THE GLOBAL DIMENSIONS OF MIXED COPRODUCT/TENSOR-PRODUCT ALGEBRAS

by George M. Bergman

If K is a field, it is known that the free associative algebra on $n > 0$ indeterminates, $K\langle X_1, \ldots, X_n \rangle$ has left global dimension 1, while if we impose the commutativity relations $X_i X_j = X_j X_i$ $(i, j \leq n)$ this becomes the commuting polynomial ring $K[X_1, \ldots, X_n]$, with global dimension n. ([6], [7], cf. [4].)

What happens if we introduce only some chosen subset of the above $n(n-1)/2$ commutativity relations? I shall show in this note that the left global dimension of the resulting algebra is as small as it can conceivably be, namely it is equal to the largest m such that some m of our indeterminates are all made to commute with each other; i.e., such that our algebra contains a commuting polynomial subalgebra $K[X_{i_1}, \ldots, X_{i_m}]$ $(i_1 < \ldots < i_m \leq n)$.

We shall get this result by induction on the number of indeterminates, showing that if R is such a ring, then <u>either</u> $R = K[X_1, \ldots, X_n]$, <u>or</u> R can be written in a nice way as a coproduct with amalgamation of smaller rings of the same sort. In this case, we apply a result of Dicks to bound the dimension of this coproduct in terms of the dimensions of the smaller rings.

Consider the following generalization of the above construction. Start with a family of K-algebras $R_1, \ldots, R_n$, form their coproduct as K-algebras, and then impose, for any subset of the $n(n-1)/2$ pairs of rings R_i, R_j, relations saying that all elements of R_i commute with all elements of R_j. The result is a kind of mixture of the coproduct and tensor product constructions on K-algebras. We shall use the same methods to prove a bound on the right

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global dimension of the resulting ring R, in terms of the global dimensions of the rings of the form $R_{i_1} \otimes \ldots \otimes R_{i_m}$ arising from families of indices $\{i_1, \ldots, i_m\}$ for which all commutativity conditions have been imposed. However, the formula we will get is not as simple as in the polynomial case, and it appears that it is not the best possible (§3).

I do not know whether these results remain true if infinitely many indeterminates X_i or algebras R_i are allowed. In §4 it is shown that they will be true whenever the graph describing the commutativity relations we impose is finitely colorable!

All rings and algebras are associative with 1.

Note: I would appreciate any references on global dimensions of tensor product algebras!

§1. Statement and proof of the main results (with one step deferred).

Let I be a set, and $A \subseteq I \times I$ a symmetric antireflexive relation. (I.e., $(i,j) \in A \Rightarrow (j,i) \in A$; (i,i) never in A). If K is a commutative ring and $(R_i)_{i \in I}$ a family of K-algebras, let us form the coproduct $\coprod_I R_i$ of the R_i in the category of K-algebras, and factor out the ideal generated by $\{ xy - yx \mid x \in R_i,\ y \in R_j,\ (i,j) \in A \}$. We shall denote the resulting ring by $*_I^A R_i$. Given $H \subseteq I$, we shall abbreviate $*_H^{A \cap (H \times H)} R_i$ to $*_H^A R_i$.

e ted $*_I^A R_i$

If U, V and W are rings, given with homomorphisms $U \to V$, $U \to W$, then V and W are called U-rings. In this situation, the coproduct of V and W in the category of U-rings will be denoted $V \coprod_U W$. This is also the pushout of the diagram $V \leftarrow U \rightarrow W$ in the category of rings, sometimes known as the "free product of V and W with amalgamation of U", or an "amalgamation of V and W." One easily sees the following:

Lemma 1. Let I, A, R_i be as in the first paragraph above, and suppose $I = G \cup H$, with $A \subseteq (G \times G) \cup (H \times H)$. Then

$$*_I^A R_i \cong (*_G^A R_i) \coprod_{*_{G \cap H}^A R_i} (*_H^A R_i). \|$$

We shall use the above observation in conjunction with a result of Warren Dicks, which says that if V and W are U-rings, such that $V \coprod_U W$ is flat as a right U, V and W-module, then

$\text{l.gl.dim.}(V \coprod_U W) \leq \max(\text{l.gl.dim. } V,\ \text{l.gl.dim. } W,\ 1 + \text{l.gl.dim. } U)$. Actually, Dicks' result [5, Cor. 7] concerns "colimits of trees of rings", and the above is the simplest nontrivial case, where the tree has the form $\overset{V}{\circ} \overset{U}{\text{———}} \overset{W}{\circ}$. We shall prove in §2 that if every R_i is free as a K-module, on a basis containing 1 (e.g., if R_i is a polynomial or group algebra over K, or if K is a field) then $*_I^A R_i$ is in fact free as a right module over every $*_H^A R_i$ $(H \subseteq I)$, so that Dicks' Theorem is applicable. In this section, we will study l.gl.dim. $*_I^A R$ assuming this result.

To help in visualization, let us note that a symmetric antireflexive relation $A \subseteq I \times I$ corresponds to an unoriented graph I_A with vertex-set I: we define I_A to have an edge connecting i and j $(i, j \in I)$ if and only if $(i,j) \in A$. If J is any subset of I, the subgraph $J_{A\cap(J \times J)} \subseteq I_A$ will be abbreviated J_A.

Recall that a <u>complete graph</u> is one with an edge connecting every pair of vertices. Thus, the <u>complete</u> subgraphs of I_A correspond to the subsets $J \subseteq I$ such that every two vertices in J are connected by an edge (in I_A).

Trivial observations: Any nonempty graph contains a complete 1-vertex subgraph (one vertex, no edges), and <u>any</u> graph contains a complete 0-vertex subgraph (the empty subgraph).

<u>Theorem 2.</u> Let K be a commutative ring, I a finite set, say of cardinality $n \geq 0$, $A \subseteq I \times I$ a symmetric antireflexive relation, and m the largest number of vertices in a complete subgraph of I_A.

Let $(X_i)_{i\in I}$ be indeterminates, and

$$R = K < X_i(i\in I) \mid X_iX_j = X_jX_i \ ((i,j)\in A) >$$

$$= *_I^A K[X_i].$$

Then

(1) $\quad$ $\text{l.gl.dim. } R = m + \text{gl.dim. } K.$

<u>Proof.</u> We suppose inductively that the Theorem is true for all K-algebras constructed in this manner on fewer than n indeterminates.

Now if I_A is the complete graph on the vertex-set I, then R is the commuting polynomial ring $K[X_i\ (i\in I)]$, and the result is classical [6]. (In particular, this observation covers the case $m = 0$, since this implies $I = \emptyset$.)

In the contrary case, we may choose a vertex $j \in I$ which is not connected with all other vertices. Let us write I as a disjoint union $\{j\} \cup P \cup Q$, where P is the set of vertices connected by an edge to j, and Q the set of vertices other than j which are not so connected. Q is nonempty by choice of j.

Since no edge connects a point of Q to j, we see that $A \subseteq (\{j\} \cup P)^2 \cup (P \cup Q)^2$. Hence by Lemma 1 and the result of Dicks' quoted, the left global dimension of R will be less than or equal to the maximum of the three numbers:

(2) $\qquad$ l.gl.dim. $*^A_{\{j\}\cup P}\, K[X_i]$,

(3) $\qquad$ l.gl.dim $*^A_{P\cup Q}\, K[X_i]$,

(4) $\qquad 1 +$ l.gl.dim $*^A_P\, K[X_i]$.

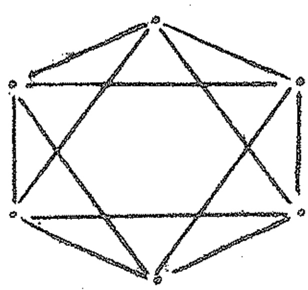
Now the sets $\{j\} \cup P$, $P \cup Q$ and P appearing in (2), (3), (4) each have fewer than n elements, so we can apply our inductive hypothesis to bound the left global dimensions of the algebras in question. We see that each of these dimensions is $\leq m +$ gl.dim. K. But in the case of (4), we need, and can get, a better estimate. I claim that every complete subgraph of P_A has strictly fewer than m vertices. Indeed, if P_A had a complete m-vertex subgraph, then since j is joined to each vertex of P, I_A would have a complete $m+1$-vertex subgraph, contradicting our choice of m. Hence the global dimension appearing in (4) is $\leq m-1 +$ gl.dim. K. (Note that this argument is valid even in the minimal case $m = 1$, where it implies that $P = \emptyset$, and the ring in (4) is just K itself.) Hence, adding 1, we see that (4), like (2) and (3), is $\leq m +$ gl. dim. K, so l.gl.dim. $R \leq m +$ gl.dim. K.

To get precise equality, let $i_1, \ldots, i_m$ be vertices of an m-element complete subgraph of I_A. Note that there exist K-algebra maps

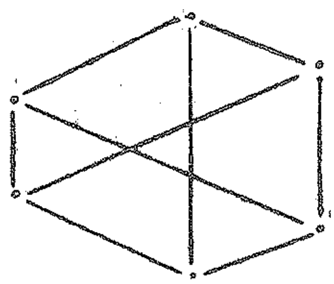
$$K[X_{i_1},\ldots, X_{i_m}] \to R \to K[X_{i_1},\ldots, X_{i_m}]$$

which compose to the identity. (For the second map, one can send all indeterminates X_i ($i \neq i_1,\ldots,i_m$) to 0, or to 1.) This means that $K[X_{i_1},\ldots, X_{i_m}]$ is a retract of R in the category of K-algebras, and this implies ([1] §1) that l.gl.dim R $\geq$ l.gl.dim. $K[X_{i_1},\ldots, X_{i_m}]$ = m + gl.dim. K. This completes the proof of (1). ||

<u>Example:</u> Let K be a field, and I = {1,2,3,4,5,6}. For I_A the complete graph on 6 vertices, with 15 edges, R is $K[X_1,\ldots,X_6]$, which has global dimension 6. If we delete any one edge, the global dimension goes down to 5. If we remove another edge not having a vertex in common with the first, the global dimension becomes 4, and if we now delete the unique edge having no vertex in common with those two, it goes down to 3 (see (5a)), though this ring has 12 of the original 15 commutativity relations! We can get global dimension 2 still keeping 9 of our 15 relations (5b), but to bring the dimension down to 1, we would have to discard all commutativity relations.



(5a)



(5b).

If we replace the $K[X_i]$ by arbitrary K-algebras R_i, the development of Theorem 2 almost seems to go through, replacing the definition "m = the largest number of vertices in a complete subgraph of I_A" by "m = the maximum, over all complete subgraphs $J_A \subseteq I_A$, of the left global dimension of $*_J^A R_i = \otimes_J R_i$." The one difficulty that comes up concerns the terms corresponding to (4). We can still note that the maximal complete subgraphs of P_A are all non-maximal in I_A, but this no longer insures that the global dimensions of the associated tensor products $\otimes_J R_i$ are strictly less than the largest dimension of such a tensor product. So to make our inductive proof work, we need a correction term.

If J_A is a complete subgraph of I_A, let us define the <u>depth</u> of J in I by $d_I(J)$ = the maximum over all complete subgraphs $J'_A \subseteq I_A$ containing J_A, of $|J' - J|$. In particular, the depth of the <u>empty</u> subgraph is the maximum number of vertices in a complete subgraph of I_A.

<u>Theorem 3.</u> Let K be a commutative ring, I a finite set, and $A \subseteq I \times I$ a symmetric antireflexive relation. For each $i \in I$, let R_i be a K-algebra which is free as a K-module on a basis containing 1. Then for $R = *_I^A R_i$,

(6) $\quad$ $\text{l.gl.dim. } R \leq \sup_J (d_I(J) + \text{l.gl.dim. } \otimes_J R_i)$,

where J_A ranges over all <u>complete</u> subgraphs of I_A (including the empty graph.)

<u>Proof.</u> As for Theorem 1. In considering (4), we note that our inductive estimate of the global dimension appearing there will be strictly less than the desired estimate for l.gl.dim. R, since every complete subgraph of P will have depth in P smaller by at least one than its depth in J. Hence the whole term (4) is no greater than our desired estimate of l.gl.dim. R. ||

We note that for a family R_i whose tensor products have global dimension given by the same formula as those of the rings $K[X_i]$, the same formula as in Theorem 2 will bound the global dimension of R. We easily deduce:

Corollary 4. Given K, I, and A as in Theorem 2, define the group

$$G = (X_i \ (i \in I) \mid X_i X_j = X_j X_i ((i,j) \in A)).$$

Then the group algebra $KG = \coprod_I^A K[X_i, X_i^{-1}]$ has left global dimension $m +$ gl.dim. K. More generally, this is the left global dimension of any ring R' obtained from the R of Theorem 2 by universally inverting some subset of the X_i. (I.e., the ring $\coprod_I^A R_i$, where some R_i are $K[X_i]$, and others are $K[X_i, X_i^{-1}]$.) ‖

More generally, Theorem 3 can clearly be applied to any family of group rings KG_i.

We now turn to our neglected homework of verifying that the flatness hypothesis of Dicks' Theorem on the global dimension of coproducts was satisfied when we needed it.

§2. Normal forms in $\coprod_I^A R_i$, and module-freeness.

Let I be a set (not necessarily finite), and $A \subseteq I \times I$ a symmetric antireflexive relation. Let K be a commutative ring, and for each $i \in I$, let R_i be a K-algebra which as a K-module is free on a basis $B_i \cup \{1\}$ $(1 \notin B_i)$. We shall assume these data fixed throughout this section.

Let B denote the disjoint union of the B_i. Each element $x \in B$ will be said to be "associated to" the index $i \in I$ such that $x \in B_i$. S will denote the free semigroup with 1 on the set B.

Let $R = \coprod_I^A R_i$. Since R is generated by the images of the R_i, it will be spanned as a K-module by products of the images of the elements of B (counting the empty product 1), i.e., by the natural image of S. We shall call these products "monomials", and denote them by the same symbols as elements of S of which they are images, though the map $S \to R$ is generally not 1-1.

But we will be careful to distinguish between speaking of two monomials as being "equal in R", or being "equal", which will mean "in S".

Note that if a monomial $x_1 \dots x_r$ has two successive terms $x_p x_{p+1}$ both associated with the same index $i \in I$, then by writing the product $x_p x_{p+1} \in R_i$ as a K-linear combination of elements of $B_i \cup \{1\}$, we can reduce $x_1 \dots x_r$ in R to a linear combination of monomials of shorter length. More generally, if $x_1 \dots x_r$ has two terms x_p and x_q $(p<q)$, associated with the same index $i \in I$, and if all terms x_s occurring between these (i.e., $p < s < q$) are associated with indices j such that $(i,j) \in A$, then in R we can commute x_p past these terms till it is adjacent to x_q, and then reduce our monomial as above to a linear combination of shorter monomials.

We deduce that R will be spanned as a K-module by those monomials $x_1 \dots x_r$ with the property that any two terms x_p and x_q therein, that are associated with the same index $i \in I$, are separated by at least one intermediate term x_s associated with an index j such that $(i,j) \notin A$. We shall call such $x_1 \dots x_r$ "acceptable monomials", and denote the set of acceptable monomials $S' \subseteq S$.

Note that an acceptable monomial can still have adjacent terms $x_p x_{p+1}$ associated with indices i and j such that $(i,j) \in A$, and in this case, it will be equal in R to the (also acceptable) monomial obtained by transposing these terms. To obtain invariants of acceptable monomials under such transposition, let us associate to any acceptable monomial $x_1 \dots x_n$ a <u>partial ordering</u> of its terms, setting

(7) $x_p \prec x_q$ if $p < q$ and there exists a sequence $p = s_1 < \dots < s_u = q$ such that, writing $i(v)$ for the index associated with x_{s_v}, we have $(i(v), i(v+1)) \notin A$ $(v < u)$.

Again, we are being sloppy in our notation, since a monomial may repeat terms of B, so that it is not really the terms x_p (members of B) that are being partially ordered, but, if you will, their subscripts p; or, if you prefer, the pairs (p, x_p). In any case, our point is that we obtain from our monomial a finite partially ordered set, with its vertices labeled with certain elements of B, possibly with repetitions. This partially ordered set will (by (7) and the definition of acceptable monomial) have the properties that any two vertices labeled with elements of B associated to indices i, j such that $(i,j) \notin A$ must be related under our ordering (one $\preccurlyeq$ the other; note that this includes the case $i = j$); and when one vertex covers another (is a minimal vertex $\succ$ than it), the associated indices in I must be distinct.

Lemma 5. Let $x_1 \ldots x_p$ and $y_1 \ldots y_p$ be acceptable monomials of the same length. Then the following conditions are equivalent:

(a) $y_1 \ldots y_p$ can be obtained from $x_1 \ldots x_p$ by a series of transpositions of adjacent terms x_s, x_{s+1}, associated to indices i, j such that $(i,j) \in A$.

(b) There is an isomorphism between the partially ordered sets associated with these two monomials, which preserves the B-labels on the vertices. Equivalently: there exists a permutation $\pi \in S_p$ such that

$x_s = y_{\pi(s)}$, and $x_s \prec x_t$ in $x_1 \ldots x_p \Rightarrow y_{\pi(s)} \preccurlyeq y_{\pi(t)}$ in $y_1 \ldots y_p$.

Further, when this is true, the isomorphism of (b) (equivalently, the π) is unique.

Proof. (a) $\Rightarrow$ (b): We easily see that each transposition leaves the isomorphism class of labeled partially ordered set unchanged.

(b) $\Rightarrow$ (a): If π is not the identity, there will be some s such that $\pi(s) > \pi(s+1)$. We see that x_s and x_{s+1} must be unrelated under $\prec$ (otherwise π would not respect the partial ordering), hence they must be associated with a pair of indices $(i, j) \in A$. Hence we may tranpose them, transforming $x_1 \dots x_p$ to a monomial the order of whose terms is "closer" to that of $y_1 \dots y_p$ (fewer pairs of terms x_s, x_t occurring in different orders). Repeating this procedure, we see that $x_1 \dots x_p$ must be transformed in a finite number of steps into $y_1 \dots y_q$.

To see the last assertion of the Lemma, note that in our partially ordered sets, any two vertices bearing the same label in B must be related under $\succcurlyeq$. Since the sets are finite, there cannot therefore be more than one order-preserving and label-preserving bijection. ||

Let us write the equivalent conditions of the above Lemma $x_1 \dots x_p \sim y_1 \dots y_p$. This gives an equivalence relation on the set S' of acceptable monomials. We shall write $S'/\sim = S''$, and represent the equivalence class of $x_1 \dots x_p$ by $[x_1 \dots x_p] \in S''$. Clearly, the map $S' \to R$ factors through S''. We shall soon show that this map is 1-1, and its image is a K-basis of R. But first we need a result on the structure of S''.

For any subset $J \subseteq I$, let us define S''_J to be the set of all $[x_1 \dots x_p] \in S''$ such that all $x_r \in \bigcup_J B_i$. Let us also define $S''_{\neg J}$ to be the set of all $[x_1 \dots x_p] \in S''$ such that in the partially ordered set associated with this element, no <u>maximal</u> vertex is labeled with a member of any B_i for $i \in J$. We note that the maximal vertices of the partially ordered set associated with $[x_1 \dots x_p]$ correspond to those terms that can be transposed to rightmost position. (E.g., if $x \in B_i$, $y \in B_j$, $z \in B_k$, $(i,j), (i,k) \in A$, $j \neq k$, then in the partially ordered set associated with $[xyz]$, x and z are both maximal.)

Note that if $[y_1 \dots y_q] \in S''_{\neg J}$ and $[z_1 \dots z_r] \in S''_J$, then $y_1 \dots y_q z_1 \dots z_r$ will be an acceptable monomial. Further, $[y_1 \dots y_q z_1 \dots z_r]$ will be determined by the equivalence classes $[y_1 \dots y_q]$ and $[z_1 \dots z_r]$, since any transposition of terms that can be performed in the latter elements can certainly be duplicated in the product. In fact, we have:

Lemma 6. Let $J \subseteq I$. Then for any element $[x_1 \ldots x_p] \in S''$, there exist unique elements $[y_1 \ldots y_q] \in S''_{\neg J}$, $[z_1 \ldots z_r] \in S''_J$, such that $[x_1 \ldots x_p] = [y_1 \ldots y_q z_1 \ldots z_r]$.

Proof. To get the existence of such a decomposition, simply look for a maximal term of our given element associated with an index in J; if there is one, transpose it to the last position. Then treat the remaining string of p-1 terms the same way (it may contain maximal terms that were not maximal in the original element, because they were "covered" by the first term extracted). Iterate the procedure until we are left with a string $y_1 \ldots y_q$ $(q \geq 0)$ with no maximal terms associated with an index in J, followed by a string with all terms associated to indices in J.

To get uniqueness, note that $[z_1 \ldots z_r]$ must consist of precisely those terms of $x_1 \ldots x_p$ which are associated to indices in J, and are not $\not<$ any terms associated with indices in $I - J$. ‖

We can now prove:

Proposition 7. The images in R of the distinct elements of S'' (are distinct and) form a K-basis of R.

Proof. Let M be an abstract free K-module on the basis S''. We shall show that M may be made a right R-module in a natural way, and that the actions on this module of images of distinct elements of S'' are K-linearly independent. (The idea here goes back to a trick of van der Waerden's. Cf. [3] §11.2 (28").)

For any $i \in I$, consider Lemma 6, with $J = \{i\}$. We see that every member of $S''_{\{i\}}$ will be of the form $[z]$ $(z \in B_i)$ or $[1]$, so the Lemma says that we get a bijection $S''_{\neg\{i\}} \times (B_i \cup \{1\}) \to S''$, given by $([y_1 \ldots y_q], z) \mapsto [y_1 \ldots y_q z]$. But $B_i \cup \{1\}$ is a K-basis for R_i, so this decomposition allows us to give the free K-module M on S'' a structure of free right R_i-module on the basis

$S''_{\neg\{i\}}$, which extends our given K-module structure. Doing this for all $i \in I$, we get a structure of right $\coprod_K R_i$-module.

Now take any $(i,j) \in A$. We see that $S''_{\{i,j\}}$ will consist of elements $[zz'] = [z'z]$, $z \in B_i \cup \{1\}$, $z' = B_j \cup \{1\}$. (Because elements of B_i and B_j are transposable with each other, we can form no acceptable monomial of length > 2 from them — see definition of acceptable monomial.) This gives us a bijection $S''_{\neg\{i,j\}} \times (B_i \cup \{1\}) \times (B_j \cup \{1\}) \to S''$. Since $(B_i \cup \{1\}) \times (B_j \cup \{1\})$ is a K-basis of the K-algebra $R_i \otimes R_j$, this allows us to define a structure of (free) right $R_i \otimes R_j$-module on M. This clearly extends the structures of R_i- and R_j-module already defined. That means that the R_i-module operations and the R_j-module operations on M must commute with one another (since the images of R_i and R_j in $R_i \otimes R_j$ commute.) Since we have this for all pairs $(i,j) \in A$, our $\coprod_K R_i$ -module structure must in fact give a $*^A_I R_i$-module structure, by the definition of $*^A_I R_i$.

Now note that for any $[x_1 \ldots x_p] \in S''$, the module-action on the element $[1] \in M$ of the corresponding element $x_1 \ldots x_p \in R$ (my sloppy notation!) will give $[x_1 \ldots x_p] \in M$. It follows that the action of any nontrivial K-linear combination in R of the images of elements of S'' will send $[1]$ to a nonzero element of M. Hence such a nontrivial linear combination is nonzero in R, i.e., the images of the elements of S'' are linearly independent. Since we already know that they span R, this completes the proof of our Theorem. ||

Lemma 6 now clearly gives our desired freeness result:

<u>Proposition 8.</u> For any $J \subseteq I$, $*^A_I R_i$ is free as a right module over $*^A_J R_i$, with basis (the injective image of) $S''_{\neg J}$. ||

This completes the proof of the results of the preceding section.

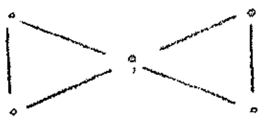
3. Improving Theorem 3.

Let us remark that the "depth" terms $d_I(J)$ that we had to introduce in Theorem 3 cannot be dropped. For the simplest example, let $I = \{1, 2\}$, $A = \emptyset$, so that we are looking at l.gl.dim. $R_1 \coprod_K R_2$. Dicks' result says that this is $\leq$ max(l.gl.dim. R_1, l.gl.dim. R_2, 1 + l.gl.dim. K). The last term is the term of (6) corresponding to $J = \emptyset$! The result is false without it. For instance if K is a field and R_1, R_2 are nontrivial extension fields, the above inequality reads $1 \leq \max(0, 0, 1+0)$. If $K = \mathbb{Z}$ and $R_1 \cong R_2 = \mathbb{Z}[i]$ it reads $2 \leq \max(1, 1, 1+1)$ ([2, Ex. 12.1]). For an example of this sort where the maximum is determined by a nonempty J, let I_A be the graph 1——2——3, K a field, R_1 and R_3 nontrivial field extensions of K, and R_2 polynomial ring $K[t_1, \ldots, t_r]$. Then $R = (R_1 \coprod_K R_3)[t_1, \ldots, t_r]$, and since $R_1 \coprod R_3$ has global dimension 1, this polynomial ring has global dimension $1+r = d_I(\{2\})$ + l.gl.dim. R_2 ([6]). The tensor-product terms with the largest global dimensions are $R_1 \otimes R_2$, $R_2 \otimes R_3$, and R_2, all of which just have global dimension r.

On the other hand, it is easy to see that in some cases Theorem 6 is not the best estimate we can make. For instance, if I_A is itself a complete graph, then $R = \otimes_I R_i$, so l.gl.dim. R is equal to the $J = I$ term of (6), and the others may be discarded, even if they are larger! For a less trivial but similar example, if I_A is the union of two complete subgraphs G_A and H_A ((8a) below) then Dicks' result (plus our Lemma 1) immediately bounds l.gl.dim. R by

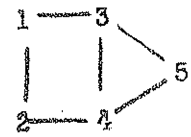
$$\max(\text{l.gl.dim. } *^A_G R_i, \ \text{l.gl.dim. } *^A_H R_i, \ 1 + \text{l.gl.dim. } *^A_{G\cap H} R_i),$$

so most terms of (6) do not appear, and the $G \cap H$ term appears with correction term 1 which is (in general) less than its depth.



(8a)

(8b)



(8c)

The problem is to organize the above sort of observation into a general result strengthening Theorem 3! The problem can be formulated purely combinatorially: Suppose I_A is a finite graph, and g is a nonnegative integer-valued function on the set of all subsets $J \subseteq I$ (corresponding to $\text{l.gl.dim.} \ast^A_J R_i$). Assume that for any two subsets $G, H \subseteq I$, such that no edge of I_A connects a vertex of $G - H$ to a vertex of $H - G$, the function g has the property

(9) $$g(G \cup H) \leq \max(g(G), g(H), 1 + g(G \cap H)).$$

Then the argument of Theorem 3 tells us that $g(I) \leq \sup_J(d_I(J) + g(J))$, where the supremum is over all J which are vertex-sets of complete subgraphs of I, but we would like to know, what is the <u>best</u> estimate one can make of $g(I)$, in terms of the $g(J)$ for these subsets?

Consideration of (8a) and similar examples suggests that perhaps in (6) we can restrict the supremum to those J which are intersections of maximal complete subgraphs of I, and replace $d_I(J)$ by the function which assigns to such a J the length n of the longest chain of intersections yielding J:

$$J_0 \supsetneqq J_0 \cap J_1 \supsetneqq \dots \supsetneqq (J_0 \cap \dots \cap J_n) = J. \quad \text{(Each } J_i \text{ a maximal complete subgraph).}$$

But an attempt to prove such a result suggest that it may just be a first approximation. For instance, if we put together figure (8b) from the subgraphs determined by the vertex-sets $\{1,2,3,4\}$ and $\{3,4,5\}$, we find that the maximal complete subgraph of their intersection, namely the whole subgraph determined by $\{3,4\}$, is not an intersection of maximal complete subgraphs of our original graph. Nevertheless, we can get the formula suggested above (in fact, without the $J = \emptyset$ term!) by tacking onto $\{3,4,5\}$, <u>first</u> $\{1,3\}$, and <u>then</u> $\{2,4\}$. We cannot do this sort of thing in (8c), but we can get our desired estimate (this time with the $J = \emptyset$ term) by building the graph up from $\{1,2,3\}$ and $\{2,3,4,5\}$. But such arguments seem arbitrary, and might not exist for sufficiently complicated graphs..

It would be interesting to know whether, at least, for every finite graph I there exists unique best formula of the form $g(I) \leq \max(n(J) + g(J))$ (n an integer-valued function depending just on the lattice structure of I, J ranging over some subset of the complete subgraphs of I), valid for all g satisfying (9). Certainly, at least, one can construct from I a finite list of such formulas, whose "min" (not necessarily an expression of the same sort) will be truly the best estimate for $g(I)$ in terms of these $g(J)$.

§4. Infinite index-sets I.

Do Theorems 2 and 4 remain true if the hypothesis "I finite" is deleted? When the bounds they give on the global dimension are ∞, they trivially hold. If the bound is finite, I do not know in general, but there is a weaker assumption than finiteness of I_A under which we can prove these results.

Recall that the chromatic number $\chi(I_A)$ of a graph I_A is defined as the least cardinal α such that I can be partitioned into α disjoint subsets ("colored with α colors") so that no two vertices in the same subset are connected by an edge. Note that a graph with chromatic number α cannot contain a complete subgraph with more than α vertices. The converse is false; in fact, there exist graphs with no complete 3-vertex subgraphs, that have infinite chromatic number.

Proposition 9. Theorems 2 and 4 remain true if the assumption "$|I| = n < \infty$" is replaced by "$\chi(I_A) = n < \infty$".

Proof. It will suffice to show this for Theorem 4, since this generalizes Theorem 2. We will use induction on $\chi(I_A)$, but our inductive statement will be a little stronger than the formulation of Theorem 4; namely, it will say that for any graph I_A, any family of K-algebras R_i, and any K-algebra S, such that S and the R_i are free as K-modules on bases containing 1, we have

$$(10) \qquad \text{l.gl.dim. } S \otimes (*_I^A R_i) \leq \sup_J (d_I(J) + \text{l.gl.dim. } S \otimes (*_J^{(A)} R_i)).$$

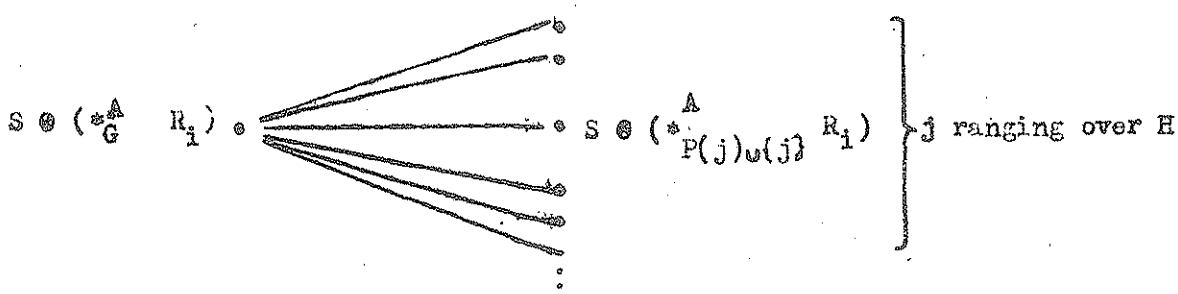
where J_A ranges over all complete subgraphs of I_A. We shall now prove (10) assuming the corresponding result for all chromatic numbers $< n$.

For $n = 0$, (10) reduces to l.gl.dim. S = l.gl.dim. S. Assuming $n > 0$, we decompose I into n sets as in the definition of chromatic number, and let G denote the union of $n-1$ of these, and H the remaining one. Then G_A is a graph of chromatic number $n-1$, so by inductive hypothesis, l.gl.dim. $S \otimes (*_G^A R_i)$ is bounded by the right hand side of (10). Now for each $j \in H$, let $P(j)$ denote the set of vertices of I to which j is connected by an edge. As no two points of H are so connected, $P(j) \subseteq G$, so our inductive hypothesis applies to this set. By considering the depths of complete subgraphs, as in the proof of Theorem 4, we see that l.gl.dim. $S \otimes (*_{P(j)}^A R_i)$ is strictly less than the right hand side of (10). Finally, we look at $P(j) \cup \{j\}$. The subgraph $(P(j)\ \{j\})_A \subseteq I_A$ may not have chromatic number less than n, but we note that because j is connected to all other vertices of this subgraph,

$$S \otimes (*_{P(j)\cup\{j\}}^A R_i) \cong (S \otimes R_j) \otimes (*_{P(j)}^A R_i).$$

Since $P(j)$ has chromatic number $< n$, we may apply our inductive hypothesis — with $S \otimes R_j$ in place of S. One verifies (I leave this to the reader) that the expression we got is bounded by the right-hand-side of (10).

We now note that the ring we want to study, $S \otimes (*^A_I R_i)$, can be expressed as the colimit of the tree of rings



where the edges are the rings $S \otimes (*^A_{P(j)} R_i)$. By Dick's result (a more general case than that used in §1) the left global dimension of this ring will be less than or equal to the supremum of the left global dimensions of the vertices of this tree, and $1 +$ the left global dimensions of the edges. By the preceding calculations, this supremum is bounded by the right hand side of (10).||

Conceivably, one might be able to extend this result to more general graphs by applying Dicks' result for still more general trees, and/or by more subtle graph-theoretic analysis, but I don't see how.

We remark that for an infinite graph I_A, a ring $*^A_I R_i$ will be the direct limit of the rings determined by finite subgraphs (or graphs with finite chromatic numbers), so we can apply results on homological dimensions of direct limits ([8] Theorem 2.3, or [9]). But for $|I| = \aleph_n$, these add a "direct limit tax" of $n+1$ to our estimate of the global dimension, which I would like to somehow avoid paying; and they are of no help starting at $\aleph_\omega$.

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