square

3.2. Definitions. By a *well-ordered set*, we mean a set S together with a total order \sqsubset such that, for each nonempty subset T of S , there exists some $x \in T$ such that, for each $t \in T$, $x \sqsubseteq t$. It is then usual to treat the total order as “less than”, and to use phrases such as “all strictly descending sequences are finite”.

An *ordinal* is a set β such that, first, each element of β is equal to some subset of β , and, secondly, β is well-ordered by \in ; see [12, 2.10].

The three lower-case Greek letters α , β , and γ will be used to denote ordinals.

We let \mathcal{Ord} denote the class of all ordinals, and, for $\alpha, \beta \in \mathcal{Ord}$, we define $\alpha < \beta$ to mean $\alpha \in \beta$. Thus, for each $\beta \in \mathcal{Ord}$, $\beta = \{\alpha \in \mathcal{Ord} \mid \alpha < \beta\}$.

Let S be any set. By the axiom of choice, S can be well-ordered, and, hence, there exists some $\alpha \in \mathcal{Ord}$ such that there exists some bijective map of sets from α to S . The minimum of the set consisting of such α is denoted $|S|$. We write $\omega_0 := | [0 \uparrow \infty[|$; thus, ω_0 is the smallest infinite ordinal, the set of finite ordinals. By abuse of notation, we view the elements of $[0 \uparrow \infty[$ as finite ordinals. \square

3.3. Definitions. Let V be any set.

We denote by $\mathcal{P}(V)$ the set of all subsets of V , and view $\mathcal{P}(V)$ as a Boolean algebra in the usual way.

Let A and B be any elements of $\mathcal{P}(V)$.

We write $A^c := \{v \in V \mid v \notin A\}$.

We say that A and B are *nested* if $\emptyset \in \{A \cap B, A \cap B^c, A^c \cap B, A^c \cap B^c\}$.

We write $A-B := A \cap B^c$.

We write $A \vee B$ to denote $A \cup B$ in the situation where $A \cap B = \emptyset$.

We write $A \nabla B := (A-B) \vee (B-A)$. If $A \nabla B$ is a finite set, we say that A and B are *almost equal*, and write $A =_a B$.

For any subset \mathcal{E} of $\mathcal{P}(V)$, we denote by $\langle \mathcal{E} \rangle_{\mathcal{B}}$ the Boolean subalgebra of $\mathcal{P}(V)$ generated by \mathcal{E} .

For each $v \in V$, we write $v^{**} := \{A \in \mathcal{P}(V) \mid v \in A\} \in \mathcal{P}(\mathcal{P}(V))$. \square

3.4. Definitions. We define the *rank of the group G* by

$$\text{rank}(G) := \min\{|S| : S \text{ is a subset of } G \text{ which generates } G\}.$$

For any subgroup H of G , we define the *rank of G relative to H* by

$$\text{rank}(G \text{ rel } H) := \min\{|S| : S \text{ is a subset of } G \text{ such that } S \cup H \text{ generates } G\}.$$

By a *G -set*, we mean a set V together with a map $G \times V \rightarrow V$, $(g, v) \mapsto gv$, such that, for each $v \in V$, $1v = v$ and, for each $(g_1, g_2) \in G \times G$, $g_1(g_2v) = (g_1g_2)v$.

By a *right G -set*, we mean a set V together with a map $V \times G \rightarrow V$, $(v, g) \mapsto vg$, such that, for each $v \in V$, $v1 = v$ and, for each $(g_1, g_2) \in G \times G$, $(vg_1)g_2 = v(g_1g_2)$. (Here, V is a G -set with $gv := vg^{-1}$.) All concepts defined for G -sets are understood to have analogues for right G -sets.

Consider any G -set V . For each subset W of V , we write

$$GW := \{gw \mid g \in G, w \in W\}.$$

If $GW = W$, we say that W is a *G -stable subset of V* and a *G -subset of V* . For each $v \in V$, we define the *G -stabilizer of v* to be $G_v := \{g \in G \mid gv = v\} \leq G$, and the *G -orbit of v* to be $Gv := \{gv \mid g \in G\}$, a G -subset of V . We let $G \setminus V := \{Gv \mid v \in V\}$, a partition of V . We say that V is *G -finite* if $G \setminus V$ is finite. A *G -transversal in V* is a subset of V which contains exactly one element of each G -orbit of V . A subgroup H of G is said to *stabilize* an element v of V if $Hv = \{v\}$ or, equivalently, $H_v = H$ or, equivalently, $H \leq G_v$; if H stabilizes some element of V , we say that H is a *G -stabilizer for V* . We let $G\text{-substabs}(V)$ denote the set of G -stabilizers for V .

Consider any G -sets V and W . By a *G -map $\varphi: V \rightarrow W$* , $v \mapsto \varphi(v)$, we mean a map of sets such that, for each $(g, v) \in G \times V$, $\varphi(gv) = g\varphi(v)$. There exists some G -map from V to W if and only if $G\text{-substabs}(V) \subseteq G\text{-substabs}(W)$.

A G -set V is said to be *G -incompressible* if each self G -map of V is bijective, or, equivalently, for each $(v, w) \in V \times V$, if $G_v \leq G_w$, then $G_v = G_w$ and $Gv = Gw$. If V is not G -incompressible, we say that V is *G -compressible*.

We now consider the right G -set G . Any $A \in \mathcal{P}(G)$ is said to be *almost right G -stable* if, for each $g \in G$, $Ag = {}_a A$. We write $\mathcal{B}(G)$ to denote the Boolean subalgebra of $\mathcal{P}(G)$ consisting of all the almost right G -stable elements. For each subgroup H of G , any $A \in \mathcal{P}(G)$ is said to be *almost a right H -set* if A is almost equal to some right H -subset of G . For each subset \mathcal{E} of $\mathcal{P}(G)$, we let $\text{Almosts}(\mathcal{E})$ denote the set consisting of all those subgroups H of G which have the property that each element of \mathcal{E} is almost a right H -set. \square

3.5. Definitions. By a *graph X* , we mean a quadruple $(V(X), E(X), \iota_X, \tau_X)$ where $V(X)$ and $E(X)$ are two disjoint sets and ι_X and τ_X are maps from $E(X)$ to $V(X)$. Where X is clear from the context, we write ι for ι_X and τ for τ_X . We define $|X| := |V(X) \vee E(X)|$. We say that $V(X)$ is the *vertex set* of X and that $E(X)$ is the *edge set* of X , and that ι and τ are the *incidence maps* of X . We say that the elements of $V(X)$ are the *vertices* of X , and the elements of $E(X)$ are the *edges* of X . For each edge e of X , we say that e is *incident to ιe* and τe , and that ιe is the *initial vertex* of e and that τe is the *terminal vertex* of e .

A *G -graph X* is a graph for which $V(X)$, $E(X)$ are G -sets, and ι , τ are G -maps. Passing to G -orbits gives a *quotient graph $G \setminus X$* . Here X is *G -finite* if $G \setminus X$ is finite, that is, $|G \setminus X|$ is finite.

For any subset S of G , the *Cayley graph $X(G, S)$* is defined as the G -graph with vertex G -set G , edge G -set $G \times S$ with G -action $g_1(g_2, s) := (g_1g_2, s)$, and incidence maps assigning to each edge $(g, s) \in G \times S$ the initial vertex g and the terminal vertex gs .

Let X be any graph.

A *subgraph of X* is a graph whose vertex set and edge set are subsets of the vertex set and edge set of X , respectively, and whose incidence maps agree with those of X .

For each vertex v of X , the *valence* of v in X is

$$|\{e \in E(X) : \iota e = v\}| + |\{e \in E(X) : \tau e = v\}|.$$

We say that X is *locally finite* if each vertex's valence is finite.

We create a set $E^{-1}(X)$ together with a bijective map $E(X) \rightarrow E^{-1}(X)$, $e \mapsto e^{-1}$, called *inversion*. We define $E^{\pm 1}(X) := E(X) \vee E^{-1}(X)$. We extend ι to a map $\iota: E^{\pm 1}(X) \rightarrow V(X)$ by setting $\iota(e^{-1}) := \tau e$ for each $e \in E(X)$. Similarly, we extend τ to a map $\tau: E^{\pm 1}(X) \rightarrow V(X)$ by setting $\tau(e^{-1}) := \iota e$ for each $e \in E(X)$. We extend inversion to a map $E^{\pm 1}(X) \rightarrow E^{\pm 1}(X)$, $e \mapsto e^{-1}$, by defining $(e^{-1})^{-1} := e$ for each $e \in E(X)$.

By an *X-path*, we shall mean any sequence $p = (v_0, e_1, v_1, e_2, \dots, e_n, v_n)$ such that $n \in [0 \uparrow \infty[$, $v_{[0 \uparrow n]}$ is a sequence in $V(X)$, $e_{[1 \uparrow n]}$ is a sequence in $E^{\pm 1}(X)$, and, for each $i \in [1 \uparrow n]$, $\iota(e_i) = v_{i-1}$ and $\tau(e_i) = v_i$. We define the *inverse* of p to be $p^{-1} := (v_n, e_n^{-1}, \dots, e_2^{-1}, v_1, e_1^{-1}, v_0)$. We say that p is *reduced* if, for each $i \in [2 \uparrow n]$, $e_i \neq e_{i-1}^{-1}$. We say that p *joins* v_0 to v_n , and that the pair (v_0, v_n) is *X-joined*. We define $\text{length}(p) := n$. If there exists no *X-path* joining v_0 to v_n of smaller length, then we say that the *X-distance* between v_0 and v_n is n . For each finite subset S of $E(X)$, we define the *number of times p crosses S* to be $|\{i \in [1 \uparrow n] : e_i \in S^{\pm 1}\}|$; if this number is positive, we say that p *crosses S*. Where S consists of a single edge, we shall usually speak of paths crossing that edge rather than crossing S .

We say that X is *connected* if each pair of vertices of X is *X-joined*. The maximal nonempty connected subgraphs of X are called the *components of X*.

For any subset E' of $E(X)$, the *graph obtained from X by collapsing E'*, denoted X/E' , is the graph with edge set $E'^c := E(X) - E'$, vertex set the set of components of $X - E'^c$, and the induced incidence maps. For example, $X/E(X)$ maps bijectively to the set of components of X , and here every edge of X gets collapsed.

For each $A \in \mathcal{P}(V(X))$, we define the *coboundary of A* (in X) as

$$\begin{aligned} \delta_X(A) &:= \{e \in E(X) \mid (\iota e, \tau e) \in ((A \times A^c) \vee (A^c \times A))\} \\ &= \{e \in E(X) \mid A \in (\iota e)^{**} \nabla (\tau e)^{**}\}; \end{aligned}$$

where X is clear from the context we write δA in place of $\delta_X(A)$.

The *Boolean algebra of X*, denoted $\mathcal{B}(X)$, is defined as the Boolean subalgebra of $\mathcal{P}(V(X))$ consisting of all the elements with finite coboundary in X .

We say that X is a *tree* if, for each $(v, w) \in V(X) \times V(X)$, there exists a unique reduced *X-path* that joins v to w . A *G-tree* is a *G-graph* which is a tree. We say that X is a *forest* if, for each $(v, w) \in V(X) \times V(X)$, there exists at most one reduced *X-path* that joins v to w . A *G-forest* is a *G-graph* which is a forest.

Let T be any *G-tree*. We say that T is *G-incompressible* if the *G-set* $V(T)$ is *G-incompressible*. An edge e of T is said to be *G-compressible* if there exists some $(v, w) \in \{(\iota e, \tau e), (\tau e, \iota e)\}$ such that $Gv \neq Gw$ and $G_v \leq G_w$; here, $G_w = G_e$. \square

In the following, the important conclusion $\mathcal{B}(X) = \mathcal{B}(G)$ is due to Specker [16].

3.6. Lemma. *Let S be any generating set of G, and set X := X(G, S). Then X is a nonempty, connected, G-free G-graph, and V(X) = G. Moreover, if S is finite, then X is locally finite, X is G-finite, and B(X) = B(G).*

Proof. Clearly, X is a nonempty, *G-free G-graph*, and $V(X) = G$. Also, $X/E(X)$ is a *G-set* with one *G-orbit*, and the image of 1 in $X/E(X)$ is stabilized by S . Since S generates G , we see that the image of 1 in $X/E(X)$ is stabilized by G . Hence, X has exactly one component.

Now suppose that S is finite. Then X is locally finite and *G-finite*. It remains to verify that $\mathcal{B}(X) = \mathcal{B}(G)$.

Consider any $A \in \mathcal{P}(G)$. For each $s \in S$,

$$\begin{aligned} A \nabla (As^{-1}) &= \{g \in G \mid g \in A - As^{-1} \text{ or } g \in As^{-1} - A\} \\ &= \{g \in G \mid g \in A, gs \in A^c \text{ or } g \in A^c, gs \in A\}, \end{aligned}$$

and, hence,

$$(3.6.1) \quad A \nabla (As^{-1}) = \{g \in G \mid (g, s) \in \delta(A)\}.$$

Suppose that $A \in \mathcal{B}(G)$. For each element s of the finite set S , $A \nabla (As^{-1})$ is a finite set. It follows from (3.6.1) that $\delta(A)$ is a finite set, that is, $A \in \mathcal{B}(X)$.

Suppose that $A \in \mathcal{B}(X)$. For each $s \in S$, by (3.6.1), $A \nabla As^{-1}$ is a finite set, that is, $A =_a As^{-1}$. Since S generates G , it follows that $A \in \mathcal{B}(G)$. \square

4. BUILDING TREES FROM NESTED SETS

This section reviews results of Dunwoody [6] with modifications by Dicks and Dunwoody [5] and Roller [14].

4.1. Notation. Let V be any set, and \mathcal{E} be any subset of the Boolean algebra $\mathcal{P}(V)$.

We say that \mathcal{E} is *c-stable* if, for each $A \in \mathcal{E}$, we have $A^c \in \mathcal{E}$.

We say that \mathcal{E} is *finitely separating* if, for all $v, w \in V$, $v^{**} \cap \mathcal{E} =_a w^{**} \cap \mathcal{E}$; note that $v^{**} \cap \mathcal{E} = \{d \in \mathcal{E} \mid v \in d\}$.

We say that \mathcal{E} is *nested* if, for each $(e, f) \in \mathcal{E} \times \mathcal{E}$, e and f are nested in V , that is, $\emptyset \in \{e \cap f, e \cap f^c, e^c \cap f, e^c \cap f^c\}$.

For each $e \in \mathcal{E}$, we define

$$\iota e := \{d \in \mathcal{E} \mid d \supseteq e \text{ or } d \supset e^c\}, \tau e := \{e, e^c\}, \text{ and } \tau' e := \{d \in \mathcal{E} \mid d \supset e \text{ or } d \supseteq e^c\}.$$

If $\mathcal{E} \neq \emptyset$, we define $T(\mathcal{E})$ to be the graph for which the edge set is \mathcal{E} , the vertex set is $\{\iota e, \tau e \mid e \in \mathcal{E}\} \subseteq \mathcal{P}(\mathcal{P}(V))$, and each $e \in \mathcal{E}$ has initial vertex ιe and terminal vertex τe .

If $\mathcal{E} \neq \emptyset$, we define $U(\mathcal{E})$ to be the graph for which the edge set is \mathcal{E} , the vertex set is $\{\iota e, \tau' e \mid e \in \mathcal{E}\} \subseteq \mathcal{P}(\mathcal{P}(V))$, and each $e \in \mathcal{E}$ has initial vertex ιe and terminal vertex $\tau' e$.

If $\mathcal{E} = \emptyset$, we define both $T(\mathcal{E})$ and $U(\mathcal{E})$ to be the graph for which the edge set is the empty set \mathcal{E} and the vertex set is $\{\mathcal{E}\} \subseteq \mathcal{P}(\mathcal{P}(V))$. \square

4.2. Example. Let T be a tree. It is sometimes natural to think of the vertices of T as certain sets of edges of T , and it is sometime natural to think of the edges of T as certain sets of vertices of T ; to achieve this formally, we create ‘double duals’ of the edges of T . For each $e \in E(T)$, we set

$$e^{**} := \{v \in V(T) \mid \text{the reduced } T\text{-path from } v \text{ to } \tau e \text{ crosses } e\};$$

then e^{**} is the vertex set of that component of $T - \{e\}$ which contains ιe ; hence, $\delta_T(e^{**}) = \{e\}$; hence, $e^{**} \in \mathcal{B}(T)$. Set $\mathcal{E}(T) := \{e^{**} \mid e \in E(T)\}$. For each $e \in E(T)$,

$$\begin{aligned} \iota_{U(\mathcal{E}(T))}(e^{**}) &= \{d^{**} \in \mathcal{E}(T) \mid d^{**} \supseteq e^{**} \text{ or } d^{**} \supset (e^{**})^c\} \\ &= \{d^{**} \in \mathcal{E}(T) \mid \iota_T e \in d^{**}\} = (\iota_T e)^{**} \cap \mathcal{E}(T). \end{aligned}$$

There is a natural identification $T = U(\mathcal{E}(T))$. \square

The following is due to Dunwoody [6, 2.1] with modifications from [5, II.1.5]. The proof given here incorporates the approach of Roller [14].

4.3. Theorem. *With Notation 4.1, if V is any set and \mathcal{E} is any c-stable, finitely separating, nested subset of $\mathcal{P}(V)$ such that $\emptyset \notin \mathcal{E}$, then the following hold.*

- (i) $T(\mathcal{E})$ is a tree with edge set \mathcal{E} , and, for any $e, f \in \mathcal{E}$, the $T(\mathcal{E})$ -distance between ιe and ιf equals $|\iota e \nabla \iota f|$.
- (ii) There exists a natural map $V \rightarrow V(T(\mathcal{E}))$, $v \mapsto v^{**} \cap \mathcal{E}$. In detail, if $\mathcal{E} \neq \emptyset$, then, for each $v \in V$, there exists some \subseteq -minimal element e of $v^{**} \cap \mathcal{E}$, and then $v^{**} \cap \mathcal{E} = \iota e$.

Proof. The case where $\mathcal{E} = \emptyset$ is straightforward, and we shall assume that $\mathcal{E} \neq \emptyset$.

(i). Here, $U(\mathcal{E})$ is the graph for which the edge set is \mathcal{E} , the vertex set is $\{\iota e \mid e \in \mathcal{E}\}$, and each $e \in \mathcal{E}$ has initial vertex ιe and terminal vertex $\tau' e = \iota(e^c)$. For each $e \in E(U(\mathcal{E})) = \mathcal{E}$, $e^c \neq e$, $e^{cc} = e$, $\iota(e^c) = \tau(e)$, and $\tau(e^c) = \iota(e)$. By a *restricted* $U(\mathcal{E})$ -*path*, we shall mean any sequence $p = e_{[1 \uparrow n]}$ in \mathcal{E} such that $n \in [1 \uparrow \infty[$ and, for each $i \in [2 \uparrow n]$, $\iota(e_i) = \tau(e_{i-1})$ and $e_i \neq e_{i-1}^c$.

For each $e \in \mathcal{E}$, $(\iota e) \nabla (\iota(e^c)) = \{e, e^c\}$, and ιe is a **c**-transversal in \mathcal{E} , or ‘orientation’, in the sense of a subset \mathcal{O} of \mathcal{E} such that $\mathcal{E} = \mathcal{O} \vee \{f^c \mid f \in \mathcal{O}\}$.

Consider any $e, f \in \mathcal{E}$. Set $[e, f[:= \{d \in \mathcal{E} \mid e \subseteq d \subset f\}$, and define $]e, f]$, $]e, f[$ and $]e, f]$ analogously. These sets are finite by the finitely separating condition, since we may choose $v \in e$ and $w \in f^c$, and find that $[e, f] \subseteq v^{**} - w^{**}$. We write $e \prec f$ to mean $[e, f[= \{e\}$ or, equivalently, $]e, f] = \{f\}$. Now

$$\iota e = \{d \in \mathcal{E} \mid e \subseteq d \text{ or } e^c \subset d\}, \quad \mathcal{E} - \iota f = \{d \in \mathcal{E} \mid d \subset f \text{ or } d \subseteq f^c\},$$

and also $e = f$ or $e \subset f$ or $e \subseteq f^c$ or $e^c \subset f$ or $e^c \subset f^c$. We then see that

$$\iota e - \iota f = \iota e \cap (\mathcal{E} - \iota f) = [e, f[\cup [e, f^c] \cup]e^c, f[\cup]e^c, f^c],$$

and that the latter union is empty if and only if $e = f$ or $e^c \prec f$. Thus, $\iota e \subseteq \iota f$ if and only if $e = f$ or $e^c \prec f$. Since $([e^c, f[)^c = [f^c, e[$, the condition $e^c \prec f$ is invariant under interchanging e and f , and we see that $\iota e = \iota f$ if and only if $e = f$ or $e^c \prec f$. By interchanging e and e^c , we see that $\tau' e = \iota f$ if and only if $f = e^c$ or $e \prec f$. (This paragraph is based on the elegant presentation of Roller [14, §2.6-§2.7] and is simpler than the discussion in [5, II.1.5].)

A restricted $U(\mathcal{E})$ -path $p = e_{[1 \uparrow n]}$, $n \in [1 \uparrow \infty[$, may then be viewed as an unrefinable increasing sequence $e_1 \prec e_2 \prec \dots \prec e_n$ in \mathcal{E} . Since $e_1 \subseteq e_n$, neither $e_1 = e_n^c$ nor $e_n \prec e_1$ are possible; thus, $\tau' e_n \neq \iota e_1$. Hence, in $U(\mathcal{E})$, no vertex is joined to itself by a restricted $U(\mathcal{E})$ -path.

We shall now see that, in $U(\mathcal{E})$, any vertex is joined to any other vertex by a restricted $U(\mathcal{E})$ -path. By the nestedness of \mathcal{E} , for any $e, f \in \mathcal{E}$, there exist $e' \in \{e, e^c\}$ and $f' \in \{f, f^c\}$ such that $e' \subseteq f'$. Since the set $[e', f']$ is finite, there exists some unrefinable increasing sequence

$$e' = e_1 \prec e_2 \prec \dots \prec e_n = f' \text{ in } \mathcal{E}, n \in [1 \uparrow \infty[;$$

this gives a restricted $U(\mathcal{E})$ -path which meets the vertices of e and f , as desired.

We may pass from the graph $U(\mathcal{E})$ to the graph $T(\mathcal{E})$ by detaching each edge from its terminal vertex and giving the elements of each unordered pair of edges $\{e, e^c\}$, $e \in \mathcal{E}$, a new common terminal vertex. Hence, $T(\mathcal{E})$ is a tree.

The $T(\mathcal{E})$ -distance formula follows since, for each $e \in \mathcal{E}$, $(\iota e) \nabla (\iota(e^c)) = \{e, e^c\}$ and $(\iota e, e, \tau e, (e^c)^{-1}, \iota(e^c))$ is a reduced $T(\mathcal{E})$ -path.

(ii). We show first that $v^{**} \cap \mathcal{E}$ has \subseteq -minimal elements. Since $\mathcal{E} \neq \emptyset$, there exists some $f \in \mathcal{E}$. We may assume that $v \in f$, for otherwise we may replace f with f^c . Since $\emptyset \notin \mathcal{E}$, there exists some $w \in f^c$, and we have

$$f \in \{e \in \mathcal{E} \mid v \in e \subseteq f\} \subseteq \{e \in \mathcal{E} \mid v \in e, w \in e^c\} = (v^{**} \cap \mathcal{E}) - (w^{**} \cap \mathcal{E}).$$

The latter set is finite, since \mathcal{E} is finitely separating. Thus $\{e \in \mathcal{E} \mid v \in e \subseteq f\}$ is finite and nonempty, and hence has a \subseteq -minimal element, which is then a \subseteq -minimal element of $\{e \in \mathcal{E} \mid v \in e\}$, as desired.

Let e be a \subseteq -minimal element of $v^{**} \cap \mathcal{E}$. We shall show that $v^{**} \cap \mathcal{E} = \iota e$.

Let $d \in \iota e$. Then either $d \supseteq e$ or $e \supset d^c$. If $d \supseteq e$ then $d \supseteq e \supseteq \{v\}$ and, hence, $d \in v^{**} \cap \mathcal{E}$. If $e \supset d^c$, then, by the \subseteq -minimality of e , $v \in (d^c)^c = d$, and, hence $d \in v^{**} \cap \mathcal{E}$. Thus, $\iota e \subseteq v^{**} \cap \mathcal{E}$.

Conversely, suppose that $d \in \mathcal{E} - \iota e$. Then $d^c \in \iota e \subseteq v^{**} \cap \mathcal{E}$. Hence, $d \in \mathcal{E} - v^{**}$. Thus, $\mathcal{E} - \iota e \subseteq \mathcal{E} - v^{**}$.

Now $v^{**} \cap \mathcal{E} = \iota e$, as desired. \square

4.4. Corollary. *Let V be any set, and \mathcal{E} be any finitely separating, nested subset of $\mathcal{P}(V)$ such that $\emptyset \notin \mathcal{E}$, $V \notin \mathcal{E}$, and, for each $e \in \mathcal{E}$, $e^c \notin \mathcal{E}$. With Notation 4.1, the following hold.*

- (i). $U(\mathcal{E})$ is a tree with edge set \mathcal{E} , and, for any $v, w \in \mathcal{E}$, the $U(\mathcal{E})$ -distance between v and w equals $|v \nabla w|$.
- (ii). There exists a natural map $V \rightarrow V(U(\mathcal{E}))$, $v \mapsto v^{**} \cap \mathcal{E}$.

Proof. Set $\mathcal{E}^c := \{e^c \mid e \in \mathcal{E}\}$. By Theorem 4.3, $T(\mathcal{E} \vee \mathcal{E}^c)$ is a tree with edge set $\mathcal{E} \vee \mathcal{E}^c$. Now $T(\mathcal{E} \vee \mathcal{E}^c)/\mathcal{E}^c = U(\mathcal{E})$, and the result follows. (Alternatively, $U(\mathcal{E}) = U(\mathcal{E} \vee \mathcal{E}^c) - \mathcal{E}^c$, the tree obtained from $U(\mathcal{E} \vee \mathcal{E}^c)$ by choosing the orientation \mathcal{E} .) \square

5. BUILDING NESTED SETS FROM GRAPHS

We now review theory developed by Bergman in [1].

5.1. Definitions. We introduce a new symbol \mathbf{t} , and view the power-series ring $\mathbb{Z}[[\mathbf{t}]]$ as an ordered abelian group with the total order \sqsubset such that

$$\sum_{\ell \in [0 \uparrow \infty[} c_\ell \mathbf{t}^\ell \sqsubset \sum_{\ell \in [0 \uparrow \infty[} d_\ell \mathbf{t}^\ell$$

if and only if there exists some $\ell_0 \in [0 \uparrow \infty[$ such that $c_{\ell_0} < d_{\ell_0}$ and, for each $\ell \in [0 \uparrow \ell_0[$, $c_\ell = d_\ell$. We view the polynomial ring $\mathbb{Z}[\mathbf{t}]$ as a subset of $\mathbb{Z}[[\mathbf{t}]]$.

Let X be any connected, locally finite graph.

For any set P of X -paths with the property that, for each $\ell \in [0 \uparrow \infty[$, the set $P_\ell := \{p \in P : \text{length}(p) = \ell\}$ is finite, we write

$$\Sigma(P) := \sum_{p \in P} \mathbf{t}^{\text{length}(p)} = \sum_{\ell \in [0 \uparrow \infty[} |P_\ell| \mathbf{t}^\ell \in \mathbb{Z}[[\mathbf{t}]].$$

For any element A of $\mathcal{B}(X)$, we let $P(A)$ denote the set of all X -paths which begin in A and end in A^c , necessarily crossing δA . Since X is locally finite and δA is finite, we see that $P(A)$ has only finitely many elements of any given length. We write $\Sigma P(A) := \Sigma(P(A))$. Inversion of paths carries $P(A)$ bijectively to $P(A^c)$; hence, $\Sigma P(A) = \Sigma P(A^c)$. We write

$$\Sigma P(\mathcal{B}(X)) := \{\Sigma P(A) \mid A \in \mathcal{B}(X)\} \subseteq \mathbb{Z}[[\mathbf{t}]].$$

For any Boolean subalgebra \mathcal{A} of $\mathcal{B}(X)$, any element C of \mathcal{A} is said to be \mathcal{A} -reducible if

$$C \in \langle \{D \in \mathcal{A} : \Sigma P(D) \sqsubset \Sigma P(C)\} \rangle_{\mathcal{B}};$$

otherwise, C is said to be \mathcal{A} -irreducible. We let $\text{irr}(\mathcal{A})$ denote the set of all \mathcal{A} -irreducible elements of \mathcal{A} . Notice that \emptyset and $V(X)$ are \mathcal{A} -reducible. \square

The following is the G -finite case of a result of Bergman [1, Lemma 2].

5.2. Theorem. *If X is any connected, locally finite, G -finite G -graph, then $\Sigma P(\mathcal{B}(X))$ is a well-ordered subset of $\mathbb{Z}[[\mathbf{t}]]$.*

Proof. We shall show that a larger subset of $\mathbb{Z}[[\mathbf{t}]]$ is well-ordered.

Let \mathbb{S} denote the set of all finite subsets of $E(X)$.

Consider any $S \in \mathbb{S}$. We denote by $P(S)$ the set of all those X -paths that cross S an odd number of times. For each $\ell \in [0 \uparrow \infty[$, we denote by $P_\ell(S)$ the set of all elements of $P(S)$ whose length equals ℓ . Since S is finite and X is locally finite, $P_\ell(S)$ is finite; $|P_\ell(S)|$ is an even number since $P_\ell(S)$ is stable under path inversion. Clearly, $|P_0(S)| = 0$ and $|P_1(S)| = 2|S|$. We write

$$\Sigma P(S) := \Sigma(P(S)) = \sum_{\ell \in [0 \uparrow \infty[} |P_\ell(S)| \mathbf{t}^\ell \text{ and } \Sigma P(\mathbb{S}) := \{\Sigma P(S) : S \in \mathbb{S}\} \subseteq 2\mathbf{t}\mathbb{Z}[[\mathbf{t}]].$$

For any $A \in \mathcal{B}(X)$, we have $\delta A \in \mathbb{S}$, and $P(\delta A)$, the set of X -paths that cross δA an odd number of times, equals $P(A) \vee P(A^c)$; hence, $\Sigma P(\delta A) = 2 \Sigma P(A)$. Thus, $2 \Sigma P(\mathcal{B}(X)) \subseteq \Sigma P(\mathbb{S})$, and it suffices to show that $\Sigma P(\mathbb{S})$ is well-ordered.

Consider any map $S_- : [0\uparrow\infty[\rightarrow \mathbb{S}$, $n \mapsto S_n$, such that the composite map $\Sigma P(S_-) : [0\uparrow\infty[\rightarrow \mathbb{Z}[[\mathbf{t}]$, $n \mapsto \Sigma P(S_n)$, is decreasing. It suffices to show that there exists some infinite subset \mathbf{N} of $[0\uparrow\infty[$ such that $\{\Sigma P(S_n) \mid n \in \mathbf{N}\}$ has exactly one element. Without loss of generality, we may assume that, for each $n \in [0\uparrow\infty[$, $S_n \neq \emptyset$.

Let \mathbf{K} denote the set consisting of those $k \in [1\uparrow\infty[$ for which there exist

$$(5.2.1) \quad \text{an infinite subset } \mathbf{N} \text{ of } [0\uparrow\infty[, \text{ a map } [1\uparrow k] \rightarrow \mathbb{S} - \{\emptyset\}, i \mapsto R_i, \text{ and a map } \\ \mathbf{N} \times [1\uparrow k] \rightarrow G, (n, i) \mapsto g_{n,i}, \text{ such that, for each } n \in \mathbf{N}, S_n = \bigcup_{i=1}^k g_{n,i} R_i.$$

In the case where $k = 1$,

$$\{\Sigma P(S_n) \mid n \in \mathbf{N}\} = \{\Sigma P(g_{n,1} R_1) \mid n \in \mathbf{N}\} = \{\Sigma P(R_1)\},$$

which gives the desired result. We shall show that $\mathbf{K} \neq \emptyset$ and that, for each $k \in \mathbf{K}$, either $k = 1$ or $k-1 \in \mathbf{K}$. This implies that $1 \in \mathbf{K}$, which completes the proof.

We now show that $|S_0| \in \mathbf{K}$, and, hence, $\mathbf{K} \neq \emptyset$. Let us choose a finite G -transversal R in the G -finite G -set $E(X)$. Consider any $n \in [0\uparrow\infty[$. Since $n \geq 0$ and $\Sigma P(S_-)$ is decreasing, $\Sigma P(S_n) \subseteq \Sigma P(S_0)$. Hence,

$$2|S_n|\mathbf{t} \subseteq \Sigma P(S_n) \subseteq \Sigma P(S_0) \subseteq 2(|S_0| + 1)\mathbf{t}.$$

Thus, $|S_n| < |S_0| + 1$. Set $k := |S_0|$. Then $1 \leq |S_n| \leq k$, and we may choose a surjective map $[1\uparrow k] \rightarrow S_n$, $i \mapsto s_{n,i}$. For each $i \in [1\uparrow k]$, there exists a unique $r_{n,i} \in R$ such that $s_{n,i} \in G r_{n,i}$, and we may choose some $g_{n,i} \in G$ such that $g_{n,i} r_{n,i} = s_{n,i}$. We have a map $r_{n,-} : [1\uparrow k] \rightarrow R$, $i \mapsto r_{n,i}$. Since

$$\{|r_{n,-} \mid n \in [0\uparrow\infty[\} \} \leq |R|^k < \omega_0,$$

there exists some infinite subset \mathbf{N} of $[0\uparrow\infty[$ and some map $r_- : [1\uparrow k] \rightarrow R$, $i \mapsto r_i$, such that, for each $n \in \mathbf{N}$, $r_{n,-} = r_-$, and, hence,

$$S_n = \bigcup_{i=1}^k g_{n,i} \{r_{n,i}\} = \bigcup_{i=1}^k g_{n,i} \{r_i\}.$$

We have (5.2.1), and $k \in \mathbf{K}$.

For any $R, S \in \mathbb{S} - \{\emptyset\}$, we let $\mathbf{d}(R, S)$ denote the length of the minimum-length X -paths that cross both R and S . Set $d := \mathbf{d}(R, S) \in [1\uparrow\infty[$. The X -distance, in the usual sense, from R to S equals $\max\{d-2, 0\}$. It may be seen that $P_d(R) \cap P_d(S)$ is nonempty and consists of the minimum-length X -paths with the properties that exactly one edge (the first or last) lies in R and exactly one edge (the last or first) lies in S . For each $\ell \in [0\uparrow d[$,

$$P_\ell(R \cup S) = P_\ell(R) \vee P_\ell(S) \text{ and } |P_\ell(R \cup S)| = |P_\ell(R)| + |P_\ell(S)|,$$

while

$$P_d(R \cup S) \subseteq P_d(R) \cup P_d(S) \text{ and } |P_d(R \cup S)| < |P_d(R)| + |P_d(S)|.$$

(If $d = 1$, then $R \cap S \neq \emptyset$ and $P_d(R \cup S) = P_d(R) \cup P_d(S)$, while if $d \geq 2$, then $R \cap S = \emptyset$ and $P_d(R \cup S) \subset P_d(R) \cup P_d(S)$.)

Now suppose that we have some $k \in \mathbf{K}$ with $k \geq 2$; we shall show that $k-1 \in \mathbf{K}$. Here, we have (5.2.1). Consider any $n \in \mathbf{N}$. We set

$$d_n := \min\{\mathbf{d}(g_{n,i} R_i, g_{n,j} R_j) \mid i, j \in [1\uparrow k] \text{ with } i < j\} \in [1\uparrow\infty[.$$

We first prove that $\{d_n \mid n \in \mathbf{N}\}$ is finite. For each $\ell \in [0\uparrow\infty[$, set

$$c_\ell := \sum_{i=1}^k |P_\ell(R_i)| = \sum_{i=1}^k |P_\ell(g_{n,i} R_i)|.$$

Then

$$|P_{d_n}(S_n)| = |P_{d_n}(\bigcup_{i=1}^k g_{n,i}R_i)| \leq |\bigcup_{i=1}^k P_{d_n}(g_{n,i}R_i)| < \sum_{i=1}^k |P_{d_n}(g_{n,i}R_i)| = c_{d_n}.$$

For each $\ell \in [0 \uparrow d_n[$,

$$|P_\ell(S_n)| = |P_\ell(\bigcup_{i=1}^k g_{n,i}R_i)| = |\bigvee_{i=1}^k P_\ell(g_{n,i}R_i)| = \sum_{i=1}^k |P_\ell(g_{n,i}R_i)| = c_\ell.$$

Hence,

$$\sum_{\ell=0}^{d_n-1} c_\ell \mathbf{t}^\ell \sqsubseteq \Sigma P(S_n) \sqsubset \sum_{\ell=0}^{d_n} c_\ell \mathbf{t}^\ell.$$

Now consider any $m \in \mathbf{N}$ such that $m \geq n$. Then $\Sigma P(S_m) \sqsubseteq \Sigma P(S_n)$, since $\Sigma P(S_-)$ is decreasing. Hence,

$$\sum_{\ell=0}^{d_m-1} c_\ell \mathbf{t}^\ell \sqsubseteq \Sigma P(S_m) \sqsubseteq \Sigma P(S_n) \sqsubset \sum_{\ell=0}^{d_n} c_\ell \mathbf{t}^\ell.$$

Thus, $d_m - 1 < d_n$. Hence, $d_m \leq d_n$. It follows that $\{d_N \mid N \in \mathbf{N}\}$ is finite, and we may assume it has exactly one element, d_* , by replacing \mathbf{N} with a suitable infinite subset.

Fix $i_n, j_n \in [1 \uparrow k]$ such that $i_n < j_n$ and

$$\mathbf{d}(g_{n,i_n}R_{i_n}, g_{n,j_n}R_{j_n}) = d_n = d_*.$$

Now $\{(i_N, j_N) : N \in \mathbf{N}\}$ is finite, and we may assume it has exactly one element, (i_*, j_*) , by replacing \mathbf{N} with a suitable infinite subset. By renumbering the R_i , we may assume that $(i_*, j_*) = (1, k)$. Now

$$d_* = \mathbf{d}(g_{n,1}R_1, g_{n,k}R_k) = \mathbf{d}(R_1, g_{n,1}^{-1}g_{n,k}R_k).$$

Since X is locally finite and R_1 and R_k are finite sets of edges, there exist only finitely many elements in the G -orbit of R_k whose X -distance from R_1 equals $\max\{d_* - 2, 0\}$. Thus, $\{g_{N,1}^{-1}g_{N,k}R_k \mid N \in \mathbf{N}\}$ is finite, and we may assume that it has exactly one element, R_* , by replacing \mathbf{N} with a suitable infinite subset. Now $g_{n,1}^{-1}g_{n,k}R_k = R_*$, and then $g_{n,1}R_1 \cup g_{n,k}R_k = g_{n,1}(R_1 \cup R_*)$. Here, we may replace R_1 with $R_1 \cup R_*$ and k with $k-1$, and we see that $k-1 \in \mathbf{K}$.

This completes the proof of Theorem 5.2. \square

The following is the locally finite case of a result of Bowditch and Dunwoody [2, §8], which was based on work of Bergman [1, Lemma 1] and Dunwoody and Swenson [7, Lemma 3.3].

5.3. Theorem. *Let X be any connected, locally finite, G -finite G -graph, and \mathcal{A} be any Boolean G -subalgebra of $\mathcal{B}(X)$. Then $\text{irr}(\mathcal{A})$ is a \mathbf{c} -stable, nested G -subset of \mathcal{A} such that $\emptyset \notin \text{irr}(\mathcal{A})$ and $\langle \text{irr}(\mathcal{A}) \rangle_{\mathcal{B}} = \mathcal{A}$.*

Proof. It is clear that $\text{irr}(\mathcal{A})$ is a \mathbf{c} -stable G -subset of \mathcal{A} such that $\emptyset \notin \text{irr}(\mathcal{A})$. By Theorem 5.2, $\Sigma P(\mathcal{A})$ is well-ordered, and then a standard argument shows that $\langle \text{irr}(\mathcal{A}) \rangle_{\mathcal{B}} = \mathcal{A}$. It remains to show that $\text{irr}(\mathcal{A})$ is nested.

Consider any $A', B' \in \text{irr}(\mathcal{A})$. It suffices to show that A' and B' are nested. Let us choose $(A, B) \in \{A', A'^c\} \times \{B', B'^c\}$ to make $\Sigma P(A \cap B)$ as \sqsubset -small as possible. In particular, $\Sigma P(A \cap B) \sqsubseteq \Sigma P(A \cap B^c)$, and we see that

$$A = (A \cap B) \cup (A \cap B^c) \in \langle \{C \in \mathcal{A} \mid \Sigma P(C) \sqsubseteq \Sigma P(A \cap B^c)\} \rangle_{\mathcal{B}}.$$

Since A is \mathcal{A} -irreducible, it is not the case that $\Sigma P(A \cap B^c) \sqsubset \Sigma P(A)$. Thus,

$$(5.3.1) \quad \Sigma P(A) \sqsubseteq \Sigma P(A \cap B^c).$$

For any elements C, D of $\mathcal{B}(X)$, let us define $P(C, D) := P(C) \cap P(D^c)$ and $\Sigma P(C, D) := \Sigma(P(C, D)) \in \mathbb{Z}[[\mathbf{t}]]$. If $C \cap D = \emptyset$, then $P(C, D)$ is the set of all

X -paths which begin in C and end in D , and, here, $\Sigma P(C, D) = \Sigma P(D, C)$. Set

$$a := \Sigma P(A \cap B, A^c \cap B), \quad b := \Sigma P(A \cap B, A^c \cap B^c), \quad c := \Sigma P(A \cap B^c, A^c), \\ d := \Sigma P(A \cap B^c, A \cap B).$$

It is not difficult to see that

$$(a + b) + c = \Sigma P(A, A^c) = \Sigma P(A) \stackrel{(5.3.1)}{\sqsubseteq} \Sigma P(A \cap B^c) = \Sigma P(A \cap B^c, A^c \cup B) = c + d.$$

Hence, $b \sqsubseteq d - a$. By interchanging A and B , we see that $b \sqsubseteq a - d$, also. Thus, $b \sqsubseteq 0$. Hence, there exists no X -path which begins in $A \cap B$ and ends in $A^c \cap B^c$; hence, $A \cap B$ or $A^c \cap B^c$ is empty; and, hence, A' and B' are nested, as desired. \square

5.4. Remarks. Throughout this section, we have considered connected, locally finite, G -finite G -graphs; these include the Cayley graphs of finitely generated groups, which are the graphs we shall be using. Both Bergman [1] and Bowditch [2] consider more general situations. Bergman obtains similar results about connected, locally finite G -graphs. Bowditch obtains results about countable groups. We have not seen any way to use these generalizations for our narrow objective of improving the proof of Theorem 1.2. In [5, II.2.20], nested generating sets were constructed for Boolean algebras of arbitrary connected graphs. \square

6. BUILDING TREES FROM THE BOOLEAN ALGEBRA OF A GROUP

In this section we shall prove a substantial part of the finitely generable case of Theorem 1.2.

Recall Definitions 3.4 and 3.5. The following result is implicit in the finitely generable case of the Almost Stability Theorem. Dunwoody [6, 4.7] showed that $G\text{-substabs}(\mathcal{V}(T)) \subseteq \text{Almosts}(\mathcal{F})$.

6.1. Theorem. *Suppose that $\text{rank}(G) < \omega_0$. For each G -finite G -subset \mathcal{F} of $\mathcal{B}(G)$, there exists some G -finite G -tree T such that $G\text{-substabs}(\mathcal{V}(T)) = \text{Almosts}(\mathcal{F})$ and $E(T)$ is G -quasifree.*

Proof. Let S be any finite generating set of G , and set $X := X(G, S)$. By Lemma 3.6, X is a connected, locally finite, G -finite, G -free G -graph, $\mathcal{V}(X) = G$, and $\mathcal{B}(X) = \mathcal{B}(G)$.

Set $\mathcal{A} := \langle \mathcal{F} \rangle_{\mathcal{B}}$ in $\mathcal{B}(G) = \mathcal{B}(X)$.

By Theorem 5.3, $\langle \text{irr}(\mathcal{A}) \rangle_{\mathcal{B}} = \mathcal{A}$, and $\text{irr}(\mathcal{A})$ is a \mathbf{c} -stable, nested G -subset of \mathcal{A} such that $\emptyset \notin \text{irr}(\mathcal{A})$.

Since \mathcal{F} is G -finite, there exists some \mathbf{c} -stable, G -finite G -subset \mathcal{E} of $\text{irr}(\mathcal{A})$ such that $\mathcal{F} \subseteq \langle \mathcal{E} \rangle_{\mathcal{B}}$, and then \mathcal{E} is nested, $\emptyset \notin \mathcal{E}$, and $\langle \mathcal{E} \rangle_{\mathcal{B}} = \mathcal{A}$.

We shall now see that \mathcal{E} is finitely separating. Consider any edge e of X . For each $A \in \mathcal{E}$, since $G_e = \{1\}$, there exist only finitely many $g \in G$ such that $ge \in \delta(A)$, or, equivalently, $e \in \delta(g^{-1}A)$. Since \mathcal{E} is G -finite, we then see that there exist only finitely many $B \in \mathcal{E}$ such that $e \in \delta B$, that is, $(\iota e)^{**} \cap \mathcal{E} =_{\mathbf{a}} (\tau e)^{**} \cap \mathcal{E}$. Since X is connected, it follows that \mathcal{E} is finitely separating.

Now \mathcal{E} is a \mathbf{c} -stable, finitely separating, nested G -subset of $\mathcal{P}(G)$ and $\emptyset \notin \mathcal{E}$. By Theorem 4.3(i), $T(\mathcal{E})$ is a G -tree with edge G -set \mathcal{E} . The edge G -set of X is G -free, and, hence, $\mathcal{B}(X) - \{\emptyset, \mathcal{V}(X)\}$ is G -quasifree. Hence, \mathcal{E} is G -quasifree, that is, $E(T(\mathcal{E}))$ is G -quasifree. Since \mathcal{E} is G -finite, we see that $T(\mathcal{E})$ is G -finite.

It remains to show that $G\text{-substabs}(\mathcal{V}(T(\mathcal{E}))) = \text{Almosts}(\mathcal{F})$. Since

$$\langle \mathcal{E} \rangle_{\mathcal{B}} = \mathcal{A} = \langle \mathcal{F} \rangle_{\mathcal{B}},$$

it is not difficult to show that $\text{Almosts}(\mathcal{E}) = \text{Almosts}(\mathcal{A}) = \text{Almosts}(\mathcal{F})$, and it suffices to show that $G\text{-substabs}(\mathcal{V}(T(\mathcal{E}))) = \text{Almosts}(\mathcal{E})$. Notice that if $\mathcal{E} = \emptyset$, then $T(\mathcal{E})$ is a single vertex stabilized by all subgroups of G , in which case it is clear that $G\text{-substabs}(\mathcal{V}(T(\mathcal{E}))) = \text{Almosts}(\mathcal{E})$. Thus, we may assume that $\mathcal{E} \neq \emptyset$.

We shall use the following observations. Consider any $e, f \in \mathcal{E}$. Notice that, for each $g \in G$,

$$(gf \in 1^{**}) \Leftrightarrow (1 \in gf) \Leftrightarrow (g^{-1} \in f) \Leftrightarrow (g \in f^{-1} := \{g^{-1} \mid g \in f\} \in \mathcal{P}(G)).$$

With careful interpretation, we may write $G\{f\} \cap 1^{**} = f^{-1}\{f\}$ in $\mathcal{P}(\mathcal{P}(G))$. By Theorem 4.3(i),(ii), $\iota e =_a 1^{**} \cap \mathcal{E}$ in $\mathcal{P}(\mathcal{P}(G))$, and, hence, $G\{f\} \cap \iota e =_a G\{f\} \cap 1^{**}$. We record

$$(6.1.1) \quad f^{-1}\{f\} =_a G\{f\} \cap \iota e \text{ in } \mathcal{P}(\mathcal{P}(G)).$$

Now consider any $H \in G\text{-substabs}(\mathcal{V}(\mathcal{T}(\mathcal{E})))$. Then there exists some $e \in \mathcal{E}$ such that H stabilizes ιe or τe . Consider any $f \in \mathcal{E}$. Notice that (6.1.1) implies that

$$f^{-1} =_a \{g \in G \mid gf \in \iota e\} \text{ in } \mathcal{P}(G),$$

since these are right G_f -sets and G_f is finite. Hence, $f =_a \{g \in G \mid f \in g\iota e\}$, which is a right $G_{\iota e}$ -set. Similarly, $f =_a \{g \in G \mid f \in g\iota(e^c)\}$ and, hence,

$$f =_a \{g \in G \mid f \in \iota ge \text{ and } f \in \iota(ge^c)\},$$

which is a right $G_{\tau e}$ -set. Thus, f is almost a right H -set. Thus, $H \in \text{Almosts}(\mathcal{E})$.

For the converse, we now consider any $H \in \text{Almosts}(\mathcal{E})$. Consider any $e \in \mathcal{E}$ and any finite G -transversal F in the G -finite G -set \mathcal{E} . For each $f \in F$, we have $f \in \mathcal{E}$, and, hence, there exists some right H -subset A_f of G such that $f =_a A_f$ in $\mathcal{P}(G)$. We may then form the H -set

$$w := \bigcup_{f \in F} (A_f^{-1}\{f\}) =_a \bigcup_{f \in F} (f^{-1}\{f\}) \stackrel{(6.1.1)}{=} \bigcup_{f \in F} (G\{f\} \cap \iota e) = \iota e \text{ in } \mathcal{P}(\mathcal{P}(G)).$$

Set $d := |(\iota e)\nabla(w)| \in [0\uparrow\infty[$. For each $h \in H$,

$$d = |(h\iota e)\nabla(hw)| = |(h\iota e)\nabla(w)| \text{ and } |(h\iota e)\nabla(\iota e)| \leq |(h\iota e)\nabla(w)| + |(w)\nabla(\iota e)| = 2d.$$

By Theorem 4.3(i), the $\mathcal{T}(\mathcal{E})$ -distance between ιe and $h\iota e$ is at most $2d$. Hence, the subtree of $\mathcal{T}(\mathcal{E})$ spanned by $H\iota e$ has finite diameter. Consider any H -subtree T of $\mathcal{T}(\mathcal{E})$ of minimum possible diameter. Then T has at most one edge, for, otherwise, deleting from T all vertices of valence one and the edges incident thereto leaves an H -subtree of smaller diameter. It follows that H stabilizes some vertex of $\mathcal{T}(\mathcal{E})$. Thus, $H \in G\text{-substabs}(\mathcal{V}(\mathcal{T}(\mathcal{E})))$. \square

7. DIGRESSION 2: STALLINGS' ENDS THEOREM

In this section, we shall deduce a form of Stallings' celebrated Ends Theorem [18, 4.1], a result which inspired much subsequent work in combinatorial group theory, including all the theory discussed in this article.

In the case where G is finitely generable and S is any finite generating set of G , let \mathbb{S} denote the set of finite subsets of $\mathcal{E}(X(G, S))$, and, for each $E \in \mathbb{S}$, let $\varphi(E)$ denote the set of infinite components of $X(G, S) - E$. Then $(\varphi(E) \mid E \in \mathbb{S})$ forms an inverse directed system, and, by a 1945 argument of Freudenthal [10, 6.16.1], the resulting inverse limit is independent of the choice of finite generating set S . The elements of this inverse limit are called the *ends* of the group G .

We wish to consider the graph-theoretical conditions

- (a) G is finitely generable and has more than one end.
- (b) There exists some G -tree such that the edge G -set is G -quasifree and no vertex is G -stable.

and the group-theoretical conditions

- (a') $\mathcal{B}(G)$ has some element A such that both A and $G - A$ are infinite.

- (b') Either G is countably infinite and locally finite, or there exists some finite subgroup B of G such that G is a free product with amalgamation $C *_B D$ where $B < C$ and $B < D$ or G is an HNN extension $C *_B \varphi$ where $B \leq C$ and $\varphi : B \rightarrow C$ is a monomorphism.

In 1949, Specker [16] showed that if G is finitely generable, then (a) \Leftrightarrow (a'). Subsequently, it became a common practice to use some cohomological form of (a') as a definition for ' G has more than one end' even if G is not finitely generable and, hence, ends of G are not defined.

By Bass-Serre theory, (b) \Leftrightarrow (b'); see [15], [5, I.4.12].

It is not difficult to show that (b)+(b') \Rightarrow (a'); see [5, IV.6.10].

In 1970, Stallings [18, 4.1] proved that (a) \Rightarrow (b'); notice that no finitely generable group is both infinite and locally finite. He remarked that a communication from Dunwoody inspired his short proof of his key lemma [18, 1.5] (which is actually the Cayley-graph case of a result of Bergman [1, Theorem 1]). In 1979, Dunwoody [6, 4.4] proved directly that if G is finitely generable, then (a') \Rightarrow (b); this is the restatement of Stallings' result in which the graph-theoretic hypothesis (a) is replaced with the group-theoretic condition (a') and the group-theoretic conclusion (b') is replaced with the graph-theoretic condition (b).

7.1. Stallings' Ends Theorem. *If G is finitely generable and there exists some element A in $\mathcal{B}(G)$ such that A and $G-A$ are infinite, then there exists some G -tree such that no vertex is G -stable, the edge G -set is G -quasifree, and the number of G -orbits of edges equals 1.*

Proof. Set $\mathcal{F} := \{gA \mid g \in G\} \subseteq \mathcal{B}(G)$. By Theorem 6.1, there exists some G -finite G -tree T such that G -substabs($V(T)$) = $\text{Almosts}(\mathcal{F})$ and $E(T)$ is G -quasifree. Notice that $G \notin \text{Almosts}(\{A\}) = \text{Almosts}(\mathcal{F}) = G$ -substabs($V(T)$). Now we collapse G -orbits of edges of T , one G -orbit at a time. At some first stage, a G -stable vertex appears, and then the G -orbit of edges that has just been collapsed is the edge G -set of a G -tree which has the desired properties. \square

7.2. Remarks. In 1968, Stallings [17] had proved a special case of (a) \Rightarrow (b'), and had written the following: "Since "ends" are, after all, a topological kind of thing, there is no need to make a profuse apology for a topological kind of proof. However, maybe there is some algebraic translation of this which will go over to infinitely generated groups." An algebraic translation which went over to all groups was given in 1989 when Dicks and Dunwoody proved in [5, IV.6.10] that (a') \Rightarrow (b'). An important advance in the theory had been made by Holt in 1981, who showed in [11] that if G is locally finite, then (a') \Rightarrow (b'); notice that no locally finite group is an HNN extension or a proper free product with amalgamation. \square

8. THE FINITELY GENERABLE CASE OF THE ALMOST STABILITY THEOREM

Recall Definitions 1.1. We may now prove the case of the Almost Stability Theorem 1.2 where G is finitely generable.

8.1. Theorem. *Suppose that $\text{rank}(G) < \omega_0$. If E and Z are any G -sets such that E is G -quasifree and each element's G -stabilizer stabilizes some element of Z , then each G -stable almost equality class in the G -set $\text{Maps}(E, Z)$ is the vertex G -set of some G -tree.*

Proof. Let V be any G -stable almost equality class in $\text{Maps}(E, Z)$. We shall prove a sequence of three equalities which will relate V to a G -incompressible G -tree.

Let S be any finite generating set of G , and set $X := X(G, S)$. Then X is a connected, locally finite, G -finite, G -free G -graph, $V(X) = G$, and $\mathcal{B}(X) = \mathcal{B}(G)$.

As V is nonempty, we may choose an element v of V .

Consider any $e \in E$. We then have a map $\langle_{-v}, e\rangle: G \rightarrow Z$, $g \mapsto \langle gv, e\rangle$, and we shall be interested in the set of fibres thereof, $\{\langle_{-v}, e\rangle^{-1}(\{z\}) \mid z \in Z\}$. The set of edges of X which are *broken* by this same map $\langle_{-v}, e\rangle: V(X) \rightarrow Z$ is

$$\delta(\langle_{-v}, e\rangle) := \{(g, s) \in E(X) \mid \langle \iota_X(g, s)v, e\rangle \neq \langle \tau_X(g, s)v, e\rangle\}.$$

Thus,

$$((g, s) \in \delta(\langle_{-v}, e\rangle)) \Leftrightarrow (\langle gv, e\rangle \neq \langle gsv, e\rangle) \Leftrightarrow ((e \in (gv)\nabla(gsv) = g(v\nabla(sv))).$$

Hence,

$$\delta(\langle_{-v}, e\rangle) = \{(g, s) \in E(X) \mid g^{-1}e \in v\nabla(sv)\}.$$

For each $s \in S$, $v\nabla(sv)$ is finite, since $v =_a sv$ in $\text{Maps}(E, Z)$. Since G_e and S are finite, we see that $\delta(\langle_{-v}, e\rangle)$ is finite. Since X is connected, the set of fibres of $\langle_{-v}, e\rangle$ is finite, and each fibre of $\langle_{-v}, e\rangle$ is then an element of $\mathcal{B}(X) = \mathcal{B}(G)$.

Set $E_G := \{e \in E : \langle_{-v}, e\rangle \text{ is not constant}\}$. Then

$$E_G = \{e \in E : \delta(\langle_{-v}, e\rangle) \neq \emptyset\} = \bigcup_{g \in G} \bigcup_{s \in S} (g(v\nabla(sv))).$$

Since $\bigcup_{s \in S} (v\nabla(sv))$ is finite, we see that E_G is G -finite.

For $g \in G$, $e \in E$, $z \in Z$, we have $g(\langle_{-v}, e\rangle^{-1}(\{z\})) = \langle_{-v}, ge\rangle^{-1}(\{gz\})$. Set $\mathcal{F} := \{\langle_{-v}, e\rangle^{-1}(\{z\}) \mid z \in Z, e \in E\}$. Then \mathcal{F} is a G -finite G -subset of $\mathcal{B}(G)$.

We shall now prove

$$(8.1.1) \quad G\text{-substabs}(V) \subseteq \text{Almosts}(\mathcal{F}),$$

that is, for each $H \in G\text{-substabs}(V)$, each $\langle_{-v}, e\rangle^{-1}(\{z\}) \in \mathcal{F}$ is almost equal to some right H -set.

Proof of (8.1.1). Here, $e \in E$, $z \in Z$, and H stabilizes some element w of V . Since $w =_a v$ and G_e is finite, we see that, for all but finitely many $g \in G$, we have $\langle v, g^{-1}e\rangle = \langle w, g^{-1}e\rangle$, that is, $g^{-1}\langle gv, e\rangle = g^{-1}\langle gw, e\rangle$, that is, $\langle gv, e\rangle = \langle gw, e\rangle$. Thus $\langle_{-v}, e\rangle =_a \langle_{-w}, e\rangle$. In particular, $\langle_{-v}, e\rangle^{-1}(\{z\}) =_a \langle_{-w}, e\rangle^{-1}(\{z\})$, and the latter set is easily seen to be a right H -set. This completes the proof of (8.1.1). \square

We shall next prove

$$(8.1.2) \quad G\text{-substabs}(V) \supseteq \text{Almosts}(\mathcal{F}).$$

Proof of (8.1.2). Consider any $H \in \text{Almosts}(\mathcal{F})$. It suffices to construct some $w \in \text{Maps}(E, Z)$ such that $w =_a v$ and H stabilizes w .

Set $E_H := \{e \in E : \langle_{-v}, e\rangle|_H \text{ is not constant}\}$, an H -subset of E_G . Thus, for $h \in H$ and $e \in E - E_H$, we have $\langle hv, e\rangle = \langle v, e\rangle$, and we see that H stabilizes $\langle v, -\rangle|_{E - E_H}$.

Consider any $e \in E_G$. We saw above that $\langle_{-v}, e\rangle$ takes only finitely many values in Z , and we are assuming that, for each $z \in Z$, $\langle_{-v}, e\rangle^{-1}(\{z\})$ is almost equal to a right H -subset of G . Hence, $\langle_{-v}, e\rangle|_H$ is almost equal to a constant map, and, also, for all but finitely many g in a right H -transversal in G , $\langle_{-v}, e\rangle|_{gH}$ is constant; here, $\langle g_{-v}, e\rangle|_H$ is constant, $g^{-1}\langle g_{-v}, e\rangle|_H$ is constant, $\langle_{-v}, g^{-1}e\rangle|_H$ is constant, $g^{-1}e \in E - E_H$, and $Hg^{-1}e \cap E_H = \emptyset$. It follows that $Ge \cap E_H$ is H -finite.

We also saw above that E_G is G -finite. It now follows that E_H is H -finite.

Let us deal first with the case where H is infinite. For each $e \in E$, as $\langle_{-v}, e\rangle|_H$ is almost equal to a constant map and H is infinite, there exists a unique $z_e \in Z$ such that, for all but finitely many $h \in H$, $\langle hv, e\rangle = z_e$. For each $e \in E$ and $h_0 \in H$, we see that, for all but finitely many $h \in H$, $\langle h_0hv, e\rangle = z_e$, and then

$$\langle hv, h_0^{-1}e\rangle = h_0^{-1}\langle h_0hv, e\rangle = h_0^{-1}z_e;$$

thus, $z_{h_0^{-1}e} = h_0^{-1}z_e$. Set $w : E \rightarrow Z$, $e \mapsto \langle w, e \rangle := z_e$; then H stabilizes w , since

$$\langle h_0 w, e \rangle = h_0 \langle w, h_0^{-1}e \rangle = h_0 z_{h_0^{-1}e} = z_e = \langle w, e \rangle.$$

For each $e \in E - E_H$, $\langle v, e \rangle = z_e = \langle w, e \rangle$; thus, $\langle v, - \rangle|_{E - E_H} = \langle w, - \rangle|_{E - E_H}$. For each $e \in E_H$, for all but finitely many $h \in H$, $\langle hv, e \rangle = z_e = \langle w, e \rangle$, and here

$$\langle v, h^{-1}e \rangle = h^{-1} \langle hv, e \rangle = h^{-1} \langle w, e \rangle = \langle h^{-1}w, h^{-1}e \rangle = \langle w, h^{-1}e \rangle.$$

Since E_H is H -finite, we see that $\langle v, - \rangle|_E =_a \langle w, - \rangle|_E$. Hence, $w \in V$ and $H \in G\text{-substabs}(V)$.

It remains to deal with the case where H is finite. Here, the H -finite set E_H is finite. Since $G\text{-substabs}(E) \subseteq G\text{-substabs}(Z)$ by hypothesis, there exists some G -stable $u \in \text{Maps}(E, Z)$. Define w to be the element of $\text{Maps}(E, Z)$ such that $\langle w, - \rangle|_{E - E_H} = \langle v, - \rangle|_{E - E_H}$ and $\langle w, - \rangle|_{E_H} = \langle u, - \rangle|_{E_H}$. Since H stabilizes both $\langle v, - \rangle|_{E - E_H}$ and $\langle u, - \rangle|_{E_H}$, we see that H stabilizes w . Since E_H is finite, $\langle w, - \rangle|_E =_a \langle v, - \rangle|_E$. Hence $w \in V$ and $H \in G\text{-substabs}(V)$.

This completes the proof of (8.1.2). \square

By combining (8.1.1) and (8.1.2), we find that

$$(8.1.3) \quad G\text{-substabs}(V) = \text{Almosts}(\mathcal{F}).$$

By Theorem 6.1, since \mathcal{F} is a G -finite G -subset of $\mathcal{B}(G)$, there exists some G -finite G -tree T_1 such that $E(T_1)$ is G -quasifree and

$$(8.1.4) \quad \text{Almosts}(\mathcal{F}) = G\text{-substabs}(V(T_1)).$$

By successively collapsing G -orbits of any G -compressible edges of T_1 , we arrive at a G -incompressible G -tree T_2 such that

$$(8.1.5) \quad G\text{-substabs}(V(T_1)) = G\text{-substabs}(V(T_2)).$$

In summary,

$$G\text{-substabs}(V) \stackrel{(8.1.3)}{=} \text{Almosts}(\mathcal{F}) \stackrel{(8.1.4)}{=} G\text{-substabs}(V(T_1)) \stackrel{(8.1.5)}{=} G\text{-substabs}(V(T_2)).$$

As $G\text{-substabs}(V) = G\text{-substabs}(V(T_2))$, there exist G -maps $\varphi : V \rightarrow V(T_2)$ and $\psi : V(T_2) \rightarrow V$. Since $V(T_2)$ is G -incompressible, the G -map $\varphi \circ \psi : V(T_2) \rightarrow V(T_2)$ must be bijective. Hence ψ is injective, and we may identify $V(T_2)$ with a G -subset of V , and T_2 with a G -subtree of the G -graph $\text{Complete}(V)$. The G -subgraph T_3 of $\text{Complete}(V)$ with vertex G -set V and edge G -set

$$E(T_3) := E(T_2) \cup \{(v, \varphi(v)) \mid v \in V - V(T_2)\}$$

is a maximal subtree of $\text{Complete}(V)$, as desired. \square

9. PRELIMINARY RESULTS ABOUT TREES

In the remainder of this article we shall describe some simplifications which may be made in the proof of the general case of the Almost Stability Theorem. We will not simplify the proofs of the preliminary results about trees. We collect together the statements of these here, for the convenience of the reader. The proofs currently known are rather technical and will not be given here.

9.1. Definitions. Let $\bar{T} = (\bar{V}, \bar{E}, \bar{\iota}, \bar{\tau})$ be any G -tree such that \bar{E} is G -quasifree. Let F be any G -forest with G -quasifree edge G -set such that the G -set of components of F is \bar{V} . Thus, $F = \bigvee_{w \in \bar{V}} T_w$, for each $w \in \bar{V}$, T_w is a G_w -tree with G_w -quasifree edge G_w -set, and, for each $g \in G$, $g(T_w) = T_{gw}$.

We shall now extend F to a G -graph $F \vee \bar{E}$ by adding \bar{E} to the edge G -set of F and extending the incidence maps ι and τ to \bar{E} as follows. Let S be any G -transversal in \bar{E} . Consider any $\bar{e} \in S$. Then $G_{\bar{e}}$ is a finite subgroup of $G_{\bar{\iota}(\bar{e})}$, and,

hence, $G_{\bar{e}}$ stabilizes some vertex of the $G_{\bar{e}}$ -tree $T_{\bar{e}}$. We take some $G_{\bar{e}}$ -stable vertex of $G_{\bar{e}}$ to be $\iota\bar{e}$. For each $g \in G$, we define $\iota(g\bar{e}) := g(\iota\bar{e})$, which is well-defined. This defines $\iota: \bar{E} \rightarrow V(F)$. We define $\tau: \bar{E} \rightarrow V(F)$ in a similar manner. This completes the definition of $F \vee \bar{E}$.

Collapsing the edges of the subforest F in $F \vee \bar{E}$ leaves the tree \bar{T} . It follows that $F \vee \bar{E}$ is a G -tree with G -quasifree edge G -set. We say that $F \vee \bar{E}$ is a G -tree obtained from \bar{T} by G -equivariantly blowing up each $w \in \bar{V}$ to T_w . \square

9.2. Definitions. Let T be any G -finite G -tree with G -quasifree edge G -set. For each $n \in [1 \uparrow \infty[$, set $E_n(T) := \{e \in E(T) : |G_e| = n\}$, and set

$$\text{size}(T) := |G \setminus E(T)| - |G \setminus V(T)| + \sum_{n \in [1 \uparrow \infty[} |G \setminus E_n(T)| \mathbf{t}^n \in \mathbb{Z}[\mathbf{t}]. \quad \square$$

9.3. Lemma. Let T be any G -tree with G -quasifree edge G -set, w be any vertex of T , and H be any subgroup of G_w . If $\text{rank}(G \text{ rel } H) < \omega_0$, then the following hold.

- (i) $\text{rank}(G_w \text{ rel } H) < \omega_0$.
- (ii) For each $v \in V(T) - Gw$, $\text{rank}(G_v) < \omega_0$.
- (iii) There exists some G -finite G -incompressible G -tree \bar{T} such $E(\bar{T})$ is G -quasifree and G -substabs($V(\bar{T})$) = G -substabs($V(T)$).

Proof. (i) and (ii) hold by [5, III.8.1], for example.

(iii). Let S be any finite subset of G such that $1 \in S$ and $H \cup S$ generates G . Let T_0 be any finite subtree of T containing Sw . Then $G(T_0)$ is a G -finite G -subforest of T . Moreover, $G(T_0)/E(G(T_0))$ consists of a single G -orbit in which the image of w is stabilized by $H \cup S$; that is, $G(T_0)$ has only one component. Thus, $G(T_0)$ is a G -finite G -subtree of T .

For each $v \in V(T)$, G_v stabilizes both v and $\{G(T_0)\}$. It follows that G_v stabilizes the (unique) vertex of $G(T_0)$ which is closest to v . Hence

$$G\text{-substabs}(V(G(T_0))) = G\text{-substabs}(V(T)).$$

Now successively collapsing G -orbits of G -compressible edges in $G(T_0)$ leaves a G -finite G -incompressible G -tree \bar{T} such that $E(\bar{T})$ is G -quasifree and

$$G\text{-substabs}(V(\bar{T})) = G\text{-substabs}(V(G(T_0))) = G\text{-substabs}(V(T));$$

see [5, III.7.2]. \square

9.4. Lemma. Let T_1 and T_2 be any G -finite, G -incompressible G -trees with G -quasifree edge G -sets. If $G\text{-substabs}(V(T_2)) \subseteq G\text{-substabs}(V(T_1))$, then the following hold.

- (i) $3|G \setminus T_2| \leq |G \setminus T_1|$.
- (ii) $\text{size}(T_2) \sqsubseteq \text{size}(T_1)$ in $\mathbb{Z}[\mathbf{t}]$.
- (iii) If $\text{size}(T_2) = \text{size}(T_1)$, then $G\text{-substabs}(V(T_2)) = G\text{-substabs}(V(T_1))$.

Proof. By [5, III.7.5], $|G \setminus E(T_1)| + |G \setminus V(T_1)| \geq |G \setminus V(T_2)|$ and (ii) and (iii) hold. By (ii), $|G \setminus E(T_1)| - |G \setminus V(T_1)| \geq |G \setminus E(T_2)| - |G \setminus V(T_2)|$. By multiplying the former inequality by 2 and adding the result to the latter inequality, we see that (i) holds. \square

10. NOTATION USED IN THE PROOF OF THE GENERAL CASE

Throughout the remainder of the article, the following will apply.

10.1. Notation. Let E and Z be any G -sets, and V be any G -stable almost equality class in the G -set $\text{Maps}(E, Z)$.

Suppose that there exists some G -stable element in $\text{Maps}(E, Z)$ and that E is G -quasifree.

By Remarks 1.3(i), the connected G -graph $\text{Complete}(V)$ has G -quasifree edge G -set.

Here V is nonempty. Let us fix $v_0 \in V$.

For any subset E' of E , we have $E = E' \vee (E - E')$, and we identify

$$\text{Maps}(E, Z) = \text{Maps}(E', Z) \times \text{Maps}(E - E', Z) = \bigvee_{w \in \text{Maps}(E - E', Z)} (\text{Maps}(E', Z) \times \{w\}),$$

where, for each $w \in \text{Maps}(E - E', Z)$, we write $\text{Maps}(E', Z) \times \{w\}$ for the fibre over w of the restriction map

$$\text{Maps}(E, Z) \rightarrow \text{Maps}(E - E', Z), \quad v \mapsto \langle v, - \rangle|_{E - E'}.$$

For each $v \in \text{Maps}(E, Z)$, we identify

$$\{v\} = \{\langle v, - \rangle|_{E'}\} \times \{\langle v, - \rangle|_{E - E'}\},$$

and by abuse of notation we shall write

$$v = \langle v, - \rangle|_{E'} \times \langle v, - \rangle|_{E - E'}.$$

We denote by $\pi_{E'}$ the self-map of $\text{Maps}(E, Z)$ defined by

$$\pi_{E'}(v) := \langle v, - \rangle|_{E'} \times \langle v_0, - \rangle|_{E - E'};$$

thus, the image of $\pi_{E'}$ equals the fibre over $\langle v_0, - \rangle|_{E - E'}$.

We denote by $\bar{V}(E')$ the image of V under the restriction/projection map $\text{Maps}(E, Z) \rightarrow \text{Maps}(E', Z)$. Then $\bar{V}(E')$ is a $G_{E'}$ -stable almost equality class in $\text{Maps}(E', Z)$. Similarly, we also have a $G_{E'}$ -stable almost equality class $\bar{V}(E - E')$ in $\text{Maps}(E - E', Z)$, and we have the identifications

$$V = \bar{V}(E') \times \bar{V}(E - E') = \bigvee_{w \in \bar{V}(E - E')} (\bar{V}(E') \times \{w\}).$$

We may construct V as $G_{E'}$ -set by blowing up each $w \in \bar{V}(E - E')$ to the $(G_{E'})_w$ -set $\bar{V}(E') \times \{w\} \subseteq V$. The restriction map $\text{Maps}(E, Z) \rightarrow \text{Maps}(E', Z)$ carries $\bar{V}(E') \times \{w\}$ bijectively to $\bar{V}(E')$ respecting the $(G_{E'})_w$ -action. For some purposes, we shall be able to identify $\bar{V}(E') \times \{w\}$ with the $(G_{E'})_w$ -stable almost equality class $\bar{V}(E')$ in $\text{Maps}(E', Z)$.

Set $w_0 := \langle v_0, - \rangle|_{E - E'} \in \bar{V}(E - E')$. We write

$$V(E') := \bar{V}(E') \times \{w_0\} = \{v \in V : \langle v, - \rangle|_{E - E'} = \langle v_0, - \rangle|_{E - E'}\},$$

the fibre over w_0 . Then $v_0 \in V(E') \subseteq V$. Also, $\pi_{E'}$ maps V to $V(E')$ fixing each element of $V(E')$ and respecting the $(G_{E'})_{w_0}$ -action.

Consider any subgroup H of G .

We define $E_H := \{e \in E : \langle \cdot, v_0, e \rangle|_H \text{ is not constant}\}$, an H -subset of E . Notice that H stabilizes $\langle v_0, - \rangle|_{E - E_H}$, and E_H is the smallest H -subset of E with this property. Also,

$$V(E_H) = \{v \in V : \langle v, - \rangle|_{E - E_H} = \langle v_0, - \rangle|_{E - E_H}\} = \bar{V}(E_H) \times \{\langle v_0, - \rangle|_{E - E_H}\},$$

and $V(E_H)$ is an H -subset of V that is isomorphic to the H -stable almost equality class $\bar{V}(E_H)$ in $\text{Maps}(E_H, Z)$.

We wish to show that some maximal subtree of $\text{Complete}(V)$ is G -stable. It suffices to show there exists some G -subtree T_G of $\text{Complete}(V)$ with vertex G -set $V(E_G)$, for then V itself is the vertex G -set of the G -subtree of $\text{Complete}(V)$ with edge G -set $E(T_G) \cup \{(v, \pi_{E_G}(v)) \mid v \in V - V(E_G)\}$.

Let \mathbf{ast} denote the class consisting of all those groups for which the Almost Stability Theorem 1.2 holds. \square

To show that $G \in \mathbf{ast}$, we may assume that Notation 10.1 holds, and it suffices to show there exists some G -subtree of $\text{Complete}(V)$ with vertex G -set $V(E_G)$.

11. FINITELY GENERABLE EXTENSIONS

The following result is a modified version of [5, III.7.6] differing mainly in the additional hypothesis that $G \in \mathbf{ast}$. The important points are that this weaker form now suffices for our purposes and the proof is simplified in two places by the additional assumption.

11.1. Theorem. *Let Notation 10.1 hold, and suppose that the following hold: $G \in \mathbf{ast}$; $\text{rank}(G \text{ rel } H) < \omega_0$; for each $g \in G-H$, $gE_H \cap E_H = \emptyset$; and, there exists some H -subtree T_H of $\text{Complete}(V)$ with vertex H -set $V(E_H)$. Then there exists some G -subtree T_G of $\text{Complete}(V)$ with vertex G -set $V(E_G)$ such that $T_H \subseteq T_G$.*

Proof. Let S be a finite subset of G such that $H \cup S$ generates G .

Set $V_\infty := \{v \in V(E_G) - G(V(E_H)) : G_v \text{ is infinite}\}$. For the moment, let W be any finite subset of V_∞ . In (11.1.6), we shall see that V_∞ is G -finite, and then take W to be a G -transversal in V_∞ .

It is not difficult to show that, for each $g \in G-H$, π_{E_H} sends every element of $gV(E_H)$ to the single point $\langle gv, - \rangle|_{E_H} \times \langle v, - \rangle|_{E-E_H}$. Recall that π_{E_H} fixes each element of $V(E_H)$. We may then use the set map π_{E_H} to construct a graph map $G(T_H) \rightarrow T_H$ which collapses each edge in $(G-H)(T_H)$ and acts as the identity map on T_H . Thus, whenever two vertices of T_H are joined by a $(G-H)(T_H)$ -path, the two vertices must be equal. It then follows that $G(T_H)$ is a G -subforest of $\text{Complete}(V)$. Set $Y := G(W) \vee G(T_H)$, also a G -subforest of $\text{Complete}(V)$.

For any subset E' of E , we have the restriction map $V(Y) \rightarrow \text{Maps}(E', Z)$, $v \mapsto \langle v, - \rangle|_{E'}$.

The map $V(Y) \rightarrow \mathcal{P}(\mathcal{P}(V(Y)))$, $v \mapsto v^{**} := \{\varepsilon \in \mathcal{P}(V(Y)) \mid v \in \varepsilon\}$, will be identified with the map $V(Y) \rightarrow \text{Maps}(\mathcal{P}(V(Y)), \mathbb{Z}_2)$, $v \mapsto \langle v, - \rangle|_{\mathcal{P}(V(Y))}$, where, for each $(v, \varepsilon) \in V(Y) \times \mathcal{P}(V(Y))$, we set

$$\langle v, \varepsilon \rangle := \begin{cases} 1 \in \mathbb{Z}_2 & \text{if } v \in \varepsilon, \\ 0 \in \mathbb{Z}_2 & \text{if } v \in \varepsilon^c := V(Y) - \varepsilon. \end{cases}$$

For each subset \mathcal{E} of $\mathcal{P}(V(Y))$, we have the restriction map $V(Y) \rightarrow \text{Maps}(\mathcal{E}, \mathbb{Z}_2)$, $v \mapsto \langle v, - \rangle|_{\mathcal{E}}$.

For each $e \in E(T_H)$, we set

$$(11.1.1) \quad e^{**} := \{v \in V(Y) \mid \text{the reduced } T_H\text{-path from } \pi_{E_H}(v) \text{ to } \tau_{T_H}(e) \text{ crosses } e\}.$$

Here, $\iota_{T_H}(e) \in e^{**}$. For each $g \in G-H$, $gV(E_H)$ is mapped to a single point in $V(E_H)$ by π_{E_H} . It follows that $\delta_Y(e^{**}) = \{e\}$. For each $h \in H$, we have $(he)^{**} = h(e^{**})$. For each $g \in G-H$, we write $(ge)^{**} := g(e^{**})$, and this is well-defined. For each subgraph Y' of Y , we define $\mathcal{E}(Y') := \{e^{**} \cap V(Y') \mid e \in E(Y')\}$. For each $e \in E(Y)$, $\delta_Y(e^{**}) = \{e\}$. It follows that $\mathcal{E}(Y')$ is isomorphic as G -set to $E(Y')$.

We shall now see the following.

$$(11.1.2) \quad \mathcal{E}(Y) \text{ is finitely separating for } V(Y).$$

Proof of (11.1.2). Consider $v, w \in V(Y)$. We shall show $\langle v, - \rangle|_{\mathcal{E}(Y)} =_a \langle w, - \rangle|_{\mathcal{E}(Y)}$.

For all but finitely many g in a right H -transversal in G , $\langle v, - \rangle|_{gE_H} = \langle w, - \rangle|_{gE_H}$, and here $\langle v, g_- \rangle|_{E_H} = \langle w, g_- \rangle|_{E_H}$, hence $g^{-1}\langle v, g_- \rangle|_{E_H} = g^{-1}\langle w, g_- \rangle|_{E_H}$, hence $\langle g^{-1}v, - \rangle|_{E_H} = \langle g^{-1}w, - \rangle|_{E_H}$, hence $\langle g^{-1}v, - \rangle|_{\mathcal{E}(T_H)} = \langle g^{-1}w, - \rangle|_{\mathcal{E}(T_H)}$ by (11.1.1), hence $g\langle g^{-1}v, - \rangle|_{\mathcal{E}(T_H)} = g\langle g^{-1}w, - \rangle|_{\mathcal{E}(T_H)}$, hence $\langle v, g_- \rangle|_{\mathcal{E}(T_H)} = \langle w, g_- \rangle|_{\mathcal{E}(T_H)}$, and hence $\langle v, - \rangle|_{g\mathcal{E}(T_H)} = \langle w, - \rangle|_{g\mathcal{E}(T_H)}$.

For all $g \in G$, $\langle g^{-1}v, - \rangle|_{\mathcal{E}(T_H)}$ and $\langle g^{-1}w, - \rangle|_{\mathcal{E}(T_H)}$ differ only on the elements of $\mathcal{E}(T_H)$ corresponding to the elements of $E(T_H)$ crossed by the reduced T_H -path from $\pi_{E_H}(g^{-1}v)$ to $\pi_{E_H}(g^{-1}w)$, hence $\langle g^{-1}v, - \rangle|_{\mathcal{E}(T_H)} =_a \langle g^{-1}w, - \rangle|_{\mathcal{E}(T_H)}$, hence

$g\langle g^{-1}v, - \rangle|_{\mathcal{E}(T_H)} =_a g\langle g^{-1}w, - \rangle|_{\mathcal{E}(T_H)}$, hence $\langle v, g_- \rangle|_{\mathcal{E}(T_H)} =_a \langle w, g_- \rangle|_{\mathcal{E}(T_H)}$, and hence $\langle v, - \rangle|_{g\mathcal{E}(T_H)} =_a \langle w, - \rangle|_{g\mathcal{E}(T_H)}$.

This completes the proof of (11.1.2). \square

Consider any $e, f \in E(Y)$ with $e \neq f$. Since $\delta_Y(e^{**}) = \{e\}$ and $\delta_Y(f^{**}) = \{f\}$, it follows that exactly one of the four sets

$$e^{**} \cap f^{**}, e^{**} \cap f^{**c}, e^{**c} \cap f^{**}, e^{**c} \cap f^{**c}$$

has empty coboundary in Y ; we denote that set by $r_{e,f}$. Thus $r_{e,f}$ is the vertex set of a union of components of Y ; also, e^{**} and f^{**} are nested in $V(Y)$ if and only if $r_{e,f} = \emptyset$.

Set $\mathcal{R} := \{r_{e,f} \mid e, f \in E(Y), e \neq f\} - \{\emptyset\}$.

We shall now prove the following crucial facts.

(11.1.3) \mathcal{R} is G -quasifree.

(11.1.4) \mathcal{R} is finitely separating for $V(Y)$.

Proof of (11.1.3) and (11.1.4). We form a G -subgraph X of $\text{Complete}(V)$ by adding to Y a G -finite G -set of edges that will be specified. We begin as follows.

Recall that $v_0 \in V(E_H)$, that S is a finite subset of G such that $H \cup S$ generates G , and that W is a finite subset of V_∞ . We take as our first approximation

$$X := Y \cup \{(gv_0, gsv_0) \mid g \in G, s \in S - G_{v_0}\} \cup \{(gv_0, gw) \mid g \in G, w \in W\},$$

a G -subgraph of $\text{Complete}(V)$ obtained by adding to Y a G -finite G -set of edges. In $X/E(X)$, each element of T_H is identified with v_0 , and the image of v_0 is stabilized by $H \cup S$ and, hence, is stabilized by G ; also, each element of W is identified with v_0 , and each element of $G(W)$ is then identified with v_0 . Hence, $X/E(X)$ consists of a single G -orbit with a single point, and, therefore, X is connected.

We shall now show that $\mathcal{E}(Y) \subseteq \mathcal{B}(X)$. By (11.1.2), $\mathcal{E}(Y)$ is finitely separating for $V(Y)$ ($= V(X)$). Hence, for each edge (v, w) in $E(X)$, there exist only finitely many $e \in E(Y)$ such that $(v, w) \in \delta_X(e^{**})$. Thus, for each edge (v, w) in $E(X) - E(Y)$, there exist only finitely many $e \in E(Y)$ such that $(v, w) \in \delta_X(e^{**})$. Hence, for each edge (v, w) in $E(X) - E(Y)$, and each $e \in E(Y)$, there exist only finitely many $g \in G$ such that $(v, w) \in \delta_X(ge^{**})$, or, equivalently, $g^{-1}(v, w) \in \delta_X(e^{**})$. Since $E(X) - E(Y)$ is G -finite, and $\delta_Y(e^{**}) = \{e\}$, we see that $\delta_X(e^{**})$ is finite. It follows that $\mathcal{E}(Y) \subseteq \mathcal{B}(X)$.

In particular, for each $e \in E(Y)$, $X - \delta_X(e^{**})$ has only a finite number of components.

Also, $\mathcal{R} \subseteq \langle \mathcal{E}(Y) \rangle_{\mathcal{B}} \subseteq \mathcal{B}(X)$. Since X is connected with G -quasifree edge G -set, while $\emptyset \notin \mathcal{R}$ and $V(X) \notin \mathcal{R}$, we see that each element of \mathcal{R} has nonempty, finite coboundary in X , and, hence, (11.1.3) holds.

We have seen that each edge in $E(X) - E(Y)$ lies in $\delta_X(e^{**})$ for only finitely many $e \in E(Y)$. Since $E(X) - E(Y)$ is G -finite, we see that the G -set

$$E' := \{e \in E(Y) \mid \delta_X(e^{**}) \cap (E(X) - E(Y)) \neq \emptyset\}$$

is G -finite. Notice that $E' = \{e \in E(Y) \mid \delta_X(e^{**}) \neq \{e\}\}$.

In particular, the G -subset of $E(Y)$ consisting of those $e \in E(Y)$ such that $X - \delta_X(e^{**})$ has more than two components is G -finite, and, for any such e , we may connect every component of $X - \delta_X(e^{**})$ to every other component using a finite set of edges of $\text{Complete}(V)$. Thus adding to X a suitable G -finite G -set of edges of $\text{Complete}(V)$ ensures that, for each $e \in E(Y)$, $X - \delta_X(e^{**})$ has exactly two components. This is the final form we want for X , and we may assume that this is the X that we had from the start, and the definition for E' now refers to the new X .

We next prove that \mathcal{R} is G -finite, and for this it suffices to prove the G -finiteness of the G -set consisting of all the pairs $(e, f) \in E(Y) \times E(Y)$ such that e^{**} and f^{**} are not nested for $V(X)$.

Consider any such (e, f) . In particular, $e \neq f$.

Consider first the case where $\delta_X(e^{**}) = \{e\}$. Since $X - \delta_X(f^{**})$ has two components, we see that $X - (\delta_X(f^{**}) \cup \delta_X(e^{**}))$ has at most three components, and, hence, e^{**} and f^{**} are nested for $V(X)$.

Thus, we may assume that e and f lie in the G -finite G -set E' , and then it remains to show that for a given e there are only finitely many possibilities for f . Let A be any finite, connected subgraph of X containing $\delta_X(e^{**})$. For any $f \in E'$, for all but finitely many $g \in G$, $\delta_X(f^{**}) \cap gA = \emptyset$, and then $\delta_X((g^{-1}f)^{**}) \cap A = \emptyset$. By the G -finiteness of E' , for all but finitely many $f \in E'$, $\delta_X(f^{**}) \cap A = \emptyset$. If $\delta_X(f^{**}) \cap A = \emptyset$, then the connected graph A lies entirely in either f^{**} or f^{**c} . Since $A \supseteq \delta_X(e^{**})$, then $f^{**c} \cap \delta_X(e^{**}) = \emptyset$ or $f^{**} \cap \delta_X(e^{**}) = \emptyset$, respectively. If $f^{**c} \cap \delta_X(e^{**}) = \emptyset$, then, since f^{**c} is the vertex set of a connected subgraph of X , f^{**c} lies entirely in either e^{**} or e^{**c} , and, hence, e^{**} and f^{**} are nested. A similar argument applies if $f^{**} \cap \delta_X(e^{**}) = \emptyset$.

Thus, \mathcal{R} is G -finite. For any edge (v, w) of X and any $r \in \mathcal{R}$, there exist only finitely many $g \in G$ such that $g(v, w) \in \delta_X(r)$, or, equivalently, $(v, w) \in \delta_X(g^{-1}r)$. By the G -finiteness of \mathcal{R} , $\langle v, - \rangle|_{\mathcal{R}} =_a \langle w, - \rangle|_{\mathcal{R}}$. Since X is connected, it follows that \mathcal{R} is finitely separating for $V(X) (= V(Y))$.

This completes the proof of (11.1.4). \square

By (11.1.3), \mathcal{R} is G -quasifree, and, by (11.1.4), the image of the map

$$V(Y) \rightarrow \text{Maps}(\mathcal{R}, \mathbb{Z}_2), \quad v \mapsto \langle v, - \rangle|_{\mathcal{R}},$$

lies in a G -stable almost equality class. Since $G \in \mathbf{ast}$, the latter G -stable almost equality class is then the vertex G -set of some G -tree with G -quasifree edge G -set, and we let T_{bottom} denote such a G -tree. Here we have a G -map

$$V(Y) \rightarrow V(T_{\text{bottom}}) \subseteq \text{Maps}(\mathcal{R}, \mathbb{Z}_2), \quad v \mapsto \langle v, - \rangle|_{\mathcal{R}}.$$

Consider any $u \in V(T_{\text{bottom}})$. Let $V_u \subseteq V(Y)$ denote the fibre over u . Since each element of \mathcal{R} is the vertex set of a union of components of Y , we see that V_u is the vertex set of some subforest Y_u of Y that is a union of components of Y . In particular, each component of Y lies entirely in a fibre, and we have a fibration of Y into unions of components.

We shall now see the following.

(11.1.5) *There exists some G_u -tree T_u and some G_u -graph map $Y_u \rightarrow T_u$ which is bijective on edges.*

Proof of (11.1.5). By (11.1.2), $\mathcal{E}(Y_u)$ is finitely separating for $V(Y_u)$.

We claim that $\mathcal{E}(Y_u)$ is nested for $V(Y_u)$. Consider any $e, f \in E(Y_u)$ with $e \neq f$. We shall show that $r_{e,f} \cap V(Y_u) = \emptyset$. This is clear if $r_{e,f} = \emptyset$. Thus we may suppose that $r_{e,f} \neq \emptyset$, and, hence $r_{e,f} \in \mathcal{R}$. For each $v \in V_u$, we see that $\langle v, - \rangle|_{\mathcal{R}} = \langle u, - \rangle|_{\mathcal{R}} = u$. In particular, $\langle v, r_{e,f} \rangle = \langle u, r_{e,f} \rangle$. In particular, $\langle \iota_Y(e), r_{e,f} \rangle = \langle u, r_{e,f} \rangle$. Now recall that $\langle \iota_Y(e), r_{e,f} \rangle = 0$. Hence $\langle v, r_{e,f} \rangle = \langle u, r_{e,f} \rangle = \langle \iota_Y(e), r_{e,f} \rangle = 0$. Thus $r_{e,f} \cap V(Y_u) = \emptyset$. This proves the claim.

Hence $\mathcal{E}(Y_u)$ is a nested, finitely separating, G_u -subset of $\mathcal{P}(V(Y_u))$. It follows from Corollary 4.4 that $T_u := U(\mathcal{E}(Y_u))$ is a G_u -tree with G_u -edge set $\mathcal{E}(Y_u) \simeq E(Y_u)$ and there exists a natural G_u -map $V(Y_u) \rightarrow V(T_u)$. For each edge e of Y_u , it follows from (11.1.1) that $(\langle \iota_Y(e), - \rangle|_{\mathcal{E}(Y)}) \nabla (\langle \tau_Y(e), - \rangle|_{\mathcal{E}(Y)}) = \{e^{**}\}$. It can then be seen that we have a G_u -graph map $Y_u \rightarrow T_u$ that is bijective on edges; it is surjective on vertices if Y_u is nonempty. This completes the proof of (11.1.5). \square

We now create a G -tree denoted T_{middle} by G -equivariantly blowing up each vertex u of T_{bottom} to the G_u -tree T_u . Then T_{middle} is a G -tree with G -quasifree edge G -set, and there is specified a G -graph map $Y \rightarrow T_{\text{middle}}$ which is injective on edges.

Since $G \in \mathbf{ast}$, there exists some G -tree T_G with vertex G -set $V(E_G)$ and G -quasifree edge G -set.

We now create a G -tree denoted T_{top} by G -equivariantly blowing up each vertex v of T_{middle} to the G_v -tree T_G using the incidence maps for Y to make each element of $E(Y) \subseteq E(T_{\text{middle}})$ incident to appropriate copies of elements of $V(Y) \subseteq V(T_G)$. Then T_{top} is a G -tree with G -quasifree edge G -set, T_{top} contains the G -forest Y as a G -subgraph, and there is specified a G -map $V(T_{\text{top}}) \rightarrow V(E_G)$ which is the identity map on $V(Y)$.

We now make some adjustments to T_{top} .

Recall that $Y = G(W) \vee G(T_H)$. We may choose a finite subtree T_0 of T_{top} which contains the finite set $\{v_0\} \cup Sv_0 \cup W$, and set $T^+ := G(T_0) \cup Y$. Then T^+ is a connected G -subgraph of T_{top} . Now T^+ is a G -tree with G -quasifree edge G -set, T^+ contains the G -forest Y as a G -subgraph, $T^+ - Y$ is G -finite, and there is specified a G -map $V(T^+) \rightarrow V(E_G)$ which is the identity map on $V(Y)$.

Then $T^+/E(Y)$ is a G -finite G -tree with G -quasifree edge G -set. While it remains possible, we successively collapse G -orbits of edges of T^+ which become G -compressible edges in $T^+/E(Y)$; we thus eventually obtain a quotient G -tree of T^+ , denoted T . Then T is a G -tree with G -quasifree edge G -set, T contains the G -forest Y as a G -subgraph, $T - Y$ is G -finite, $T/E(Y)$ is G -incompressible, and there is specified a G -map $V(T) \rightarrow V(E_G)$ which is the identity map on $V(Y)$.

Recall that $V_\infty := \{v \in V(E_G) - G(V(E_H)) : G_v \text{ is infinite}\}$ and that W is an arbitrary finite subset of V_∞ . We shall now prove the following.

$$(11.1.6) \quad V_\infty \text{ is } G\text{-finite.}$$

Proof of (11.1.6). Letting \bar{v}_0 denote the component of Y containing v_0 , we may write $Y/E(Y) = G(\bar{v}_0) \vee G(W)$.

Now $T/E(Y)$ is a G -finite, G -incompressible G -tree with G -quasifree edge G -set, $G(\bar{v}_0) \vee G(W) \subseteq V(T/E(Y))$, and there is specified a G -map

$$V(T/E(Y)) - G\bar{v}_0 \rightarrow V(E_G)$$

which is the identity map on $G(W)$.

Let W' be an arbitrary finite subset of V_∞ which contains a G -transversal in the intersection of V_∞ with the G -finite image of the G -map

$$V(T/E(Y)) - G\bar{v}_0 \rightarrow V(E_G).$$

Then $G(W') \supseteq G(W)$.

The entire foregoing argument applies with W' in place of W , and we get a G -finite, G -incompressible G -tree $T'/E(Y)$ with G -quasifree edge G -set, such that $G(\bar{v}_0) \vee G(W') \subseteq V(T'/E(Y))$ and there is specified a G -map

$$V(T'/E(Y)) - G\bar{v}_0 \rightarrow V(E_G)$$

which is the identity map on $G(W')$.

By the choice of W' , each infinite subgroup of G that stabilizes an element of $V(T'/E(Y))$ stabilizes an element of $G\bar{v}_0 \vee G(W') \subseteq V(T'/E(Y))$. Each finite subgroup of G stabilizes an element of $V(T'/E(Y))$. Hence

$$G\text{-substabs}(V(T'/E(Y))) \subseteq G\text{-substabs}(V(T'/E(Y))).$$

By Lemma 9.4(i), $3|G \setminus (T'/E(Y))| \geq |G \setminus (T/E(Y))|$. Since $V(T'/E(Y)) \supseteq G(W')$, we see that $3|G \setminus (T'/E(Y))| \geq |G \setminus G(W')|$. Thus, we have a finite upper bound on the number of G -orbits in V_∞ . This completes the proof of (11.1.6). \square

By (11.1.6), we may assume that W is taken to be a G -transversal in V_∞ from the start. Then $V_\infty = G(W) \subseteq V(T)$, and any infinite subgroup of G which stabilizes an element of $V(E_G)$ stabilizes an element of $V(T)$. Each finite subgroup of G stabilizes an element of $V(T)$. Thus,

$$G\text{-substabs}(V(E_G)) \subseteq G\text{-substabs}(V(T)),$$

and, hence, there exists a G -map $\varphi: V(E_G) \rightarrow V(T)$ which is the identity on $V(E_H)$. We already have a G -map $\psi: V(T) \rightarrow V(E_G)$ which is the identity on $V(E_H)$. Since $T/E(Y)$ is G -incompressible, the composite $V(T) \xrightarrow{\psi} V(E_G) \xrightarrow{\varphi} V(T)$ is bijective, and we may identify $V(T)$ with a G -subset of $V(E_G)$ respecting the embeddings of $V(E_H)$ in $V(T)$ and $V(E_G)$. We may then expand T to a G -subtree T_G of $\text{Complete}(V)$ with vertex G -set $V(E_G)$ and edge G -set

$$E(T) \cup \{(v, \varphi(v)) \mid v \in V(E_G) - V(T)\}.$$

This completes the proof of Theorem 11.1. \square

12. COUNTABLY GENERABLE EXTENSIONS

The following is [5, III.8.3].

12.1. Lemma. *Let Notation 10.1 hold. If $\text{rank}(G \text{ rel } H) < \omega_0$, then $E_G - G(E_H)$ is G -finite.*

Proof. Let S be a finite subset of G such that $H \cup S$ generates G , and set $F := \bigcup_{s \in S} (v_0 \nabla (sv_0))$. For each s in the finite set S , $sv_0 =_a v_0$, and, hence, F is a finite subset of E_G . Set $E' := G(E_H \cup F)$. Then E' is a G -subset of E_G . Also, $\langle v_0, - \rangle|_{E-E'}$ is stabilized by each $g \in S \cup H$, and, hence, is stabilized by G . Thus $E_G \subseteq E'$, and then $E_G - G(E_H) \subseteq G(F)$ and $E_G - G(E_H)$ is G -finite, as desired. \square

The following is part of the proof of [5, III.8.5].

12.2. Proposition. *Let Notation 10.1 hold.*

Suppose that, for each $g \in G - H$, $gE_H \cap E_H = \emptyset$.

Suppose that $H \leq K \leq G$ and $\text{rank}(K \text{ rel } H) < \omega_0$.

Then there exists some L such that $H \leq K \leq L \leq G$, $\text{rank}(L \text{ rel } H) \leq \omega_0$, and, for each $g \in G - L$, $gE_L \cap E_L = \emptyset$.

Proof. We recursively construct an ascending sequence $L_{[0 \uparrow \infty[}$ of subgroups of G such that, for each $n \in [0 \uparrow \infty[$, the following hold.

- (1) $\text{rank}(L_n \text{ rel } H) < \omega_0$.
- (2) $\{g \in G \mid gE_{L_n} \cap E_{L_n} \neq \emptyset\} \subseteq L_{n+1}$.

We set $L_0 := K$. Here (1) holds with $n = 0$.

Suppose that we are given some $m \in [0 \uparrow \infty[$ and L_m satisfying (1) with $n = m$. Let S_H be an H -transversal in E_H . Let S_m be an L_m -transversal in $E_{L_m} - L_m(E_H)$. Set $F := \{g \in G \mid gS_m \cap (S_H \cup S_m) \neq \emptyset\}$. Set $L_{m+1} := \langle L_m \cup F \rangle \leq G$.

Since, for all $g \in G - H$, $gE_H \cap E_H = \emptyset$, we see that S_H is a G -transversal in $G(E_H)$. By Lemma 12.1, S_m is finite. Hence $G(S_m) \cap S_H$ is finite. Recall that E is G -quasifree. Hence F is finite. Thus (1) holds with $n = m + 1$.

Consider any $g \in G$ such that $gE_{L_m} \cap E_{L_m} \neq \emptyset$. We wish to show that $g \in L_{m+1}$. Notice that $S_m \cup S_H$ is an L_m -transversal in E_{L_m} . Hence, on replacing g with an element of $L_m g L_m$, we may assume that $g(S_m \cup S_H) \cap (S_m \cup S_H) \neq \emptyset$. If $gS_m \cap (S_m \cup S_H) \neq \emptyset$, then $g \in F \subseteq L_{m+1}$. If $gS_H \cap S_m \neq \emptyset$, then $g \in F^{-1} \subseteq L_{m+1}$. If $gS_H \cap S_H \neq \emptyset$, then $gE_H \cap E_H \neq \emptyset$ and $g \in H \leq L_{m+1}$. Thus (2) holds with $n = m$.

This completes the recursive construction of $L_{[0 \uparrow \infty[}$.

Set $L := \bigcup_{n \in [0 \uparrow \infty[} L_n$.

Then $K = L_0 \leq L$.

By (1), $\text{rank}(L \text{ rel } H) \leq \omega_0$.

We have $E_L = \bigcup_{n \in [0 \uparrow \infty[} E_{L_n}$. For any $g \in G$ such that $gE_L \cap E_L \neq \emptyset$, there exist $m, n \in [0 \uparrow \infty[$ such that $gE_{L_m} \cap E_{L_n} \neq \emptyset$, and then, by (2), $g \in L_{\max\{m,n\}+1} \leq L$. Thus, L has all the desired properties. \square

In the remainder of the section we build a corresponding tree T_L .

The following is a modification of [5, III.8.2].

12.3. Lemma. *Let Notation 10.1 hold, and suppose that $H \leq K \leq G$.*

Suppose that $\text{rank}(G \text{ rel } H) < \omega_0$, and that, for each $g \in G - H$, $gE_H \cap E_H = \emptyset$.

Suppose that $\text{rank}(K \text{ rel } H) < \omega_0$.

Suppose that, for each L with $K \leq L \leq G$ and $\text{rank}(L \text{ rel } K) < \omega_0$,

- (a) $L \in \mathbf{ast}$, and,
- (b) if $LE_K = E_L$, then, for all $g \in L - K$, $gE_K \cap E_K = \emptyset$.

Suppose that T_H is an H -subtree of $\text{Complete}(V)$ with vertex H -set $V(E_H)$.

Suppose that T_K is a K -subtree of $\text{Complete}(V)$ with vertex K -set $V(E_K)$ such that $T_H \subseteq T_K$.

Then there exists some G -subtree T_G of $\text{Complete}(V)$ with vertex G -set $V(E_G)$ such that $T_K \subseteq T_G$.

Proof. We recursively define a descending sequence $G_{[0 \uparrow \infty[}$ of subgroups of G containing K as follows. We set $G_0 := G$, and, given $n \in [0 \uparrow \infty[$ and G_n , we define G_{n+1} to be the G_n -stabilizer of $\langle v_{0,-} \rangle|_{E-G_n(E_K)} \in \text{Maps}(E-G_n(E_K), Z)$.

We set $E_0 = E_G$ and, for each $n \in [0 \uparrow \infty[$, we set $E_{n+1} = G_n(E_K)$. Then $E_{[0 \uparrow \infty[}$ is a descending sequence of subsets of E_G containing E_K .

Consider $n \in [0 \uparrow \infty[$. Then $E_{n+1} = G_n(E_K) \subseteq E_{G_n}$. It may be shown that

$$G_n = \{g \in G : \langle gv_{0,-} \rangle|_{E-E_n} = \langle v_{0,-} \rangle|_{E-E_n}\},$$

and then that $E_{G_n} \subseteq E_n$. For each $g \in G - G_n$, $\langle gv_{0,-} \rangle|_{E-E_n} \neq \langle v_{0,-} \rangle|_{E-E_n}$, and, hence, $gV(E_n) \cap V(E_n) = \emptyset$, since $V(E_n) = \bar{V}(E_n) \times \{\langle v_{0,-} \rangle|_{E-E_n}\}$. In particular, $G_{V(E_n)} = G_n$.

We shall now show the following.

- (12.3.1) *For each $n \in [0 \uparrow \infty[$, the chain of subsets $V(E_H) \subseteq V(E_n) \subseteq V(E_G) \subseteq V$ extends to a chain of subgraphs $T_H \subseteq T_{E_n} \subseteq T^{(n)} \subseteq \text{Complete}(V)$ such that $T^{(n)}$ is a G -tree with vertex G -set $V(E_G)$, and T_{E_n} is a G_n -tree with vertex G_n -set $V(E_n)$.*

Proof of (12.3.1). Notice that $V(E_0) = V(E_G)$. By (a), $G \in \mathbf{ast}$. By Theorem 11.1, there exists some G -subtree of $\text{Complete}(V)$ with vertex G -set $V(E_G)$ containing T_H . Here we have the desired conditions for $n = 0$.

Suppose then that we are given $n \in [0 \uparrow \infty[$ and $T^{(n)}$ and T_{E_n} . Notice that $G_{T_{E_n}} = G_n$.

We have

$$\begin{aligned} V(E_n) &= \bar{V}(E_n) \times \{\langle v_{0,-} \rangle|_{E-E_n}\} = \bar{V}(E_{n+1}) \times \bar{V}(E_n - E_{n+1}) \times \{\langle v_{0,-} \rangle|_{E-E_n}\} \\ &= \bigvee_{w \in \bar{V}(E_n - E_{n+1}) \times \{\langle v_{0,-} \rangle|_{E-E_n}\}} (\bar{V}(E_{n+1}) \times \{w\}). \end{aligned}$$

Now $\bar{V}(E_n - E_{n+1})$ is a G_n -stable almost equality class in $\text{Maps}(E_n - E_{n+1}, Z)$. By (a), $G_n \in \mathbf{ast}$. Hence, there exists some G_n -tree with vertex G_n -set $\bar{V}(E_n - E_{n+1})$

and G_n -quasifree edge G_n -set. Equivalently, there exists some G_n -tree \bar{T} with vertex G_n -set $\bar{V}(E_n - E_{n+1}) \times \{\langle v_{0,-} \rangle|_{E-E_n}\}$ and with G_n -quasifree edge G_n -set.

Let $w_0 := \langle v_{0,-} \rangle|_{E-E_{n+1}} \in \bar{V}(E-E_{n+1})$.

We now take $w \in V(\bar{T}) = \bar{V}(E_n - E_{n+1}) \times \{\langle v_{0,-} \rangle|_{E-E_n}\}$ and consider two cases, where in Case 1 $w \notin G_n(w_0)$ and in Case 2 $w = w_0$, and here $(G_n)_{w_0} = G_{n+1} \geq K$.

By Lemma 9.3, in Case 1, $\text{rank}((G_n)_w) < \omega_0$, while in Case 2, we have $\text{rank}(G_{n+1} \text{ rel } K) < \omega_0$.

In Case 1, by Theorem 8.1, $(G_n)_w \in \mathbf{ast}$, and there then exists some $(G_n)_w$ -tree with $(G_n)_w$ -quasifree edge $(G_n)_w$ -set and vertex $(G_n)_w$ -set $\bar{V}(E_{n+1})$. Equivalently, there exists some $(G_n)_w$ -tree T_w with $(G_n)_w$ -quasifree edge $(G_n)_w$ -set and vertex $(G_n)_w$ -set $\bar{V}(E_{n+1}) \times \{w\}$.

In Case 2, by (a), $G_{n+1} \in \mathbf{ast}$. By Theorem 11.1, there exists some G_{n+1} -subtree $T_{w_0}^-$ of $\text{Complete}(V)$ with vertex G_{n+1} -set $V(E_{G_{n+1}})$ such that $T_H \subseteq T_{w_0}^-$. Then $T_{w_0}^-$ can be extended to some G_{n+1} -subtree, denoted T_{w_0} and $T_{E_{n+1}}$, of $\text{Complete}(V)$ with vertex G_{n+1} -set $V(E_{n+1}) = \bar{V}(E_{n+1}) \times \{w_0\}$.

We now G_n -equivariantly blow up each vertex w of \bar{T} to T_w and get a G_n -tree T with G_n -vertex set $V(E_n)$ having a G_{n+1} -subtree $T_{E_{n+1}}$ with vertex G_{n+1} -set $V(E_{n+1})$ such that $T_H \subseteq T_{E_{n+1}}$.

The G -tree $T^{(n)}$ has a G_n -subtree T_{E_n} with vertex G_n -set $V(E_n) = V(T)$. We now build $T^{(n+1)}$ from $T^{(n)}$ by G -equivariantly removing the edges in T_{E_n} and replacing them with the edges of the new G_n -tree T , which has the same vertex G_n -set as T_{E_n} . This completes the proof of (12.3.1). \square

We next show the following.

(12.3.2) *The descending sequence of subgroups $G_{[0\uparrow\infty[}$ is eventually constant.*

Proof of (12.3.2). Here we may assume that all the terms of $G_{[0\uparrow\infty[}$ are infinite subgroups.

The G -set $V(T^{(0)}/E(G(T_H)))$ is obtained from $V(E_G)$ by identifying all the elements of $gV(E_H)$ with each other, for each $g \in G$. Since $\text{rank}(G \text{ rel } H) < \omega_0$, by Lemma 9.3(iii) there exists some G -finite G -incompressible G -tree $\bar{T}^{(\omega_0)}$ such that $E(\bar{T}^{(\omega_0)})$ is G -quasifree and

$$G\text{-substabs}(V(\bar{T}^{(\omega_0)})) = G\text{-substabs}(V(T^{(0)}/E(G(T_H)))).$$

Consider any $n \in [0\uparrow\infty[$.

The G -set $V(T^{(n)}/E(G(T_{E_n})))$ is obtained from $V(E_G)$ by identifying all the elements of $gV(E_n)$ with each other, for each $g \in G$. Since $\text{rank}(G \text{ rel } H) < \omega_0$, by Lemma 9.3(iii) there exists some G -finite G -incompressible G -tree $\bar{T}^{(n)}$ such that $E(\bar{T}^{(n)})$ is G -quasifree and

$$G\text{-substabs}(V(\bar{T}^{(n)})) = G\text{-substabs}(V(T^{(n)}/E(G(T_{E_n}))).$$

Since $V(E_H) \subseteq V(E_n)$, we see that there exists a natural G -map

$$V(T^{(0)}/E(G(T_H))) \rightarrow V(T^{(n)}/E(G(T_{E_n}))).$$

Hence,

$$G\text{-substabs}(V(T^{(0)}/E(G(T_H)))) \subseteq G\text{-substabs}(V(T^{(n)}/E(G(T_{E_n}))),$$

and this is equivalent to

$$G\text{-substabs}(V(\bar{T}^{(\omega_0)})) \subseteq G\text{-substabs}(V(\bar{T}^{(n)})).$$

By Lemma 9.4(i), $3|G \setminus \bar{T}^{(\omega_0)}| \geq |G \setminus \bar{T}^{(n)}|$, and we have now shown that, as n varies over $[0\uparrow\infty[$, $|G \setminus \bar{T}^{(n)}|$ has a finite bound. By definition, $\text{size}(\bar{T}^{(n)})$ is an element of

$\mathbb{Z}[\mathbf{t}]$ with non-negative coefficients, and we now see that the sum of the coefficients has a finite bound.

Since $V(E_{n+1}) \subseteq V(E_n)$, we see that there exists a natural G -map

$$V(T^{(n+1)}/E(G(T_{E_{n+1}}))) \rightarrow V(T^{(n)}/E(G(T_{E_n}))).$$

Hence,

$$G\text{-substabs}(V(T^{(n+1)}/E(G(T_{E_{n+1}})))) \subseteq G\text{-substabs}(V(T^{(n)}/E(G(T_{E_n})))),$$

and this is equivalent to $G\text{-substabs}(V(\overline{T}^{(n+1)})) \subseteq G\text{-substabs}(V(\overline{T}^{(n)}))$. Now, by Lemma 9.4(ii), we see that $\text{size}(\overline{T}^{(n+1)}) \supseteq \text{size}(\overline{T}^{(n)})$. Thus, $\text{size}(\overline{T}^{(n)})$ increases with $n \in [0\uparrow\infty[$. It is not difficult to use the foregoing boundedness restraint to show that there exists some $n \in [0\uparrow\infty[$ such that $\text{size}(\overline{T}^{(n+1)}) = \text{size}(\overline{T}^{(n)})$.

By Lemma 9.4(iii), $G\text{-substabs}(V(\overline{T}^{(n+1)})) = G\text{-substabs}(V(\overline{T}^{(n)}))$, and this is equivalent to

$$G\text{-substabs}(V(T^{(n+1)}/E(G(T_{E_{n+1}})))) = G\text{-substabs}(V(T^{(n)}/E(G(T_{E_n}))))).$$

Since G_{n+1} is infinite and the edge G -set of $T^{(n+1)}/E(G(T_{E_{n+1}}))$ is G -quasifree, we see that $T^{(n+1)}/E(G(T_{E_{n+1}}))$ has at most one G_{n+1} -stable vertex. Since G_{n+1} stabilizes the image of $V(E_{n+1})$, we see that the image of $V(E_{n+1})$ is the unique G_{n+1} -stable vertex of $T^{(n+1)}/E(G(T_{E_{n+1}}))$. Since $T^{(n)}/E(G(T_{E_n}))$ has a G_n -stable vertex, namely the image of $V(E_n)$, we see that G_n stabilizes some vertex of $T^{(n+1)}/E(G(T_{E_{n+1}}))$, and, as such a vertex is then G_{n+1} -stable, it must be the image of $V(E_{n+1})$. Thus, $G_n \leq G_{V(E_{n+1})} = G_{n+1}$. This completes the proof of (12.3.2). \square

By (12.3.2), there exists some $n \in [1\uparrow\infty[$ such that $G_n = G_{n-1}$. Hence

$$G_n E_K = E_{n+1} \subseteq E_{G_n} \subseteq E_n = G_{n-1} E_K = G_n E_K.$$

Thus $G_n E_K = E_{G_n} = E_n$. By (b), for all $g \in G_n - K$, $gE_K \cap E_K = \emptyset$. By (a) and Theorem 11.1, there exists a G_n -subtree T of $\text{Complete}(V)$ with vertex G_n -set $V(E_n)$ such that $T_K \subseteq T$.

We now build T_G from $T^{(n)}$ by G -equivariantly removing the edges in T_{E_n} and replacing them with the edges of the new G_n -tree T , which has the same vertex G_n -set as T_{E_n} . This completes the proof. \square

The following is a modification of [5, III.8.4].

12.4. Theorem. *Let Notation 10.1 hold.*

Suppose that $\text{rank}(G \text{ rel } H) \leq \omega_0$, and that, for each $g \in G - H$, $gE_H \cap E_H = \emptyset$.

Suppose that, for each subgroup K of G , if $H \leq K$ and $\text{rank}(K \text{ rel } H) < \omega_0$, then $K \in \mathbf{ast}$.

Suppose that T_H is some H -subtree of $\text{Complete}(V)$ with vertex H -set $V(E_H)$.

Then there exists some G -subtree T_G of $\text{Complete}(V)$ with vertex G -set $V(E_G)$ such that $T_H \subseteq T_G$.

Proof. Let $g_{[1\uparrow\infty[}$ be a sequence in G such that $H \cup g_{[1\uparrow\infty[}$ generates G .

We now recursively construct an ascending sequence of subgroups $G_{[0\uparrow\infty[}$ of G such that $G_0 = H$, and, for each $n \in [1\uparrow\infty[$, the following hold.

- (1) $g_n \in G_n$.
- (2) $\text{rank}(G_n \text{ rel } H) < \omega_0$.
- (3) Whenever $G_n \leq K \leq G$ and $\text{rank}(K \text{ rel } H) < \omega_0$ and $K(E_{G_n}) = E_K$, then, for each $k \in K - G_n$, $kE_{G_n} \cap E_{G_n} = \emptyset$.

Suppose that we are given $n \in [1\uparrow\infty[$ and G_{n-1} . Let \mathbf{K} denote the set of those subgroups K of G such that K contains $G_{n-1} \cup \{g_n\}$ and $\text{rank}(K \text{ rel } H) < \omega_0$. By Lemma 12.1, for each $K \in \mathbf{K}$, $E_K - K(E_H)$ is K -finite. Hence, $K \setminus (E_K - K(E_H))$ achieves a minimum value as K ranges over \mathbf{K} . We take G_n to be an element of \mathbf{K} where this minimum is achieved. Then $G_{n-1} \leq G_n$, $g_n \in G_n$, and $\text{rank}(G_n \text{ rel } H) < \omega_0$. Consider any subgroup K of G such that K contains G_n and $\text{rank}(K \text{ rel } H) < \omega_0$ and $K(E_{G_n}) = E_K$. Then $K \in \mathbf{K}$. By minimality for G_n ,

$$\begin{aligned} |G_n \setminus (E_{G_n} - G_n(E_H))| &\leq |K \setminus (E_K - K(E_H))| \\ &= |K \setminus (K(E_{G_n}) - (KG_n)(E_H))| \\ &\leq |K \setminus (K(E_{G_n} - G_n(E_H)))| \\ &\leq |G_n \setminus (E_{G_n} - G_n(E_H))|. \end{aligned}$$

We have equality throughout, and then

$$K(E_{G_n}) - (KG_n)(E_H) = K(E_{G_n} - G_n(E_H)),$$

and, for each $k \in K - G_n$, $k(E_{G_n} - G_n(E_H)) \cap (E_{G_n} - G_n(E_H)) = \emptyset$, while, by hypothesis, $kG_n(E_H) \cap G_n(E_H) = \emptyset$. Let S be a right G_n -transversal in K . Then

$$\begin{aligned} K(E_{G_n}) &= (K(E_{G_n}) - (KG_n)(E_H)) \vee (KG_n)(E_H) \\ &= \bigvee_{s \in S} s(E_{G_n} - G_n(E_H)) \vee \bigvee_{s \in S} sG_n(E_H) = \bigvee_{s \in S} sE_{G_n}. \end{aligned}$$

Hence, for each $k \in K - G_n$, $kE_{G_n} \cap E_{G_n} = \emptyset$. This completes the recursive construction of $G_{[0\uparrow\infty[}$.

We next recursively construct an ascending sequence $T_{[0\uparrow\infty[}$ of subtrees of $\text{Complete}(V)$ containing T_H such that, for each $n \in [0\uparrow\infty[$, T_n is a G_n -subtree of $\text{Complete}(V)$ with vertex G_n -set $V(E_{G_n})$.

We take $T_0 := T_H$. Suppose that we are given $n \in [0\uparrow\infty[$ and T_n . By Lemma 12.3, there exists some G_{n+1} -subtree of $\text{Complete}(V)$ with vertex G_{n+1} -set $V(E_{G_{n+1}})$ such that $T_n \subseteq T_{n+1}$. This completes the recursive construction of $T_{[0\uparrow\infty[}$.

We now take $T_G := \bigcup_{n \in [0\uparrow\infty[} T_n$. □

We shall use two different forms of this result.

12.5. Corollary. *Let Notation 10.1 hold.*

Suppose that $|G| \leq \omega_0$.

Then there exists some G -subtree T_G of $\text{Complete}(V)$ with vertex G -set $V(E_G)$.

Proof. We take $H = \{1\}$ in Theorem 12.4. Here, $E_H = \emptyset$ and, for each subgroup K of G , if $\text{rank}(K) < \omega_0$, then $K \in \mathbf{ast}$ by Theorem 8.1. □

12.6. Corollary. *Let Notation 10.1 hold.*

Suppose that $\text{rank}(G \text{ rel } H) \leq \omega_0$, and that, for each $g \in G - H$, $gE_H \cap E_H = \emptyset$.

Suppose that every subgroup of G lies in \mathbf{ast} .

Suppose that T_H is some H -subtree of $\text{Complete}(V)$ with vertex H -set $V(E_H)$.

Then there exists some G -subtree T_G of $\text{Complete}(V)$ with vertex G -set $V(E_G)$ such that $T_H \subseteq T_G$. □

13. THE PROOF

Proof of the Almost Stability Theorem 1.2. We may assume that Notation 10.1 holds and it suffices to show that there exists some G -subtree of $\text{Complete}(V)$ with vertex G -set $V(E_G)$.

By Corollary 12.5, we may assume that $\omega_0 < |G|$.

By transfinite induction, we may assume that, for each subgroup H of G , if $|H| < |G|$, then $H \in \mathbf{ast}$.

Set $\gamma := |G|$ and choose a bijective map $\gamma \rightarrow G$, $\beta \mapsto g_\beta$. We shall recursively construct an ascending chain of subgroups $(G_\beta \mid \beta \leq \gamma)$ of G and, at the same time, an ascending chain of subtrees $(T_\beta \mid \beta \leq \gamma)$ of $\text{Complete}(V)$. For each $\beta \leq \gamma$, we shall set $E_\beta := E_{G_\beta}$, $V_\beta := V(E_\beta)$, and the following will hold.

- (1) $\{g_\alpha \mid \alpha < \beta\} \subseteq G_\beta$.
- (2) $|G_\beta| \leq \max\{\omega_0, |\beta|\}$.
- (3) For each $g \in G - G_\beta$, $gE_\beta \cap E_\beta = \emptyset$.
- (4) $V(T_\beta) = V_\beta$ and $G_\beta E(T_\beta) = E(T_\beta)$.

Suppose that we are given some $\beta \leq \gamma$ and a chain of subgroups $(G_\alpha \mid \alpha < \beta)$ and a chain of subtrees $(T_\alpha \mid \alpha < \beta)$ satisfying (1)–(4) at each step.

Case 1. $\beta = 0$.

We define $G_0 := \{1\}$ and $T_0 = \{v_0\}$. Here $E_0 = \emptyset$, $V_0 = \{v_0\}$ and conditions (1)–(4) hold in Case 1.

Case 2. β is a successor ordinal.

By Proposition 12.2, there exists some subgroup G_β of G with the properties that $G_{\beta-1} \cup \{g_{\beta-1}\} \subseteq G_\beta$ and $\text{rank}(G_\beta \text{ rel } G_{\beta-1}) \leq \omega_0$ and, for each $g \in G - G_\beta$, $gE_\beta \cap E_\beta = \emptyset$. Hence, (3) holds.

Then $\{g_\alpha \mid \alpha < \beta\} = \{g_\alpha \mid \alpha < \beta-1\} \cup \{g_{\beta-1}\} \subseteq G_{\beta-1} \cup \{g_{\beta-1}\} \subseteq G_\beta$, and (1) holds.

Since $\text{rank}(G_\beta \text{ rel } G_{\beta-1}) \leq \omega_0$, we have $|G_\beta| \leq \max\{\omega_0, |G_{\beta-1}|\}$. Now $|G_\beta| \leq \max\{\omega_0, |G_{\beta-1}|\} \leq \max\{\omega_0, |\beta-1|\} \leq \max\{\omega_0, |\beta|\}$, and (2) holds.

Since $|\beta-1| \leq \beta-1 < \beta \leq \gamma$, we also have $|G_\beta| \leq \max\{\omega_0, |\beta-1|\} < \gamma$, and then every subgroup of G_β lies in \mathbf{ast} , by the transfinite induction hypothesis. By Corollary 12.6, there exists some G_β -subtree T_β of $\text{Complete}(V)$ with vertex G_β -set V_β such that the $T_{\beta-1} \subseteq T_\beta$. Hence, (4) holds.

Now conditions (1)–(4) hold in Case 2.

Case 3. β is a limit ordinal.

Here, we set $G_\beta := \bigcup_{\alpha < \beta} G_\alpha$ and $T_\beta := \bigcup_{\alpha < \beta} T_\alpha$.

Notice that $E_\beta = \bigcup_{\alpha < \beta} E_\alpha$ and $V_\beta = \bigcup_{\alpha < \beta} V_\alpha$. Hence (4) holds.

For each $\alpha < \beta$, we have $\alpha+1 < \beta$ and $g_\alpha \in G_{\alpha+1} \subseteq G_\beta$. Hence (1) holds.

Notice that $\omega_0 \leq |\beta|$. Thus

$$|G_\beta| = \left| \bigcup_{\alpha < \beta} G_\alpha \right| \leq \sum_{\alpha < \beta} |G_\alpha| \leq \sum_{\alpha < \beta} \max\{\omega_0, |\alpha|\} \leq \sum_{\alpha < \beta} |\beta| \leq |\beta|^2 = |\beta|;$$

see, for example, [12, Theorem 3.5] and [12, Lemma 5.2]. Hence (2) holds.

For each $g \in G$, if $gE_\beta \cap E_\beta \neq \emptyset$, then there exist $\alpha_1 < \beta$ and $\alpha_2 < \beta$ such that $gE_{\alpha_1} \cap E_{\alpha_2} \neq \emptyset$, and then $g \in G_{\max\{\alpha_1, \alpha_2\}} \subseteq G_\beta$. Hence (3) holds.

Thus conditions (1)–(4) hold in Case 3.

This completes the recursive construction.

By (1), $G_\gamma = G$. By (4), T_γ is a G -subtree of $\text{Complete}(V)$ with vertex G -set $V(E_G)$. This completes the proof. \square

14. ARBITRARY EXTENSIONS

With a similar argument, we get the relative version, [5, III.8.5].

14.1. Theorem. *Let Notation 10.1 hold.*

Suppose that, for each $g \in G-H$, $gE_H \cap E_H = \emptyset$, and that there exists some H -subtree T_H of $\text{Complete}(V)$ with vertex H -set $V(E_H)$. Then there exists some G -subtree T_G of $\text{Complete}(V)$ with vertex G -set $V(E_G)$ such that $T_H \subseteq T_G$.

Proof. Set $\gamma := |G|$, and choose a bijective map $\gamma \rightarrow G$, $\beta \mapsto g_\beta$. We shall recursively construct an ascending chain of subgroups $(G_\beta \mid \beta \leq \gamma)$ of G and, at the same time, an ascending chain of subtrees $(T_\beta \mid \beta \leq \gamma)$ of $\text{Complete}(V)$. For each $\beta \leq \gamma$, we shall write $E_\beta := E_{G_\beta}$, $V_\beta := V(E_\beta)$, and the following will hold.

- (1) $\{g_\alpha \mid \alpha < \beta\} \subseteq G_\beta$.
- (2) For each $g \in G - G_\beta$, $gE_\beta \cap E_\beta = \emptyset$.
- (3) $V(T_\beta) = V_\beta$ and $G_\beta E(T_\beta) = E(T_\beta)$.

Suppose that we are given some $\beta \leq \gamma$ and a chain of subgroups $(G_\alpha \mid \alpha < \beta)$ and a chain of subtrees $(T_\alpha \mid \alpha < \beta)$ satisfying (1)–(3) at each step.

Case 1. $\beta = 0$.

We define $G_0 := H$ and $T_0 = T_H$. Now conditions (1)–(3) hold in Case 1.

Case 2. β is a successor ordinal.

By Proposition 12.2, there exists some subgroup G_β of G with the properties that $G_{\beta-1} \cup \{g_{\beta-1}\} \subseteq G_\beta$ and $\text{rank}(G_\beta \text{ rel } G_{\beta-1}) \leq \omega_0$ and, for each $g \in G - G_\beta$, $gE_\beta \cap E_\beta = \emptyset$. Hence, (2) holds.

Then $\{g_\alpha \mid \alpha < \beta\} \subseteq G_{\beta-1} \cup \{g_{\beta-1}\} \subseteq G_\beta$, and (1) holds.

By Theorem 1.2, every subgroup of G_β lies in **ast**. By Corollary 12.6, there exists some G_β -subtree T_β of $\text{Complete}(V)$ with vertex G_β -set V_β such that the $T_{\beta-1} \subseteq T_\beta$. Hence, (3) holds.

Now conditions (1)–(3) hold in Case 2.

Case 3. β is a limit ordinal.

Here, we define $G_\beta := \bigcup_{\alpha < \beta} G_\alpha$ and $T_\beta := \bigcup_{\alpha < \beta} T_\alpha$.

Notice that $E_\beta = \bigcup_{\alpha < \beta} E_\alpha$ and $V_\beta = \bigcup_{\alpha < \beta} V_\alpha$. Hence (3) holds.

For each $\alpha < \beta$, we have $\alpha+1 < \beta$ and $g_\alpha \in G_{\alpha+1} \subseteq G_\beta$. Hence (1) holds.

For each $g \in G$, if $gE_\beta \cap E_\beta \neq \emptyset$, then there exist $\alpha_1 < \beta$ and $\alpha_2 < \beta$ such that $gE_{\alpha_1} \cap E_{\alpha_2} \neq \emptyset$, and then $g \in G_{\max\{\alpha_1, \alpha_2\}} \subseteq G_\beta$. Hence (2) holds.

Thus conditions (1)–(3) hold in Case 3.

This completes the recursive construction.

By (1), $G_\gamma = G$. By (3), T_γ is a G -subtree of $\text{Complete}(V)$ with vertex G -set $V(E_G)$. Since $T_0 = T_H$, this completes the proof. \square

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