Isomorphisms of Brin-Higman-Thompson groups

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Abstract. Let m, m', r, r', t, t' be positive integers with r, $r' \geqslant 2$. Let $\mathbb{L}_r$ denote the ring that is universal with an invertible $1{\times}r$ matrix. Let $\mathrm{M}_m(\mathbb{L}_r^{\otimes t})$ denote the ring of $m \times m$ matrices over the tensor product of t copies of $\mathbb{L}_r$. In a natural way, $\mathrm{M}_m(\mathbb{L}_r^{\otimes t})$ is a partially ordered ring with involution. Let $\mathrm{PU}_m(\mathbb{L}_r^{\otimes t})$ denote the group of positive unitary elements. We show that $\mathrm{PU}_m(\mathbb{L}_r^{\otimes t})$ is isomorphic to the Brin-Higman-Thompson group $tV_{r,m}$; the case $t{=}1$ was found by Pardo, that is, $\mathrm{PU}_m(\mathbb{L}_r)$ is isomorphic to the Higman-Thompson group $V_{r,m}$.

We survey arguments of Abrams, Ánh, Bleak, Brin, Higman, Lanoue, Pardo, and Thompson that prove that $t'V_{r',m'} \cong tV_{r,m}$ if and only if $r'{=}r$, $t'{=}t$ and $\gcd(m', r'{-}1) = \gcd(m, r{-}1)$ (if and only if $\mathrm{M}_{m'}(\mathbb{L}_{r'}^{\otimes t'})$ and $\mathrm{M}_m(\mathbb{L}_r^{\otimes t})$ are isomorphic as partially ordered rings with involution).

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1 Introduction

The notation we use will be explained in the next section.

Throughout, fix r, $r' \in [2{\uparrow}\infty[$, m, m', t, $t' \in [1{\uparrow}\infty[$, and fix symbols x and y, and let

$$\mathbb{L}_r := \mathbb{Z}\langle\, x_{[1\uparrow r]}, y_{[1\uparrow r]} \mid x^{\mathrm{transp}}_{[\![1\uparrow r]\!]} \cdot y_{[\![1\uparrow r]\!]} = \mathbf{I}_r \text{ and } y_{[\![1\uparrow r]\!]} \cdot x^{\mathrm{transp}}_{[\![1\uparrow r]\!]} = 1\,\rangle.$$

Thus, for example, $\mathbb{L}_2 = \mathbb{Z}\langle\, x_1, x_2, y_1, y_2 \mid \begin{pmatrix} x_1y_1 & x_1y_2 \\ x_2y_1 & x_2y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, y_1x_1 + y_2x_2 = 1\,\rangle$. We use the symbol $\mathbb{L}$ in recognition of Leavitt's pioneer work on these rings in [14], [15]. We let $\mathbb{L}_r^{\otimes t} := \mathbb{L}_r \otimes_{\mathbb{Z}} \mathbb{L}_r \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{L}_r$, the ring obtained by forming the tensor product over $\mathbb{Z}$ of

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t copies of $\mathbb{L}_r$. We shall be interested in the $m\times m$ matrix ring $\mathrm{M}_m(\mathbb{L}_r^{\otimes t})$. The mnemonic is that r is for 'ring', t is for 'tensor', and m is for 'matrix'. In a natural way, $\mathrm{M}_m(\mathbb{L}_r^{\otimes t})$ is a partially ordered ring with involution. We then let $\mathrm{PU}_m(\mathbb{L}_r^{\otimes t})$ denote the subgroup of positive unitary elements in the group of units of $\mathrm{M}_m(\mathbb{L}_r^{\otimes t})$.

Abrams, Ánh and Pardo [1], [17] found that if $\gcd(m', r-1) = \gcd(m, r-1)$, then $\mathrm{M}_{m'}(\mathbb{L}_r)$ and $\mathrm{M}_m(\mathbb{L}_r)$ are isomorphic as partially ordered rings with involution; we shall observe that it then follows easily that $\mathrm{M}_{m'}(\mathbb{L}_r^{\otimes t})$ and $\mathrm{M}_m(\mathbb{L}_r^{\otimes t})$ are isomorphic as partially ordered rings with involution, and that the groups $\mathrm{PU}_{m'}(\mathbb{L}_r^{\otimes t})$ and $\mathrm{PU}_m(\mathbb{L}_r^{\otimes t})$ are isomorphic. We shall give self-contained proofs of all these isomorphisms.

Pardo [17] discovered a connection between these rings and certain famous groups. In [11], Higman constructed a group $V_{r,m}$ with the properties that the abelianization of $V_{r,m}$ has order $\gcd(2, r-1)$ and the derived group of $V_{r,m}$ is a finitely presentable, infinite, simple group; see also [20]. The group $V_{2,1}$ is Thompson's group V. In [8], Brown showed that $V_{r,m}$ is of type FP_∞. Pardo [17] found that $V_{r,m} \cong \mathrm{PU}_m(\mathbb{L}_r)$; hence, if $\gcd(m', r-1) = \gcd(m, r-1)$, then the above isomorphism $\mathrm{PU}_{m'}(\mathbb{L}_r) \cong \mathrm{PU}_m(\mathbb{L}_r)$ gives the converse of Higman's result that $V_{r',m'} \cong V_{r,m}$ only if $r' = r$ and $\gcd(m', r'-1) = \gcd(m, r-1)$; see [11, Theorem 6.4].

In [6, Section 4.2], Brin constructed a group $tV_{r,m}$ which can be considered as a t-dimensional analogue of the Higman-Thompson group $V_{r,m}$ $(= 1V_{r,m})$. In [6], he proved that $2V_{2,1}$ is simple and that $2V_{2,1} \not\cong V_{r,m}$ and other results. In [7], he proved that $tV_{2,1}$ is simple. In [10], Hennig and Matucci gave a finite presentation of $tV_{2,1}$. In [4], Bleak and Lanoue showed that $t'V_{2,1} \cong tV_{2,1}$ if and only if $t' = t$. In [13], a description of $tV_{r,m}$ along the lines of Higman's construction [11], [20], was given, and it was used to show that $2V_{2,1}$ and $3V_{2,1}$ are of type FP_∞.

The main purpose of this article is to show that $tV_{r,m} \cong \mathrm{PU}_m(\mathbb{L}_r^{\otimes t})$. Straightforward adaptations of known arguments then show that the following are equivalent.

(a) $r' = r$, $t' = t$ and $\gcd(m', r'-1) = \gcd(m, r-1)$.

(b) $\mathrm{M}_{m'}(\mathbb{L}_{r'}^{\otimes t'})$ and $\mathrm{M}_m(\mathbb{L}_r^{\otimes t})$ are isomorphic as partially ordered rings with involution.

(c) $t'V_{r',m'} \cong tV_{r,m}$.

Thus, with r and t fixed and m varying, the set of isomorphism classes of the groups $tV_{r,m}$ is in bijective correspondence with the set of positive divisors of $r-1$.

The structure of the article is as follows. In the first part, we work exclusively with $\mathrm{M}_m(\mathbb{L}_r^{\otimes t})$.

In Section 2, we summarize the notation that we shall be using, and endow $\mathrm{M}_m(\mathbb{L}_r^{\otimes t})$ with the structure of a partially ordered ring with involution.

In Section 3, we give a streamlined proof of the crucial Abrams-Ánh-Pardo result [1] that if $m > r \geqslant 3$ and $\gcd(m, r-1) = 1$, then $\mathrm{M}_m(\mathbb{L}_r)$ and $\mathbb{L}_r$ are isomorphic as partially ordered rings with involution.

In Section 4, following Pardo [17], we show that if $\gcd(m', r-1) = \gcd(m, r-1)$ then $\mathrm{M}_{m'}(\mathbb{L}_r^{\otimes t})$ and $\mathrm{M}_m(\mathbb{L}_r^{\otimes t})$ are isomorphic as partially ordered rings with involution, and, hence, $\mathrm{PU}_{m'}(\mathbb{L}_r^{\otimes t}) \cong \mathrm{PU}_m(\mathbb{L}_r^{\otimes t})$.

In the second part of the article, we concentrate on $tV_{r,m}$.

In Section 5, we prove our main result that the Brin-Higman-Thompson group $tV_{r,m}$ is isomorphic to $\mathrm{PU}_m(\mathbb{L}_r^{\otimes t})$; the case $t=1$ was found by Pardo [17], that is, the Higman-Thompson group $V_{r,m}$ is isomorphic to $\mathrm{PU}_m(\mathbb{L}_r)$. It then follows that if $\gcd(m', r{-}1) = \gcd(m, r{-}1)$, then $tV_{r,m'} \cong tV_{r,m}$.

In Section 6, we find that arguments of Higman show that if $t'V_{r',m'} \cong tV_{r,m}$, then $r' = r$ and $\gcd(m', r'{-}1) = \gcd(m, r{-}1)$.

In Section 7, we find that arguments of Bleak, Brin, Lanoue and Rubin show that if $t'V_{r',m'} \cong tV_{r,m}$, then $t' = t$.

In Section 8, we summarize much of the foregoing by recording the above equivalence (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c). We conclude with a sketch of unpublished results of Ara, Bell and Bergman that show that t, r and $\gcd(m, r{-}1)$ are invariants of the isomorphism class of $\mathrm{M}_m(\mathbb{L}_r^{\otimes t})$ *as ring*, and thus the foregoing equivalent conditions are further equivalent to
(b$'$) $\mathrm{M}_{m'}(\mathbb{L}_{r'}^{\otimes t'})$ and $\mathrm{M}_m(\mathbb{L}_r^{\otimes t})$ are isomorphic *as rings*.

2 Notation

We will find it useful to have a vocabulary for intervals in $\mathbb{Z}$.

2.1 Notation. Let i, $j \in \mathbb{Z}$. We define the vector

$$[\![i{\uparrow}j]\!] := \begin{cases} (i, i+1, \ldots, j-1, j) \in \mathbb{Z}^{j-i+1} & \text{if } i \leqslant j, \\ () \in \mathbb{Z}^0 & \text{if } i > j. \end{cases}$$

The underlying subset of $\mathbb{Z}$ will be denoted $[i{\uparrow}j]$. Similar notation applies for $[i{\uparrow}\infty[$.

Let v_k be an integer-indexed symbol. We define the vector

$$v_{[\![i\uparrow j]\!]} := \begin{cases} (v_i, v_{i+1}, \cdots, v_{j-1}, v_j) & \text{if } i \leqslant j, \\ () & \text{if } i > j. \end{cases}$$

The underlying set will be denoted $v_{[i\uparrow j]}$.

When $v_{k,k'}$ is a doubly indexed symbol, we write

$$v_{[i\uparrow j]\times[i'\uparrow j']} := \{v_{k,k'} \mid k \in [i{\uparrow}j], k' \in [i'{\uparrow}j']\}.$$

Let R be a ring with a unit.

For any subset Z of R, we let $\langle\langle Z\rangle\rangle$ denote the multiplicative submonoid of R generated by Z.

Suppose that m, $n \in [1{\uparrow}\infty[$.

We let ${}^mR^n$ denote the set of $m\times n$ matrices over R and we write $\mathrm{M}_m(R) \coloneqq {}^mR^m$.

For $i \in [1{\uparrow}m]$ and $j \in [1{\uparrow}n]$, we let $e_{i,j} \in {}^m\mathbb{Z}^n$ denote the $m \times n$ matrix whose (i, j) coordinate is 1, and all other coordinates are zero. This notation applies only where the ranges of i and j are clearly specified. We think of ${}^mR^n$ as ${}^m\mathbb{Z}^n \otimes_{\mathbb{Z}} R$ and use the same symbol $e_{i,j}$ to denote the image in ${}^mR^n$.

We define an additive *transpose map* ${}^m\mathbb{Z}^n \to {}^n\mathbb{Z}^m$, $U \mapsto U^*$, such that $e^*_{i,j} \coloneqq e_{j,i}$. We endow ${}^m\mathbb{Z}^n$ with the structure of a partially ordered abelian group in which the *positive cone* $\mathrm{P}({}^m\mathbb{Z}^n)$ (the set of elements $\geqslant 0$) is the additive monoid generated by $e_{[1\uparrow m]\times[1\uparrow n]}$.

In particular, $\mathrm{M}_m(\mathbb{Z})$ has the structure of a ring with involution $p \mapsto p^*$, and the structure of a partially ordered abelian group. We note that the positive cone $\mathrm{P}_m(\mathbb{Z}) \coloneqq \mathrm{P}({}^m\mathbb{Z}^m)$ contains 1 and is closed under multiplication and the involution. Thus $\mathrm{M}_m(\mathbb{Z})$ has the structure of a partially ordered ring with involution.

2.2 Notation. Throughout, fix $r \in [2{\uparrow}\infty[$, and fix symbols x and y, and let

$$\mathbb{L}_r \coloneqq \mathbb{Z}\langle\, x_{[1\uparrow r]}, y_{[1\uparrow r]} \mid x^{\text{transp}}_{[\![1\uparrow r]\!]} \cdot y_{[\![1\uparrow r]\!]} = \mathbf{I}_r \text{ and } y_{[\![1\uparrow r]\!]} \cdot x^{\text{transp}}_{[\![1\uparrow r]\!]} = 1 \,\rangle.$$

Here $x_{[\![1\uparrow r]\!]}$ and $y_{[\![1\uparrow r]\!]}$ are $1 \times r$ row vectors, $x^{\text{transp}}_{[\![1\uparrow r]\!]}$ denotes the $r \times 1$ transpose of $x_{[\![1\uparrow r]\!]}$, and $\mathbf{I}_r$ denotes the $r \times r$ identity matrix.

Leavitt [14] showed that each element of $\mathbb{L}_r$ has a unique *normal form*, which is an expression as a $\mathbb{Z}$-linear combination of elements of $\langle\langle x_{[1\uparrow r]} \cup y_{[1\uparrow r]}\rangle\rangle$ which do not contain any contiguous subword of the form $x_s y_{s'}$ $(= \delta_{s,s'})$, s, $s' \in [1{\uparrow}r]$, or $y_r x_r$ $(= 1 - \sum_{s\in[1\uparrow(r-1)]} y_s x_s)$. By Leavitt's normal-form result, the multiplicative monoid $\langle\langle x_{[1\uparrow r]}\rangle\rangle$ is freely generated by $x_{[1\uparrow r]}$, and similarly for $\langle\langle y_{[1\uparrow r]}\rangle\rangle$.

We endow $\mathbb{L}_r$ with the involution $p \mapsto p^*$ which is the unique anti-automorphism which interchanges $x_{[\![1\uparrow r]\!]}$ and $y_{[\![1\uparrow r]\!]}$.

We endow $\mathbb{L}_r$ with the structure of a partially ordered abelian group in which the positive cone $\mathrm{P}(\mathbb{L}_r)$ is the additive monoid generated by the set of monomials $\langle\langle x_{[1\uparrow r]} \cup y_{[1\uparrow r]}\rangle\rangle$. This is a partial order since a nonempty sum of monomials is not zero. To see this notice that for any row vector of zeros and monomials, some of which have positive x-degree, multiplying on the right by a suitable y_i leaves a nonzero vector of smaller largest x-degree, and the result follows by induction. We note that the positive cone contains 1 and is closed under multiplication and the involution. Thus $\mathbb{L}_r$ has been endowed with the structure of a partially ordered ring with involution.

Let m, n, $t \in [1{\uparrow}\infty[$.

We extend the involutions on each of the $t{+}1$ factors to the *conjugate-transpose map* ${}^m\mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{L}_r^{\otimes t} \to {}^n\mathbb{Z}^m \otimes_{\mathbb{Z}} \mathbb{L}_r^{\otimes t}$, $U \mapsto U^*$. Recall that we identify ${}^m\mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{L}_r^{\otimes t} = {}^m(\mathbb{L}_r^{\otimes t})^n$. Let $\mathrm{U}({}^m(\mathbb{L}_r^{\otimes t})^n)$ denote the set of $Y \in {}^m(\mathbb{L}_r^{\otimes t})^n$ such that $Y \cdot Y^* = \mathbf{I}_m$ and $Y^* \cdot Y = \mathbf{I}_n$. The elements of $\mathrm{U}({}^m(\mathbb{L}_r^{\otimes t})^n)$ are called the *unitary* $m \times n$ matrices over $\mathbb{L}_r^{\otimes t}$. We write $\mathrm{U}_m(\mathbb{L}_r^{\otimes t}) \coloneqq \mathrm{U}({}^m(\mathbb{L}_r^{\otimes t})^m)$, a subgroup of the group of units of $\mathrm{M}_m(\mathbb{L}_r^{\otimes t})$.

We extend the partial order on each of the $t{+}1$ factors to all of ${}^m\mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{L}_r^{\otimes t}$ by taking as the positive cone $\mathrm{P}({}^m\mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{L}_r^{\otimes t})$ the additive submonoid generated by the product of the positive cones of the factors; as before, a nonempty sum of monomials is not zero. We write $\mathrm{P}_m(\mathbb{L}_r^{\otimes t}) \coloneqq \mathrm{P}({}^m(\mathbb{L}_r^{\otimes t})^m)$, a multiplicative submonoid with involution in the ring with involution $\mathrm{M}_m(\mathbb{L}_r^{\otimes t})$. Thus $\mathrm{M}_m(\mathbb{L}_r^{\otimes t})$ has been endowed with the structure of a partially ordered ring with involution.

Let $\mathrm{PU}({}^m(\mathbb{L}_r^{\otimes t})^n) \coloneqq \mathrm{P}({}^m(\mathbb{L}_r^{\otimes t})^n) \cap \mathrm{U}({}^m(\mathbb{L}_r^{\otimes t})^n)$ and $\mathrm{PU}_m(\mathbb{L}_r^{\otimes t}) \coloneqq \mathrm{PU}({}^m(\mathbb{L}_r^{\otimes t})^m)$. Then $\mathrm{PU}_m(\mathbb{L}_r^{\otimes t}) = \mathrm{P}_m(\mathbb{L}_r^{\otimes t}) \cap \mathrm{U}_m(\mathbb{L}_r^{\otimes t})$, an intersection of multiplicative monoids with involution, and hence itself a multiplicative monoid with involution. Since $\mathrm{PU}_m(\mathbb{L}_r^{\otimes t})$ lies in $\mathrm{U}_m(\mathbb{L}_r^{\otimes t})$ the involution acts as inversion and $\mathrm{PU}_m(\mathbb{L}_r^{\otimes t})$ is a multiplicative group. We call $\mathrm{PU}_m(\mathbb{L}_r^{\otimes t})$ the *group of positive unitary* $m \times m$ *matrices over* $\mathbb{L}_r^{\otimes t}$.

3 The crucial ring isomorphism

This following beautiful result of Abrams, Ánh and Pardo has the unusual property that it shows that two naturally defined rings are isomorphic without giving a natural reason, and there may not be one. We shall be giving their proof but shall incorporate a permutation of $\mathbb{Z}$ that will automate much of their book-keeping. Although the proof we shall give uses $r \neq 2$ and $r < m$, we shall see in the next section that the result holds without these restrictions.

3.1 Theorem [1, Theorem 4.14]. *Let* $r \in [3{\uparrow}\infty[$ *and* $m \in [(r{+}1){\uparrow}\infty[$ *with* $\gcd(m, r{-}1) = 1$. *Then* $\mathbb{L}_r$ *and* $\mathrm{M}_m(\mathbb{L}_r)$ *are isomorphic as partially ordered rings with involution.*

Proof. Let $\mathbb{L} \coloneqq \mathbb{L}_r$. Define $\pi\colon \mathbb{Z} \to \mathbb{Z}$ by

$$i \mapsto i^\pi \coloneqq \begin{cases} i{+}r & \text{if } i \equiv 0 \pmod{m}, \\ i{+}r{-}2 & \text{if } i \equiv 1 \pmod{m}, \\ i{+}r{-}1 & \text{if } i \not\equiv 0, 1 \pmod{m}. \end{cases}$$

Thus π shifts every element of $\mathbb{Z}$ up by $r{-}1$, except that certain adjacent pairs $(\ell m, \ell m{+}1)$ are carried to $(\ell m{+}r, \ell m{+}r{-}1)$, that is, they are shifted by $r{-}1$ and then interchanged. Notice that π is bijective.

We claim that $[2{\uparrow}r]$ is a set of $\langle \pi \rangle$-orbit representatives in $\mathbb{Z}$. Because π shifts every element of $\mathbb{Z}$ up by at most r and by at least $r{-}2\,(\geqslant 1)$, it follows that each $\langle \pi \rangle$-orbit meets $[1{\uparrow}r]$. Now $1^\pi = r{-}1\,(\geqslant 2)$ which lies in $[2{\uparrow}r]$. Hence, each $\langle \pi \rangle$-orbit meets $[2{\uparrow}r]$. Since $2^\pi = r{+}1$ and π shifts every element of $\mathbb{Z}$ up by at least $r{-}2$, we see that no $\langle \pi \rangle$-orbit meets $[2{\uparrow}r]$ twice. This proves the claim.

Now any sequence of $r{-}1$ consecutive integers is a set of $\langle \pi \rangle$-orbit representatives unless two elements are in the same $\langle \pi \rangle$-orbit, and the only pairs in the same $\langle \pi \rangle$-orbit that are at distance less than $r{-}1$ are of the form $\ell m{+}1 \mapsto \ell m{+}r{-}1$. If our sequence of $r{-}1$

consecutive integers does not start at an $\ell m{+}1$, then it cannot contain two elements in the same $\langle\,\pi\,\rangle$-orbit. Hence, for each $k \in \mathbb{Z}$, $[(k{+}1){\uparrow}(k{+}r{-}1)]$ is a set of $\langle\,\pi\,\rangle$-orbit representatives in $\mathbb{Z}$ if and only if $k \not\equiv 0 \bmod m$.

Let $s \in [2{\uparrow}r]$ and $j \in [1{\uparrow}(m{-}1)]$. Since $\gcd(m, r{-}1) = 1$, we have $(r{-}1)j \not\equiv 0 \bmod m$. Hence, $[(1{+}(r{-}1)j){\uparrow}((r{-}1)(j{+}1))]$ is a set of $\langle\,\pi\,\rangle$-orbit representatives in $\mathbb{Z}$ and therefore contains a unique element in the $\langle\,\pi\,\rangle$-orbit of s. We denote that element by $s\#j$. Thus $s\#j \in [(1{+}(r{-}1)j){\uparrow}((r{-}1)(j{+}1))]$ and $(s\#j)^{\langle\,\pi\,\rangle} = s^{\langle\,\pi\,\rangle}$. In $\mathbb{L}$, define $y_{(s\#j)+m} \coloneqq y_1^{j-1}y_s$. Define $y_{r+m-1} \coloneqq y_1^{m-1}$. For each $k \in [(r{+}m{-}1){\uparrow}(mr)]$, define $x_k \coloneqq y_k^*$. We claim that we have defined $y_{[\![(r+m-1)\uparrow(mr)]\!]}$ with underlying set $y_1^{[0\uparrow(m-2)]}y_{[2\uparrow r]} \cup \{y_1^{m-1}\}$. For each $j \in [1{\uparrow}(m{-}1)]$, varying $s \in [2{\uparrow}r]$, we see that

$$[2{\uparrow}r]\#j = [(1{+}(r{-}1)j){\uparrow}(r{-}1)(j{+}1)],$$

and then

$$[2{\uparrow}r]\#j + m = [(1{+}(r{-}1)j{+}m){\uparrow}((r{-}1)(j{+}1){+}m)],$$

and we have defined $y_{[\![(1+(r-1)j+m)\uparrow((r-1)(j+1)+m)]\!]}$ with underlying set $y_1^{j-1}y_{[2\uparrow r]}$. By then varying $j \in [1{\uparrow}(m{-}1)]$, we obtain $y_{[\![(r+m)\uparrow(mr)]\!]}$ with underlying set $y_1^{[0\uparrow(m-2]}y_{[2\uparrow r]}$. Thus we have defined $y_{[\![(r+m-1)\uparrow(mr)]\!]}$ with underlying set $y_1^{[0\uparrow(m-2)]}y_{[2\uparrow r]} \cup \{y_1^{m-1}\}$. It is easy to see that $y_{[\![(r+m-1)\uparrow(mr)]\!]} \in \mathrm{PU}(\,{}^{1}\mathbb{L}^{(m-1)(r-1)+1})$.

Let

$$Y \coloneqq y_{[\![1\uparrow r]\!]} \oplus \mathbf{I}_{m-2} \oplus y_{[\![(m+r-1)\uparrow(mr)]\!]} = \begin{pmatrix} y_{[\![1\uparrow r]\!]} & 0 & 0 \\ 0 & \mathbf{I}_{m-2} & 0 \\ 0 & 0 & y_{[\![(m+r-1)\uparrow(mr)]\!]} \end{pmatrix} \in \mathrm{PU}(\,{}^{m}\mathbb{L}^{mr}).$$

We identify ${}^{m}\mathbb{L}^{mr} = (\mathrm{M}_m(\mathbb{L}))^r$, and let $Y_{[\![1\uparrow r]\!]}$ denote the resulting partition of Y, that is, $Y = Y_{[\![1\uparrow r]\!]} \in (\mathrm{M}_m(\mathbb{L}))^r$.

We then have a well-defined homomorphism $\mathbb{L} \to \mathrm{M}_m(\mathbb{L})$ that sends $y_{[\![1\uparrow r]\!]}$ to $Y_{[\![1\uparrow r]\!]}$ and sends $x^{\mathrm{transp}}_{[\![1\uparrow r]\!]} (= (y_{[\![1\uparrow r]\!]})^{-1})$ to $X^{\mathrm{transp}}_{[\![1\uparrow r]\!]} \coloneqq Y^{-1} = Y^*$.

This homomorphism is nonzero with torsion-free image, and hence is injective on $\mathbb{Z}$, and hence is injective, by the following argument of Leavitt [15, Theorem 2]. Consider any nonzero element of the kernel. By multiplying on the left by a suitable x-monomial, we get a nonzero element in the free x-subalgebra. By multiplying on the right by a suitable y-monomial, we get a nonzero element of $\mathbb{Z}$, which is the desired contradiction.

Let S denote the image of $\mathrm{P}_1(\mathbb{L})$, that is, the additive monoid that is generated by the multiplicative monoid that is generated by $Y_{[1\uparrow r]} \cup Y^*_{[1\uparrow r]}$ in $\mathrm{M}_m(\mathbb{L})$. Clearly $S^* = S \subseteq \mathrm{P}_m(\mathbb{L})$. It remains to show that $\mathrm{P}_m(\mathbb{L}) \subseteq S$, for then the injective map $\mathbb{L} \to \mathrm{M}_m(\mathbb{L})$ is surjective and the resulting inverse map carries $\mathrm{P}_m(\mathbb{L})$ into $\mathrm{P}_1(\mathbb{L})$.

Since $m > r > 2$,

$$Y_1 = \sum_{j\in[1\uparrow r]} y_j e_{1,j} + \sum_{j\in[(r+1)\uparrow m]} e_{j-r+1,j}, \tag{1}$$

$$Y_2 = \sum_{j\in[1\uparrow(r-2)]} e_{j+m-r+1,j} + \sum_{j\in[(r-1)\uparrow m]} y_{j+m}e_{m,j}, \tag{2}$$

$$Y_s = \sum_{j\in[1\uparrow m]} y_{j+(s-1)m}e_{m,j} \text{ for each } s \in [3\uparrow r], \tag{3}$$

$$Y_1^* \overset{(1)}{=} \sum_{j\in[1\uparrow r]} x_j e_{j,1} + \sum_{j\in[(r+1)\uparrow m]} e_{j,j-r+1}, \tag{4}$$

$$Y_1Y_1^* \overset{(1),(4)}{=} \sum_{j\in[1\uparrow(m-r+1)]} e_{j,j}, \tag{5}$$

$$Y_1e_{j,j}Y_1^* \overset{(1),(4)}{=} e_{j-r+1,j-r+1} \text{ for each } j \in [(r+1)\uparrow m], \tag{6}$$

$$e_{1,1}Y_1e_{j,j} \overset{(1)}{=} y_je_{1,j} \text{ for each } j \in [1\uparrow r], \tag{7}$$

$$e_{j-r+1,j-r+1}Y_1e_{j,j} \overset{(1)}{=} e_{j-r+1,j} \text{ for each } j \in [(r+1)\uparrow m], \tag{8}$$

$$Y_2^* \overset{(2)}{=} \sum_{j\in[1\uparrow(r-2)]} e_{j,j+m-r+1} + \sum_{j\in[(r-1)\uparrow m]} x_{j+m}e_{j,m}, \tag{9}$$

$$Y_2e_{j,j}Y_2^* \overset{(2),(9)}{=} e_{j+m-r+1,j+m-r+1} \text{ for each } j \in [1\uparrow(r-2)], \tag{10}$$

$$e_{j+m-r+1,j+m-r+1}Y_2e_{j,j} \overset{(2)}{=} e_{j+m-r+1,j} \text{ for each } j \in [1\uparrow(r-2)], \tag{11}$$

$$e_{m,m}Y_2e_{j,j} \overset{(2)}{=} y_{j+m}e_{m,j} \text{ for each } j \in [(r-1)\uparrow m], \tag{12}$$

$$e_{m,m}Y_se_{j,j} \overset{(3)}{=} y_{j+(s-1)m}e_{m,j} \text{ for each } j \in [1\uparrow m],\ s \in [3\uparrow r]. \tag{13}$$

3.1.1 Definition. The m-cycle $j \mapsto (j-r+1)[\textbf{mod } m]$.

For each $j \in \mathbb{Z}$, let $j[\text{mod } m]$ denote the representative of $j+m\mathbb{Z}$ in $[1\uparrow m]$.

Since it is a unit in $\mathbb{Z}_m$, $(r-1)+m\mathbb{Z}$ additively generates a subgroup of order m in $\mathbb{Z}_m$, and hence shifting down by $r-1$ determines an m-cycle on $\mathbb{Z}_m$. Hence $j \mapsto (j-r+1)[\text{mod } m]$ determines an m-cycle on $[1\uparrow m]$. We think of this m-cycle as an m-gon with two distinguished sides, $r-1 \mapsto m$ and $r \mapsto 1$.

$$\begin{array}{ccccc} m & \mapsto & \cdots & \mapsto & r \\ \upmapsto & & & & \downmapsto \\ r-1 & \mapsfrom & \cdots & \mapsfrom & 1 \end{array} \tag{14}$$

3.1.2 Claim. Both $e_{1,1}$ and $e_{m,m}$ lie in S.

For each $i \in [1\uparrow m]$, let $E_i := \sum\limits_{j\in[1\uparrow i]} e_{j,j}$ and $E_i' := \sum\limits_{j\in[(i+1)\uparrow m]} e_{j,j} = \mathbf{I}_m - E_i$. We shall show that $E_i \in S$ and $E_i' \in S$ by letting i travel around (14) from m to $r-1$. This claim is clear for $i = m$.

Now suppose that $i \in [1\uparrow(r-2)] \cup [r\uparrow m]$ such that E_i, $E_i' \in S$.

If $i \in [1{\uparrow}(r{-}2)]$, then

$$E_i = \sum_{j\in[1\uparrow i]\subseteq[1\uparrow(r-2)]} e_{j,j}, \text{ and } (i{-}r{+}1)[\text{mod } m] = i{+}m{-}r{+}1,$$

$$E_{i+m-r+1} \overset{(10)}{=} Y_2E_iY_2^* + E_{m-r+1} \overset{(5)}{=} Y_1Y_1^* + Y_2E_iY_2^* \in S,$$

$$E'_{i+m-r+1} = \mathbf{I}_m{-}E_{i+m-r+1} = \sum_{s\in[1\uparrow r]} Y_sY_s^* - Y_1Y_1^* - Y_2E_iY_2^* = Y_2E'_iY_2^* + \sum_{s\in[3\uparrow r]} Y_sY_s^* \in S.$$

If $i \in [r{\uparrow}m]$, then

$$E'_i = \sum_{j\in[i+1,m]\subseteq[(r+1)\uparrow m]} e_{j,j}, \text{ and } (i{-}r{+}1)[\text{mod } m] = i{-}r{+}1,$$

$$E'_{i-r+1} \overset{(6)}{=} Y_1E'_iY_1^* + E'_{m-r+1} \overset{(5)}{=} Y_1E'_iY_1^* + \sum_{s\in[2\uparrow r]} Y_sY_s^* \in S,$$

$$E_{i-r+1} = \mathbf{I}_m{-}E'_{i-r+1} = \sum_{s\in[1\uparrow r]} Y_sY_s^* - Y_1E'_iY_1^* - \sum_{s\in[2\uparrow r]} Y_sY_s^* = Y_1E_iY_1^* \in S.$$

It now follows by induction on path-length in (14) that, for each $i \in [1{\uparrow}m]$, $E_i \in S$ and $E'_i \in S$. Hence, $e_{1,1} = E_1 \in S$ and $e_{m,m} = E'_{m-1} \in S$. We could have stopped when we had reached whichever came later of 1 and $m{-}1$.

3.1.3 Claim. Each $e_{j,j}$ lies in S.

Let $j \in [1{\uparrow}m]$. We shall show that $e_{j,j} \in S$ by letting j travel along the top of (14) from m to r and along the bottom of (14) from 1 to $r{-}1$. We have proved the claim for $j = m$ and $j = 1$. Now suppose that $j \in [1{\uparrow}(r{-}2)] \cup [(r{+}1){\uparrow}m]$ such that $e_{j,j} \in S$.

If $j \in [1{\uparrow}(r{-}2)]$, then

$$(j{-}r{+}1)[\text{mod } m] = j{+}m{-}r{+}1 \text{ and } e_{j+m-r+1,j+m-r+1} \overset{(10)}{=} Y_2e_{j,j}Y_2^* \in S.$$

If $j \in [(r{+}1){\uparrow}m]$, then

$$(j{-}r{+}1)[\text{mod } m] = j{-}r{+}1 \text{ and } e_{j-r+1,j-r+1} \overset{(6)}{=} Y_1e_{j,j}Y_1^* \in S.$$

It now follows by induction on path-length in (14) that $e_{j,j} \in S$ for all $j \in [1{\uparrow}m]$.

3.1.4 Review. The $e_{i,i}Y_se_{j,j}$ lie in S.

We have now shown that for all i, $j \in [1{\uparrow}m]$, and all $s \in [1{\uparrow}r]$, $e_{i,i}Y_se_{j,j} \in S$. This has the following consequences.

(15) For each $j \in [1{\uparrow}r]$, $y_je_{1,j} \overset{(7)}{\in} S$.

(16) For each $j \in [(r{+}1){\uparrow}m]$, $e_{j-r+1,j} \overset{(8)}{\in} S$.

(17) For each $j \in [1{\uparrow}(r{-}2)]$, $e_{j+m-r+1,j} \overset{(11)}{\in} S$.

(18) For each $j \in [(r{-}1)\uparrow m]$, $y_{j+m}e_{m,(j+m)[\text{mod } m]} = y_{j+m}e_{m,j} \overset{(12)}{\in} S$.

(19) For each $j \in [1\uparrow m]$ and $s \in [3\uparrow r]$, $y_{j+(s-1)m}e_{m,(j+(s-1)m)[\text{mod } m]} = y_{j+(s-1)m}e_{m,j} \overset{(13)}{\in} S$.

(20) For each $k \in [(r{-}1{+}m)\uparrow(mr)]$, $y_k e_{m,k[\text{mod } m]} \overset{(18),(19)}{\in} S$.

(21) $y_1^{m-1}e_{m,r-1} = y_{r-1+m}e_{m,(r-1+m)[\text{mod } m]} \overset{(20)}{\in} S$.

(22) For $j \in [1\uparrow(m{-}1)]$, $s \in [2\uparrow r]$, $y_1^{j-1}y_s e_{m,(s\#j)[\text{mod } m]}$

$$= y_{(s\#j)+m}e_{m,((s\#j)+m)[\text{mod } m]} \overset{(20)}{\in} S.$$

3.1.5 Claim. All the $e_{i,j}$ lie in S.

It follows from (16) and (17) that for each edge $j \mapsto j'$ in the top of diagram (14), we have $e_{j',j} \in S$. Since $e_{j_1,j_2}e_{j_2,j_3} = e_{j_1,j_3}$, we see that for any subpath $j \mapsto \cdots \mapsto j'$ of the top of diagram (14), we have $e_{j',j} \in S$, and $e_{j,j'} = e^*_{j',j} \in S$. Thus if j, j' are two points on the top of the diagram (14), then $e_{j',j} \in S$. The same result holds for the bottom of the diagram (14). To obtain $e_{[1\uparrow m]\times[1\uparrow m]} \subseteq S$, it now suffices to show that $e_{1,m} \in S$.

Recall that for $s \in [2\uparrow r]$ and $j \in [1\uparrow(m{-}1)]$, s and $s\#j$ lie in the same $\langle\,\pi\,\rangle$-orbit. It is clear that π induces an action modulo m, and hence induces a permutation π_m of $[1\uparrow m]$. Hence s and $(s\#j)[\text{mod } m]$ lie in the same $\langle\,\pi_m\,\rangle$-orbit. On $[2\uparrow(m{-}1)]$, π_m acts as $i \mapsto (i{+}r{-}1)[\text{mod } m]$, while $1 \mapsto r{-}1$ and $m \mapsto r$. It follows that there are two $\langle\,\pi_m\,\rangle$-orbits and they are given by the top and the bottom of the diagram (14). Hence $e_{s,(s\#j)[\text{mod } m]} \in S$.

In $\mathbb{L}$, $1 = y_1^{m-1}x_1^{m-1} + \sum\limits_{j\in[1\uparrow(m-1)]}\sum\limits_{s\in[2\uparrow r]}(y_1^{j-1}y_s x_s x_1^{j-1})$. Hence, in $\mathrm{M}_m(\mathbb{L})$,

$$\begin{aligned} e_{1,m} &= y_1^{m-1}x_1^{m-1}e_{1,m} + \sum_{j\in[1\uparrow(m-1)]}\sum_{s\in[2\uparrow r]}(y_1^{j-1}y_s x_s x_1^{j-1}e_{1,m}) \\ &= (y_1e_{1,1})^{m-1}(e_{1,r-1})(x_1^{m-1}e_{r-1,m}) \\ &\quad + \sum_{j\in[1\uparrow(m-1)]}\sum_{s\in[2\uparrow r]}(y_1e_{1,1})^{j-1}(y_se_{1,s})(e_{s,(s\#j)[\text{mod } m]})(x_s x_1^{j-1}e_{(s\#j)[\text{mod } m],m}). \end{aligned}$$

Using (15), (21), and (22), and the fact that $S = S^*$, we see that $e_{1,m} \in S$.

Now $e_{[1\uparrow m]\times[1\uparrow m]} \subseteq S$. By (15), $y_{[1\uparrow r]}e_{[1\uparrow m]\times[1\uparrow m]} \subseteq S$. Hence $\mathrm{P}_m(\mathbb{L}) \subseteq S$. This completes the proof. □

3.2 Example. Let us illustrate the proof of Theorem 3.1 by considering the case $r = 3$ and $m = 5$; here $\gcd(m, r{-}1) = \gcd(5, 2) = 1$ and $\mathrm{M}_5(\mathbb{L}_3) \cong \mathbb{L}_3$.

We find that the cycle decomposition of π is $(\ldots,0,3,5,8,10,\ldots)(\ldots,1,2,4,6,7,9,\ldots)$.

Now $y_{m+r-1} \coloneqq y_1^{m-1}$, that is, $y_7 \coloneqq y_1^4$.

Now consider $s \in [2\uparrow r] = [2\uparrow 3]$ and $j \in [1\uparrow(m{-}1)] = [1\uparrow 4]$. We defined

$$\{s\#j\} \coloneqq [(1 + (r{-}1)j)\uparrow((r{-}1)(j{+}1))] \cap s^\pi = [(1 + 2j)\uparrow(2j{+}2)] \cap s^\pi.$$

Thus $\{s\#1\} = [3{\uparrow}4]\cap s^\pi$, $\{s\#2\} = [5{\uparrow}6]\cap s^\pi$, $\{s\#3\} = [7{\uparrow}8]\cap s^\pi$ and $\{s\#4\} = [9{\uparrow}10]\cap s^\pi$. For $s = 2$, we are in the set $\{\ldots, 4, 6, 7, 9, \ldots\}$, and for $s = 3$, we are in the set $\{\ldots, 3, 5, 8, 10, \ldots\}$. Thus

$2\#1 = 4, \quad 2\#2 = 6, \quad 2\#3 = 7, \quad 2\#4 = 9,$

$3\#1 = 3, \quad 3\#2 = 5, \quad 3\#3 = 8 \quad 3\#4 = 10.$

We define $y_{(s\#j)+m} := y_1^{j-1}y_s$, that is, $y_{(s\#j)+5} := y_1^{j-1}y_s$. Thus

$y_9 = y_2,\ y_{11} = y_1y_2,\ y_{12} = y_1^2y_2,\ y_{14} = y_1^3y_2,$

$y_8 = y_3,\ y_{10} = y_1y_3,\ y_{13} = y_1^2y_3$ and $y_{15} = y_1^3y_3$. Hence

$y_7 = y_1^4,\ y_8 = y_3,\ y_9 = y_2,\ y_{10} = y_1y_3,\ y_{11} = y_1y_2,\ y_{12} = y_1^2y_2,\ y_{13} = y_1^2y_3,\ y_{14}{=}y_1^3y_2,\ y_{15}{=}y_1^3y_3.$

Thus $y_{[\![(r+m-1)\uparrow(mr)]\!]} = y_{[\![7\uparrow15]\!]} = (y_1^4, y_3, y_2, y_1y_3, y_1y_2, y_1^2y_2, y_1^2y_3, y_1^3y_2, y_1^3y_3)$. Now we take $Y := y_{[\![1\uparrow r]\!]} \oplus \mathbf{I}_{m-2} \oplus y_{[\![(r+m-1)\uparrow(mr)]\!]} = y_{[\![1\uparrow3]\!]} \oplus \mathbf{I}_3 \oplus y_{[\![7\uparrow15]\!]}$. Hence

$$Y = \begin{pmatrix} y_1 & y_2 & y_3 & 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&y_1^4&y_3&y_2&y_1y_3&y_1y_2&y_1^2y_2&y_1^2y_3&y_1^3y_2&y_1^3y_3 \end{pmatrix}.$$

Now we partition Y as

$$Y_1 = \begin{pmatrix} y_1&y_2&y_3&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix},\ Y_2 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 1&0&0&0&0\\ 0&y_1^4&y_3&y_2&y_1y_3 \end{pmatrix},\ Y_3 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ y_1y_2&y_1^2y_2&y_1^2y_3&y_1^3y_2&y_1^3y_3 \end{pmatrix}.$$

We let $X_i := Y_i^*$.

Now

$$Y_1X_1 = \begin{pmatrix} y_1&y_2&y_3&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \begin{pmatrix} x_1&0&0&0&0\\ x_2&0&0&0&0\\ x_3&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0 \end{pmatrix} = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} = E_3.$$

Thus, $E_3 = Y_1X_1$. Hence, $E_3' = Y_2X_2 + Y_3X_3$.

Now $Y_1E_3'X_1 =$

$$\begin{pmatrix} y_1&y_2&y_3&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix} \begin{pmatrix} x_1&0&0&0&0\\ x_2&0&0&0&0\\ x_3&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \begin{pmatrix} x_1&0&0&0&0\\ x_2&0&0&0&0\\ x_3&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} = E_1'{-}E_3'.$$

Hence $E_1' = Y_1E_3'X_1 + E_3' = Y_1Y_2X_2X_1 + Y_1Y_3X_3X_1 + Y_2X_2 + Y_3X_3$ which we abbreviate to $E_1' = Y_{1,2}X_{2,1} + Y_{1,3}X_{3,1} + Y_2X_2 + Y_3X_3$. Hence $E_1 = Y_{1,1}X_{1,1}$.

Similar straightforward calculations show that $Y_2E_1X_2 = E_4{-}E_3$. Hence
$E_4 = E_3 + Y_2E_1X_2 = Y_1X_1 + Y_{2,1,1}X_{1,1,2}$. Hence
$E_4' = Y_{2,1,2}X_{2,1,2} + Y_{2,1,3}X_{3,1,2} + Y_{2,2}X_{2,2} + Y_{2,3}X_{3,2} + Y_3X_3$.

Similarly, $Y_1E_4'X_1 = E_2' - E_3'$. Hence $E_2' = Y_1E_4'X_1 + E_3'$. Hence
$E_2' = Y_{1,2,1,2}X_{2,1,2,1} + Y_{1,2,1,3}X_{3,1,2,1} + Y_{1,2,2}X_{2,2,1} + Y_{1,2,3}X_{3,2,1} + Y_{1,3}X_{3,1} + Y_2X_2 + Y_3X_3$. Hence
$E_2 = Y_{1,1}X_{1,1} + Y_{1,2,1,1}X_{1,1,2,1}$.

Now

$e_{1,1} = E_1 = Y_{1,1}X_{1,1}$.

$e_{4,4} = Y_2e_{1,1}X_2 = Y_{2,1,1}X_{1,1,2}$.

$e_{2,2} = Y_1 e_{4,4} X_1 = Y_{1,2,1,1} X_{1,1,2,1}$.
$e_{5,5} = E'_4 = Y_{2,1,2} X_{2,1,2} + Y_{2,1,3} X_{3,1,2} + Y_{2,2} X_{2,2} + Y_{2,3} X_{3,2} + Y_3 X_3$.
$e_{3,3} = Y_1 e_{5,5} X_1 = Y_{1,2,1,2} X_{2,1,2,1} + Y_{1,2,1,3} X_{3,1,2,1} + Y_{1,2,2} X_{2,2,1} + Y_{1,2,3} X_{3,2,1} + Y_{1,3} X_{3,1}$.

The interested reader can calculate the expressions for the remaining $e_{i,j}$.

4 The Abrams-Ánh-Pardo Theorem

The following is a straightforward consequence of the Chinese remainder theorem; the earliest mention of it that we have found is [12, p. 466, line 9].

4.1 Lemma. *Let m_1, m_2, $s \in \mathbb{Z}$. If $\gcd(m_1, s) = \gcd(m_2, s)$, then there exists $u \in \mathbb{Z}$ such that $um_1 \equiv m_2 \bmod s$ and $\gcd(u, s) = 1$.*

Proof. Note first that if $s = 0$ then $m_1 = \pm m_2$ and we can take $u = \pm 1$. Thus we may assume $s \neq 0$.

Let $g := \gcd(m_1, s) = \gcd(m_2, s)$. There exist n_1, $n_2 \in \mathbb{Z}$ such that $n_1 g = m_1$ and $n_2 g = m_2$. By Euclid's lemma, there exist k_1, $k_2 \in \mathbb{Z}$ such that $m_1 k_1 \equiv g \bmod s$ and $m_2 k_2 \equiv g \bmod s$.

Let $R := \mathbb{Z}_s$, $a := m_1 + s\mathbb{Z}$, $b := m_2 + s\mathbb{Z}$, $c := n_1 k_2 + s\mathbb{Z}$, $d := n_2 k_1 + s\mathbb{Z}$.

We have a, b, c, $d \in R$ such that $ad = b$ and $bc = a$, and it suffices to find some unit $x = u + s\mathbb{Z} \in R$ such that $ax = b$.

Eliminating b, we then have a, c, $d \in R$ such that $a(1{-}cd) = 0$, and it suffices to find some unit $x \in R$ such that $ax = ad$.

If $R = \mathbb{Z}_{p^m}$ where p is a prime number and $m \geqslant 1$, then either $a = 0$ and here we can take $x = 1$ as a solution, or $a \neq 0$ and then $1{-}cd$ is a zerodivisor, hence $(1{-}cd)^m = 0$, hence $1 - (1{-}cd)$ is a unit, hence cd is a unit, hence d is a unit, hence $x = d$ is a solution.

By the Chinese remainder theorem, R is a direct product of a finite number of rings of the form $\mathbb{Z}_{p^m}$ where p is a prime number and $m \geqslant 1$. By the preceding paragraph, we can find a suitable unit in each of these factors, and then form a suitable unit in R. This completes the proof. □

The following is also well known.

4.2 Lemma. *Let $r \in [2{\uparrow}\infty[$ and m, $m' \in [1{\uparrow}\infty[$. If $m' \equiv m \bmod (r{-}1)$, then $\mathrm{M}_{m'}(\mathbb{L}_r)$ and $\mathrm{M}_m(\mathbb{L}_r)$ are isomorphic as partially ordered rings with involution.*

Proof. Let $\mathbb{L} := \mathbb{L}_r$.

Consider first the case where there exists some $Y \in \mathrm{PU}({}^{m'}\mathbb{L}^{m})$. We then have a map $\mathrm{M}_{m'}(\mathbb{L}) \to \mathrm{M}_m(\mathbb{L})$, $\underset{m'\times m'}{M} \mapsto \underset{m\times m'}{Y^*} \underset{m'\times m'}{M} \underset{m'\times m}{Y}$, and it is easily seen to be a homomorphism of partially order rings with involution. Using Y^* in place of Y, we get a map in the reverse direction, and the two maps are mutually inverse. Thus it suffices to prove that $\mathrm{PU}({}^{m}\mathbb{L}^{m+r-1})$

is nonempty. Now $y_{[\![1\uparrow r]\!]} \in \mathrm{PU}({}^1\mathbb{L}^r)$ and $\mathbf{I}_{m-1} \in \mathrm{PU}_{m-1}(\mathbb{L}) = \mathrm{PU}({}^{m-1}\mathbb{L}^{m-1})$. Here, we have the diagonal sum

$$\underset{1\times r}{y_{[\![1\uparrow r]\!]}} \oplus \underset{(m-1)\times(m-1)}{\mathbf{I}_{m-1}} := \begin{pmatrix} y_{[\![1\uparrow r]\!]} & 0 \\ 0 & \mathbf{I}_{m-1} \end{pmatrix} \in \mathrm{PU}({}^{m}\mathbb{L}^{m+r-1}).$$

This completes the proof. □

4.3 The Abrams-Ánh-Pardo Theorem [17]. *Let $r \in [2\uparrow\infty]$ and $m, m', t \in [1\uparrow\infty[$. If $\gcd(m', r-1) = \gcd(m, r-1)$, then $\mathrm{M}_{m'}(\mathbb{L}_r^{\otimes t})$ and $\mathrm{M}_m(\mathbb{L}_r^{\otimes t})$ are isomorphic as partially ordered rings with involution, and, hence, $\mathrm{PU}_{m'}(\mathbb{L}_r^{\otimes t}) \cong \mathrm{PU}_m(\mathbb{L}_r^{\otimes t})$.*

Proof. For the purposes of this proof, let us write $\underset{\text{porwi}}{\cong}$ to indicate "isomorphic as partially ordered rings with involution".

We claim that $\mathrm{M}_{m'}(\mathbb{L}_r) \underset{\text{porwi}}{\cong} \mathrm{M}_m(\mathbb{L}_r)$. If $r = 2$, this holds by Lemma 4.2; thus we may assume that $r \geqslant 3$. By Lemma 4.1, there exists $u \in \mathbb{Z}$ such that $m'u \equiv m \bmod (r-1)$ and $\gcd(u, r-1) = 1$. By adding some multiple of $r-1$ to u, we may further assume that $u > r$. By Theorem 3.1, $\mathbb{L}_r \underset{\text{porwi}}{\cong} \mathrm{M}_u(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{L}_r$. It is well known that $\mathrm{M}_{m'}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{M}_u(\mathbb{Z}) \underset{\text{porwi}}{\cong} \mathrm{M}_{m'u}(\mathbb{Z})$. By Lemma 4.2, $\mathrm{M}_{m'u}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{L}_r \underset{\text{porwi}}{\cong} \mathrm{M}_m(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{L}_r$. It then follows that

$$\mathrm{M}_{m'}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{L}_r \underset{\text{porwi}}{\cong} \mathrm{M}_{m'}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{M}_u(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{L}_r \underset{\text{porwi}}{\cong} \mathrm{M}_{m'u}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{L}_r \underset{\text{porwi}}{\cong} \mathrm{M}_m(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{L}_r,$$

and the claim is proved.

Now, for $t \geqslant 2$, applying $(-) \otimes_{\mathbb{Z}} \mathbb{L}_r^{\otimes(t-1)}$ gives the desired result. □

5 The Brin-Higman-Thompson group $tV_{r,m}$ is $\mathrm{PU}_m(\mathbb{L}_r^{\otimes t})$

We now consider the Brin-Higman-Thompson groups. To lead into the definition gradually, we consider first the Higman-Thompson groups.

5.1 Definitions. Let $r \in [2\uparrow\infty[$ and $m \in [1\uparrow\infty[$. We now recall one of the constructions of the Higman-Thompson group $V_{r,m}$ from [11]; see also [20].

Let $\mathbb{L} := \mathbb{L}_r$. By Leavitt's normal-form result, the multiplicative submonoid $\langle\langle y_{[1\uparrow r]}\rangle\rangle$ of $\mathbb{L}$ is the free monoid on $y_{[1\uparrow r]}$. We view the Cartesian product $e_{[1\uparrow m]\times\{1\}} \times \langle\langle y_{[1\uparrow r]}\rangle\rangle$ as the product $e_{[1\uparrow m]\times\{1\}}\langle\langle y_{[1\uparrow r]}\rangle\rangle \subseteq \mathrm{M}_m(\mathbb{L})$.

Let A be any finite subset of $e_{[1\uparrow m]\times\{1\}}\langle\langle y_{[1\uparrow r]}\rangle\rangle$. For any $a \in A$, the ath *expansion of A* is

$$\partial_a(A) := (A \setminus \{a\}) \cup a y_{[1\uparrow r]} \subseteq e_{[1\uparrow m]\times\{1\}}\langle\langle y_{[1\uparrow r]}\rangle\rangle.$$

Let $\mathfrak{B}_m$ denote the smallest set of (finite) subsets of $e_{[1\uparrow m]\times\{1\}}\langle\langle y_{[1\uparrow r]}\rangle\rangle$ such that $e_{[1\uparrow m]\times\{1\}} \in \mathfrak{B}_m$ and $\mathfrak{B}_m$ is closed under taking expansions, that is, whenever $A \in \mathfrak{B}_m$ and $a \in A$, then $\partial_a(A) \in \mathfrak{B}_m$. An element of $\mathfrak{B}_m$ is called a *basis*.

For any $A \in \mathfrak{B}_m$, we can apply suitable expansions and arrive at an element $B \in \mathfrak{B}_m$ whose elements all have the same length, and then we have *all* of the elements of $e_{[1\uparrow m]\times\{1\}}\langle\langle y_{[1\uparrow r]}\rangle\rangle$ of this length. Any such B is called a *homogeneous* element of $\mathfrak{B}_m$.

We now consider the set of maps that are bijections between elements of $\mathfrak{B}_m$,

$$\Phi \coloneqq \{\varphi : A \to B,\, a \mapsto a^{\varphi} \mid A, B \in \mathfrak{B}_m,\ \varphi \text{ bijective}\}.$$

We shall construct $V_{r,m}$ using equivalence classes in Φ.

Suppose that $A \xrightarrow{\varphi} B$ is an element of Φ, and that $a \in A$, and let $b \coloneqq a^{\varphi}$. We define $\partial_a(\varphi) : \partial_a(A) \to \partial_b(B)$ in the natural way, that is, $\partial_a(\varphi)$ acts as φ for the bijection $A \setminus \{a\} \to B \setminus \{b\}$, and sends ay_s to by_s for each $s \in [1\uparrow r]$. We call $\partial_a(\varphi)$ the ath *expansion* of φ.

We define the set $V_{r,m}$ to consist of the equivalence classes in Φ obtained by identifying each element of Φ with all of its expansions.

We define a binary operation on $V_{r,m}$ as follows. For any φ, $\psi \in \Phi$, we can take successive expansions of φ^{-1} and ψ until they have homogeneous domains of the same length, in particular until they have the same domain. We may then compose $\varphi\,\psi$. We then obtain a well-defined binary operation on $V_{r,m}$. This concludes the definition of *the Higman-Thompson group* $V_{r,m}$.

Let us mention some subgroups of $V_{r,m}$. We give $e_{[1\uparrow m]\times\{1\}}\langle\langle y_{[1\uparrow r]}\rangle\rangle$ the lexicographic ordering. If A, $B \in \mathfrak{B}_m$ have the same size and $A \xrightarrow{\varphi} B$ is the unique bijective map that respects the induced orderings, then all the expansions of $A \xrightarrow{\varphi} B$ will also respect the induced orderings. The set of elements of $V_{r,m}$ represented by order-preserving maps form a subgroup of $V_{r,m}$, denoted $F_{r,m}$. Similarly, we can allow $A \xrightarrow{\varphi} B$ to be one of the maps that respects the induced orderings cyclically. We then get the subgroup $T_{r,m}$ of $V_{r,m}$ that contains $F_{r,m}$; see [11] or [8]. Here, $F_{2,1}$ and $T_{2,1}$ are Thompson's group F and T, respectively.

5.2 Definitions. Let m, $t \in [1\uparrow\infty[$ and $r \in [2\uparrow\infty[$. We now define the Brin-Higman-Thompson group $tV_{r,m}$ along the same lines as in the above definition of the Higman-Thompson groups.

Let $\mathbb{L} \coloneqq \mathbb{L}_r$. For $\ell \in [1\uparrow t]$, $k \in [1\uparrow r]$, we define $y_{\ell,k} \coloneqq 1^{\otimes(\ell-1)} \otimes y_k \otimes 1^{\otimes(t-\ell)} \in \mathbb{L}^{\otimes t}$ and $x_{\ell,k} \coloneqq y^*_{\ell,k} = 1^{\otimes(\ell-1)} \otimes x_k \otimes 1^{\otimes(t-\ell)} \in \mathbb{L}^{\otimes t}$. We view the Cartesian product

$$e_{[1\uparrow m]\times\{1\}} \times \langle\langle y_{[1\uparrow t]\times[1\uparrow r]}\rangle\rangle$$

as the product $e_{[1\uparrow m]\times\{1\}}\langle\langle y_{[1\uparrow t]\times[1\uparrow r]}\rangle\rangle \subseteq \mathrm{M}_m(\mathbb{L}^{\otimes t})$.

We consider t different kinds of *expansions* on a finite subset A of $e_{[1\uparrow m]\times\{1\}}\langle\langle y_{[1\uparrow t]\times[1\uparrow r]}\rangle\rangle$ as follows. For each $\ell \in [1\uparrow t]$, $a \in A$, let

$$\partial_{\ell,a}(A) \coloneqq A \setminus \{a\} \cup a y_{\{\ell\}\times[1\uparrow r]}.$$

Let $\mathfrak{B}_m^{(t)}$ be the smallest set of subsets of $e_{[1\uparrow m]\times\{1\}}\langle\langle y_{[1\uparrow t]\times[1\uparrow r]}\rangle\rangle$ such that $e_{[1\uparrow m]\times\{1\}} \in \mathfrak{B}_m^{(t)}$ and $\mathfrak{B}_m^{(t)}$ is closed under taking expansions of all kinds. The elements of $\mathfrak{B}_m^{(t)}$ are called *bases*.

A subset A of $e_{[1\uparrow m]\times\{1\}}\langle\langle y_{[1\uparrow t]\times[1\uparrow r]}\rangle\rangle$ is said to be *unitary* if it satisfies $\sum\limits_{a\in A}(a\cdot a^*) = \mathbf{I}_m$, and, for all a, $b \in A$, if $a \neq b$, then $a^*\cdot b = 0$ (and thus a is not a prefix of b). It is not difficult to show that every expansion of a unitary set is unitary. Notice that the question of multiplicity does not arise since, in a unitary set, no element is a prefix of another. Since $e_{[1\uparrow m]\times\{1\}}$ is a unitary set, we see that every basis is unitary.

Each $b \in e_{[1\uparrow m]\times\{1\}}\langle\langle y_{[1\uparrow t]\times[1\uparrow r]}\rangle\rangle$ can be expressed uniquely as a product $b = e_{i1}b_1\cdots b_t$, where each b_ℓ lies in $\langle\langle y_{\{\ell\}\times[1\uparrow r]}\rangle\rangle$, for each $\ell \in [1\uparrow t]$. The length of b_ℓ is called the *ℓ-length* of b.

A finite subset A of $e_{[1\uparrow m]\times\{1\}}\langle\langle y_{[1\uparrow t]\times[1\uparrow r]}\rangle\rangle$ is *multi-homogeneous* if, for each $\ell \in [1\uparrow t]$, all the elements of A have the same ℓ-length. Clearly, any finite subset of $e_{[1\uparrow m]\times\{1\}}\langle\langle y_{[1\uparrow t]\times[1\uparrow r]}\rangle\rangle$ can be expanded to a multi-homogeneous subset. In particular, any basis can be expanded to a multi-homogeneous basis, which will then have all the elements that have the specified ℓ-length, for each ℓ. (See also [13] Lemma 3.2.)

If B is a multi-homogeneous unitary set, then B lies in a unique multi-homogeneous basis C. If $B \neq C$, then, with respect to the partial order on $\mathrm{M}_m(\mathbb{L}^{\otimes t})$, we would have $\mathbf{I}_m = \sum\limits_{b\in B}(b\cdot b^*) < \sum\limits_{c\in C}(c\cdot c^*) = \mathbf{I}_m$, which is a contradiction. Thus, $B = C$. Hence, each multi-homogeneous unitary set is a basis. Hence, each unitary set can be expanded to a multi-homogeneous basis.

We now consider the set of maps that are bijections between elements of $\mathfrak{B}_m^{(t)}$,

$$\Phi := \{A \xrightarrow{\varphi} B \mid A, B \in \mathfrak{B}_m^{(t)},\ \varphi \text{ bijective}\}.$$

We construct the Brin-Higman-Thompson group $tV_{r,m}$ as the set of equivalence classes in Φ in the same way that we defined the Higman-Thompson group $V_{r,m}$ in Definitions 5.1.

For $t \geqslant 2$, the symbols $tF_{r,m}$ and $tT_{r,m}$ have not been assigned definitions; Brin [6, Remark 4.9] discusses his unsuccessful efforts to define a $2F_{2,1}$ with desirable properties.

5.3 Remarks. If $t = 2$, then every unitary set is a basis. To see this, suppose that B is unitary. It suffices to consider the case $m = 1$. Recall that each $b \in B$ has a factorization $b = b_1b_2 = b_2b_1$ with $b_i \in \langle\langle y_{\{i\}\times[1\uparrow r]}\rangle\rangle$ for $i = 1, 2$. Consider first the case where, for some $b \in B$, we have $b_1 = 1$ and $b_2 \neq 1$. Here, for each $c \in B$, if $c \neq b$, then $c_2 \cdot b^* = 0$ and $c_2 \neq 1$. Then B is a disjoint union of $y_{2,k}B_k$ for each $k \in [1\uparrow r]$. Each B_k is unitary, and by induction is a basis. Hence B is a basis. In the remaining case, for each $b \in B$, we have $b_1 \neq 1$, and then B is a disjoint union of $y_{1,k}B_k$ for each $k \in [1\uparrow r]$, and, by the same argument, B is a basis.

Similarly, if $t = 1$, then each unitary set is a basis.

For $t = 3$, $m = 1$, and $r = 2$, $\{y_{2,1}y_{3,1},\ y_{1,1}y_{2,1}y_{3,2},\ y_{1,1}y_{2,2},\ y_{1,2}y_{2,2}y_{3,1},\ y_{1,2}y_{3,2}\}$ is a unitary set that is not a basis.

We now come to our main result. In [17], Pardo found this result for Higman-Thompson groups, i.e., in the case $t = 1$.

5.4 Theorem. *Let $r \in [2{\uparrow}\infty[$ and $m,\, t \in [1{\uparrow}\infty[$. Then $\mathrm{PU}_m(\mathbb{L}_r^{\otimes t})$ is isomorphic to the Brin-Higman-Thompson group $tV_{r,m}$.*

Proof. We use the notation of Definitions 5.2.

For each $(A \xrightarrow{\varphi} B) \in \Phi$, we define $(A \xrightarrow{\varphi} B)^\alpha \coloneqq \sum\limits_{a \in A} (a \cdot (a^\varphi)^*) \in \mathrm{P}_m(\mathbb{L}^{\otimes t})$. It is readily verified that α has the same value on all the expansions of $(A \xrightarrow{\varphi} B)$. Also,

$$(B \xrightarrow{\varphi^{-1}} A)^\alpha = \sum_{a \in A} (a^\varphi \cdot a^*) = ((A \xrightarrow{\varphi} B)^\alpha)^*.$$

Thus we have a well-defined map of sets $\alpha\colon tV_{r,m} \to \mathrm{P}_m(\mathbb{L}^{\otimes t})$. It is a morphism of multiplicative monoids, since the identity maps to the identity, and, for any $(A \xrightarrow{\varphi} B)$, $(B \xrightarrow{\psi} C) \in \Phi$,

$$\begin{aligned}(A \xrightarrow{\varphi} B)^\alpha \cdot (B \xrightarrow{\psi} C)^\alpha &= \sum_{a \in A} (a \cdot (a^\varphi)^*) \cdot \sum_{b \in B} (b \cdot (b^\psi)^*) = \sum_{a \in A} (a \cdot (a^\varphi)^*) \cdot (a^\varphi \cdot ((a^\varphi)^\psi)^*) \\ &= \sum_{a \in A} (a \cdot (a^{\varphi\psi})^*) = (A \xrightarrow{\varphi\psi} C)^\alpha.\end{aligned}$$

In particular, $(B \xrightarrow{\varphi^{-1}} A)^\alpha = ((A \xrightarrow{\varphi} B)^\alpha)^{-1}$; as we have already seen that $(B \xrightarrow{\varphi^{-1}} A)^\alpha = ((A \xrightarrow{\varphi} B)^\alpha)^*$, we see that $(A \xrightarrow{\varphi} B)^\alpha \in \mathrm{PU}_m(\mathbb{L}^{\otimes t})$. In summary, we have a well-defined homomorphism $\alpha\colon tV_{r,m} \to \mathrm{PU}_m(\mathbb{L}^{\otimes t})$ which sends the equivalence class of $(A \xrightarrow{\varphi} B)$ to $\sum\limits_{a \in A} (a \cdot (a^\varphi)^*)$.

We next prove surjectivity of $\alpha\colon tV_{r,m} \to \mathrm{PU}_m(\mathbb{L}^{\otimes t})$.

Consider an arbitrary $u \in \mathrm{PU}_m(\mathbb{L}^{\otimes t})$. Since $u \in \mathrm{P}_m(\mathbb{L}^{\otimes t})$, we have an expression of u as a sum of elements of the form $e_{i,j} \cdot w \cdot z^*$ with $w,\, z \in \langle\langle y_{[1\uparrow t]\times[1\uparrow r]}\rangle\rangle$; notice that $e_{i,j} \cdot w \cdot z^* = (e_{i,1} \cdot w) \cdot (e_{j,1} \cdot z)^*$. By repeatedly inserting $\sum\limits_{k \in [1\uparrow r]} (y_{\ell,k} \cdot x_{\ell,k})$ $(= 1)$ between suitable w and z^*, we can arrange for all the ws to have the same ℓ-length, for each $\ell \in [1{\uparrow}t]$, and obtain an expression $u = \sum\limits_{a \in A} a \cdot p_a^*$, where $A \in \mathfrak{B}_m^{(t)}$, multi-homogeneous, and, for each $a \in A$, p_a is a sum of elements from $e_{[1\uparrow m]\times\{1\}}\langle\langle y_{[1\uparrow t]\times[1\uparrow r]}\rangle\rangle$. It is not difficult to see that if p_a has at least two summands then $e_{1,1} < (p_a^*) \cdot (p_a)$. Since it is a basis, A is a unitary set. Let $B \coloneqq \{p_a \mid a \in A\}$, and let $A \xrightarrow{\varphi} B$ be given by $a \mapsto p_a$. We shall show that B is a unitary set and that φ is injective. Let $a,\, a' \in A$. Since u is a unitary matrix, $u \cdot u^* = \mathbf{I}_m$, and, hence,

$$(p_a^*) \cdot (p_{a'}) = (a^* \cdot u) \cdot (u^* \cdot a') = a^* \cdot \mathbf{I}_m \cdot a' = a^* \cdot a' = e_{1,1}\delta_{a,a'}.$$

In particular, $(p_a^*)\cdot(p_a) = e_{1,1}$, and we see that p_a has exactly one summand, that is, $B \subseteq e_{[1\uparrow m]\times\{1\}}\langle\langle y_{[1\uparrow t]\times[1\uparrow r]}\rangle\rangle$. If $a' \neq a$, we have $(p_a^*)\cdot(p_{a'}) = 0$; in particular, φ is injective and, hence, bijective. We also have

$$\sum_{b\in B}(b\cdot b^*) = \sum_{a\in A}(p_a \cdot p_a^*) = (\sum_{a\in A} p_a\cdot a^*)\cdot(\sum_{a'\in A} a'\cdot p_{a'}^*) = u^*\cdot u = \mathbf{I}_m.$$

Thus B is a unitary set. We may expand B to a multi-homogeneous set B', and B' is again a unitary set, and is then a basis; see Definitions 5.2. By considering the corresponding expansion of φ^{-1} we get an expansion A' of A, which is again a basis, and an expansion $A' \xrightarrow{\varphi'} B'$ of φ. Then $(A' \xrightarrow{\varphi'} B') \in \Phi$ and $\sum\limits_{a'\in A'}(a'\cdot(a'^{\varphi'})^*) = \sum\limits_{a\in A}(a\cdot(a^{\varphi})^*) = u$. This proves that $\alpha\colon tV_{r,m} \to \mathrm{PU}_m(\mathbb{L}^{\otimes t})$ is surjective.

It remains to show that $\alpha\colon tV_{r,m} \to \mathrm{PU}_m(\mathbb{L}^{\otimes t})$ is injective. Suppose that $(A \xrightarrow{\varphi} B) \in \Phi$ and $\sum\limits_{a\in A}(a\cdot(a^{\varphi})^*) = \mathbf{I}_m$. For each $a \in A$, right multiplying the latter equation by a^{φ} gives $a = a^{\varphi}$. This proves that $\alpha\colon tV_{r,m} \to \mathrm{PU}_m(\mathbb{L}^{\otimes t})$ is injective. □

6 Higman's proof of invariance of r and $\gcd(m, r-1)$

We use the notation of Definitions 5.2.

6.1 Background. Here we quickly review the main points of Higman's analysis of conjugacy classes of finite subgroups of $tV_{r,m}$. The arguments are easily adapted from the articles [11], [16], both of which are set in a broader framework where two bases need not have a common ancestor.

For each $n \in [1\uparrow\infty[$, there exists some $B \in \mathfrak{B}_m^{(t)}$ with $|B| = n$ if and only if $n \equiv m \bmod (r-1)$.

By working with minimal common expansions, one can show that each finite subgroup H of $tV_{r,m}$ permutes the elements of some $B \in \mathfrak{B}_m^{(t)}$; moreover, the conjugacy class of H in $tV_{r,m}$ is then determined by the decomposition of B into H-orbits modulo identifying expansions of entire H-orbits. Here we will be counting the number of isomorphic copies of an orbit modulo $r-1$ except that we must distinguish between the number of isomorphic copies of an orbit being zero and being a nonzero multiple of $r-1$.

Conversely, for any finite group H, any finite H-set of cardinal congruent to $m \bmod (r-1)$ can be identified with some $B \in \mathfrak{B}_m^{(t)}$ and hence give a homomorphism from H to $tV_{r,m}$.

6.2 Conclusions. Let us now recall Higman's recovery of r and $\gcd(m, r-1)$ from the isomorphism class of $tV_{r,m}$.

Let p be a prime number, let $a \in [1\uparrow\infty[$, and let $\mathrm{cc}(p^a, tV_{r,m})$ denote the number of conjugacy classes of cyclic subgroups of $tV_{r,m}$ whose order divides p^a. Then $\mathrm{cc}(p^a, tV_{r,m})$ is an invariant of the isomorphism class of $tV_{r,m}$. It follows from Background 6.1 that

(23) $\mathrm{cc}(p^a, tV_{r,m})$ is equal to the number of sequences $n_{[\![0\uparrow a]\!]} \in [0\uparrow(r-1)]^{a+1}$ such that $\sum_{j=0}^{a}(n_j p^j) \equiv m \bmod (r-1)$, and $n_i \neq 0$ for some $i \in [0\uparrow a]$.

(i). Let p^a be the highest power of p dividing $r-1$. Let p^b denote the highest power of p dividing $\gcd(m, r-1)$. We shall show that $\mathrm{cc}(p^a, tV_{r,m}) = \sum_{i=0}^{b}(p^i r^{a-i})$.

By rewriting (23) ignoring leading zeros in $n_{[\![0\uparrow a]\!]}$, we see that

$$\mathrm{cc}(p^a, tV_{r,m}) = \sum_{i=0}^{a} |\{n_{[\![i\uparrow a]\!]} \in ([0\uparrow(r-1)])^{a+1-i} : \sum_{j=i}^{a}(n_j p^j) \equiv m \bmod (r-1), \text{ and } n_i \neq 0\}|.$$

If $b < a$, then $xp^{b+1} \equiv m \bmod (r-1)$ has no solutions, and now, since $b \leqslant a$, we see

$$\mathrm{cc}(p^a, tV_{r,m}) = \sum_{i=0}^{b} |\{n_{[\![i\uparrow a]\!]} \in ([0\uparrow(r-1)])^{a+1-i} : \sum_{j=i}^{a}(n_j p^j) \equiv m \bmod (r-1) \text{ and } n_i \neq 0\}|.$$

Here, the solutions of $n_i p^i \equiv m - \sum_{j=i+1}^{a}(n_j p^j) \bmod (r-1)$, $n_i \neq 0$, are given by all possible r^{a-i} choices for $n_{[\![(i+1)\uparrow a]\!]} \in ([0\uparrow(r-1)])^{a-i}$, and then p^i choices for n_i in the set $[1\uparrow(r-1)]$ of representatives of $\mathbb{Z}_{r-1}$. Hence $\mathrm{cc}(p^a, tV_{r,m}) = \sum_{i=0}^{b}(p^i r^{a-i})$, as claimed.

(ii). Now suppose that p does not divide $r-1$.

By arguing as in (i), we can show that $\mathrm{cc}(p^a, tV_{r,m}) \overset{(23)}{=} \sum_{i=0}^{a} r^{a-i}$. The case $a = 1$ shows that for all but finitely many primes p, there are exactly r conjugacy classes of subgroups of order exactly p in $tV_{r,m}$. It now follows that r is an invariant of the isomorphism class of $tV_{r,m}$.

It then follows from (i) that $\gcd(m, r-1)$ is also an invariant of the isomorphism class of $tV_{r,m}$.

7 The Bleak-Brin-Lanoue proof of invariance of t

We use the notation of Definitions 5.2.

In [4], Bleak-Lanoue developed arguments of Brin [6], [5] to prove that if $t'V_{2,1} \cong tV_{2,1}$ then $t' = t$. In this section we shall give a straightforward adaptation of their arguments to our language and show that if $t'V_{r',m'} \cong tV_{r,m}$, then $t' = t$. Here, $tV_{r,m}$ will be viewed as a group of self-homeomorphisms of a Cantor set $\mathcal{E}^{(t)}_{r,m}$; since the elements of $\mathcal{E}^{(t)}_{r,m}$ involve one-sided infinite words, we follow the standard practice of using *left* actions on *right*-infinite words.

7.1 Definitions. Let X be a topological space and let G be a group of self-homeomorphisms of X acting on the left, $g : x \mapsto g \cdot x$.

Let $x \in X$. We let $\mathcal{N}(x)$ denote the set of all open neighbourhoods of x in X, a downward directed system. We write $\mathrm{Fix}(x; G) \coloneqq \{g \in G \mid g \cdot x = x\} \leqslant G$. For each subset U of X, we write $\mathrm{Fix}(U; G) \coloneqq \bigcap\limits_{u \in U} \mathrm{Fix}(u; G) \leqslant G$. We write

$$\mathrm{Fix}^\circ(x; G) \coloneqq \bigcup_{U \in \mathcal{N}(x)} \mathrm{Fix}(U; G) \trianglelefteq \mathrm{Fix}(x; G) \quad \text{and} \quad \mathrm{Germs}(x; G) \coloneqq \mathrm{Fix}(x; G)/\mathrm{Fix}^\circ(x; G),$$

called the *groups of germs of G which fix x*.

We say that G is *locally dense* if, for each nonempty, open subset U of X and each $u \in U$, the closure of the orbit $\mathrm{Fix}(X \backslash U; G) \cdot u$ contains some nonempty, open subset of U.

To recall Rubin's theorem, we *copy* the following paragraph from [6] and add to Rubin's theorem a phrase from [4] about germs.

7.2 Background ([6], [4])**.** The following is essentially Theorem 3.1 of [19] where it is described as a combination of parts (a), (b) and (c) of Theorem 3.5 of [18]. The hypothesis that there be no isolated points was inadvertently omitted from [19] where it is needed. The terminology *locally dense* is not used in either [19] or [18]. However, in the absence of isolated points, it implies the notion of *locally moving* that is used in [19]. The absence of isolated points seems to correspond to the assumption of "no atoms" in the Boolean algebras of [18].

7.2.1 Rubin's theorem [19]. *Let G, resp. H, be a locally dense group of self-homeomorphisms of a locally compact, Hausdorff topological space without isolated points X, resp. Y. For each isomorphism $\varphi\colon G \to H$, there exists a unique homeomorphism $\tau\colon X \to Y$ with the property that, for each $g \in G$, $\varphi(g) = \tau g \tau^{-1}$, and then, for each $x \in X$,* $\mathrm{Germs}(x; G) \cong \mathrm{Germs}(\tau(x); H)$. □

7.3 Remarks. We shall recall below that $tV_{r,m}$ can be viewed as a locally dense group of self-homeomorphisms of a Cantor set $\mathcal{E}^{(t)}_{r,m}$ which is a compact, Hausdorff topological space without isolated points. We shall show that the set of isomorphism classes of groups given by $\{\mathrm{Germs}(\nu; tV_{r,m}) : \nu \in \mathcal{E}^{(t)}_{r,m}\}$ equals the set of isomorphism classes of groups given by $\{\mathbb{Z}^n : n \in [0{\uparrow}t]\}$. It will then follow from Rubin's theorem that if $t'V_{r',m'} \cong tV_{r,m}$, then $t' = t$.

In [6], Brin showed that $2V_{2,1} \ncong V_{r,m}$ by using Rubin's theorem and a delicate analysis of dynamics and orbit sizes. In an earlier article [5], Brin had considered germs to study Thompson's groups F and T. Bleak-Lanoue [4] combined these two approaches. They found the set of isomorphism classes of groups given by $\{\mathrm{Germs}(\nu; tV_{2,1}) : \nu \in \mathcal{E}^{(t)}_{2,1}\}$ and deduced that if $tV_{2,1} \cong t'V_{2,1}$, then $t' = t$. Our proof closely follows theirs.

7.4 Definitions. For each $\ell \in [1{\uparrow}t]$, let $\mathcal{E}_\ell$ denote the set of right-infinite words in $y_{\{\ell\}\times[1\uparrow r]}$. We view $\mathcal{E}_\ell$ as a metric space with $d(\beta,\gamma) := (1 + |\text{ largest common prefix of } \beta,\, \gamma\,|\,)^{-1}$. We view $e_{[1\uparrow m]\times\{1\}}$ as a discrete space. Let

$$\mathcal{E}^{(t)}_{r,m} := e_{[1\uparrow m]\times\{1\}} \times \mathcal{E}_1 \times \cdots \times \mathcal{E}_t,$$

and let $\mathcal{E}^{(t)}_{r,m}$ have the product topology. Then $\mathcal{E}^{(t)}_{r,m}$ is a compact, Hausdorff space without isolated points.

We write each $\nu = (e_{i,1}, \beta_1, \ldots, \beta_t) \in \mathcal{E}^{(t)}_{r,m}$ as a formal product $\nu = e_{i,1}\beta_1 \cdots \beta_t$, thought of as a limit of elements of $e_{i,1}\langle\langle y_{[1\uparrow t]\times[1\uparrow r]}\rangle\rangle$ which have long factors in each $\langle\langle y_{\{\ell\}\times[1\uparrow r]}\rangle\rangle$. With this formal-product viewpoint, we can define the set of elements of $e_{[1\uparrow m]\times\{1\}}\langle\langle y_{[1\uparrow t]\times[1\uparrow r]}\rangle\rangle$ that are *prefixes* of ν. Let $b \in e_{[1\uparrow m]\times\{1\}}\langle\langle y_{[1\uparrow t]\times[1\uparrow r]}\rangle\rangle$. We define the *shadow* of b, denoted $(b\blacktriangleleft)$, to be the set of all elements of $\mathcal{E}^{(t)}_{r,m}$ that have b as a prefix. If B is any basis, then $\mathcal{E}^{(t)}_{r,m}$ is the disjoint union of the shadows of the elements of B. Then $(b\blacktriangleleft)$ is a closed and open subset of $\mathcal{E}^{(t)}_{r,m}$, and the set of all shadows forms a basis for the open topology on $\mathcal{E}^{(t)}_{r,m}$.

Let $\mathbb{Z}[\mathcal{E}^{(t)}_{r,m}]$ denote the free abelian group on $\mathcal{E}^{(t)}_{r,m}$, with the elements of $\mathbb{Z}[\mathcal{E}^{(t)}_{r,m}]$ expressed as formal sums $\sum\limits_{\nu\in\mathcal{E}^{(t)}_{r,m}} n_\nu \cdot \nu$, with $n_\nu = 0$ for all but finitely many $\nu \in \mathcal{E}^{(t)}_{r,m}$. We think of the elements of $\mathbb{Z}[\mathcal{E}^{(t)}_{r,m}]$ as matrices that can be approximated arbitrarily closely by elements of $\mathrm{M}_m(\mathbb{L}^{\otimes t})\, e_{1,1}$. In this way, $\mathbb{Z}[\mathcal{E}^{(t)}_{r,m}]$ has the structure of a topological left $\mathrm{M}_m(\mathbb{L}^{\otimes t})$-module. We shall see that $\mathrm{PU}_m(\mathbb{L}^{\otimes t})$ acts on the $\mathbb{Z}$-basis $\mathcal{E}^{(t)}_{r,m}$.

Let $\nu \in \mathcal{E}^{(t)}_{r,m}$ and let $a,\, b \in e_{[1\uparrow m]\times\{1\}}\langle\langle y_{[1\uparrow t]\times[1\uparrow r]}\rangle\rangle$.

If b is not a prefix of ν, then $b^* \cdot \nu = 0$.

If $\nu \in (b\blacktriangleleft)$, we have $b^* \cdot \nu \in \mathcal{E}^{(t)}_{r,m}$, with first factor $e_{1,1}$, and we have $a \cdot b^* \cdot \nu \in \mathcal{E}^{(t)}_{r,m}$. The element $a \cdot b^* \in \mathrm{M}_m(\mathbb{L}^{\otimes t})$ uniquely determines $a,\, b \in e_{[1\uparrow m]\times\{1\}}\langle\langle y_{[1\uparrow t]\times[1\uparrow r]}\rangle\rangle$. We shall be viewing $a \cdot b^*$ as a homeomorphism $(b\blacktriangleleft) \to (a\blacktriangleleft)$, $\mu \mapsto a \cdot b^* \cdot \mu$, that replaces the prefix b with the prefix a. This homeomorphism is an identity map if and only if $a = b$. As homeomorphisms, $a \cdot b^*$ and $b \cdot a^*$ are mutually inverse.

Let $u \in \mathrm{PU}_m(\mathbb{L}^{\otimes t})$. Then there exist bases $A,\, B \in \mathfrak{B}^{(t)}_m$ and a bijective map $A \xrightarrow{\varphi} B$ such that $u = \sum\limits_{b\in B} (b^{\varphi^{-1}} \cdot b^*)$. Recall that the set of all such $(A \xrightarrow{\varphi} B)$ forms a single equivalence class for the smallest equivalence relation which identifies expansions. We view u as being this equivalence class, and we write $(A \xrightarrow{\varphi} B) \in u$. Let $\nu \in \mathcal{E}^{(t)}_{r,m}$. Then there is a unique element b_0 of B which is a prefix of ν, and $u \cdot \nu = b_0^{\varphi^{-1}} \cdot b_0^* \cdot \nu \in \mathcal{E}^{(t)}_{r,m}$. Left multiplication by u then gives a self-homeomorphism of $\mathcal{E}^{(t)}_{r,m}$ which acts as $b^{\varphi^{-1}} \cdot b^*$ on $(b\blacktriangleleft)$, for each $b \in B$. The action of u on $\mathcal{E}^{(t)}_{r,m}$ is trivial only if $u = 1$. Thus $tV_{r,m}$, identified with $\mathrm{PU}_m(\mathbb{L}^{\otimes t})$, is a group of self-homeomorphisms of $\mathcal{E}^{(t)}_{r,m}$.

7.5 Lemma (Brin [6])**.** *The group $tV_{r,m}$ of self-homeomorphisms of $\mathcal{E}^{(t)}_{r,m}$ is locally dense.*

Proof. Consider an open subset of $\mathcal{E}^{(t)}_{r,m}$ and then choose a smaller open subset of the form $(b\blacktriangleleft)$. Let $H \coloneqq \mathrm{Fix}(\mathcal{E}^{(t)}_{r,m} \setminus (b\blacktriangleleft); tV_{r,m})$. Then H acts on $(b\blacktriangleleft)$. Consider any $\nu \in (b\blacktriangleleft)$. We shall show that the closure of the orbit $H \cdot \nu$ is all of $(b\blacktriangleleft)$, which will show that $tV_{r,m}$ is locally dense.

Choose any $\nu' \in (b\blacktriangleleft)$. We want to approximate ν' arbitrarily closely by various $u \cdot \nu$ with $u \in H$. Choose an open neighbourhood of ν' in $(b\blacktriangleleft)$, and then choose a smaller open neighbourhood of the form $(b'\blacktriangleleft)$. It suffices to find $u \in H$ such that $u \cdot \nu \in (b'\blacktriangleleft)$.

Now b is a prefix of b' which is a prefix of ν'. Let B be a basis containing b, and expand B to a basis B' by expanding b towards b', that is, $B \setminus \{b\} \subseteq B'$ and $b' \in B'$. There exists a unique $a \in B'$ such that a is a prefix of ν, and then b is a prefix of a. Choose a bijective map $B' \xrightarrow{\varphi} B'$ that fixes $B \setminus \{b\}$ and sends b' to a. Then $(B' \xrightarrow{\varphi} B')$ lies in a unique $u \in tV_{m,r}$. Now $u \in H$, and u carries $(a\blacktriangleleft)$ to $(b'\blacktriangleleft)$. In particular, $u \cdot \nu \in (b'\blacktriangleleft)$, as desired. □

7.6 Conclusions. Let $\nu = (e_{i,1}, \beta_1, \ldots, \beta_t) \in \mathcal{E}^{(t)}_{r,m}$. We want to analyse $\mathrm{Germs}(\nu, tV_{r,m})$.

Consider any $u \in \mathrm{Fix}(\nu; tV_{r,m})$, and consider any $(A \xrightarrow{\varphi} B) \in u$. There exist a unique $a \in A$ and a unique $b \in B$ such that a and b are prefixes of ν. Since $u \in \mathrm{Fix}(\nu; tV_{r,m})$, we have $b^{\varphi^{-1}} \cdot b^* \cdot \nu = \nu$, or, equivalently, $(b^{\varphi^{-1}})^* \cdot \nu = b^* \cdot \nu$, or, equivalently, we have two factorizations $\nu = b^{\varphi^{-1}} \cdot \nu' = b \cdot \nu'$ with the same *tail* ν'. Thus $a^\varphi = b$. Notice that u acts as $a \cdot b^*$ on $(b\blacktriangleleft)$. Moreover, any element of the coset $u \cdot \mathrm{Fix}((b\blacktriangleleft); tV_{r,m})$ will also act as $a \cdot b^*$ on $(b\blacktriangleleft)$. If in place of $(A \xrightarrow{\varphi} B)$ we choose an expansion of $(A \xrightarrow{\varphi} B)$ in u, then in place of $a \cdot b^*$ we get an element of the form $a \cdot c \cdot c^* \cdot b^*$, where c is a prefix of ν'; we then say that $a \cdot c \cdot c^* \cdot b^*$ is an *expansion of* $a \cdot b^*$ *towards* ν. Here all the elements of the coset $u \cdot \mathrm{Fix}(((b \cdot c)\blacktriangleleft); tV_{r,m})$ act as $a \cdot c \cdot c^* \cdot b^*$ on $((b \cdot c)\blacktriangleleft)$. Thus longer and longer expansions of $a \cdot b^*$ towards ν determine larger and larger cosets within the germ of u.

This leads us to consider the set $\{a \cdot b^* \mid a, b \in e_{i,1}\langle\langle y_{[1\uparrow t]\times[1\uparrow r]}\rangle\rangle,\ a \cdot b^* \cdot \nu = \nu\}$ modulo the smallest equivalence relation that identifies expansions towards ν. This set of equivalence classes is a group, denoted $\mathrm{rep}(\nu)$, with the multiplication that is induced from the multiplication of compatible representatives, $(a \cdot b^*) \cdot (b \cdot c^*) = a \cdot c^*$.

It follows from the foregoing that we have an injective homomorphism

$$\mathrm{Germs}(\nu; tV_{r,m}) \to \mathrm{rep}(\nu).$$

To see that this homomorphism is also surjective, notice that it is a straightforward matter to construct an element of $tV_{r,m}$ which carries one given prefix of ν to another as follows. We choose one basis containing each, and if the bases are not the same size, the smaller basis can be expanded without removing the specified prefix until the bases are the same size. Then we choose any bijection between the bases that carries the first chosen prefix of ν to the second chosen prefix of ν.

The next step is to compute $\mathrm{rep}(\nu)$.

Let $\ell \in [1\uparrow t]$. We say that β_ℓ is *rational* if there exist w_ℓ, $z_\ell \in \langle\langle y_{\{\ell\}\times[1\uparrow r]}\rangle\rangle$ with $z_\ell \neq 1$ such that $\beta_\ell = w_\ell \cdot z_\ell^\infty$; otherwise, β_ℓ is *irrational.*

For the purpose of exposition, we may assume that there exists some $n \in [0{\uparrow}t]$ such that β_ℓ is rational if $\ell \in [1{\uparrow}n]$ and β_ℓ is irrational if $\ell \in [(n+1){\uparrow}t]$. Thus we may write

$$\nu = (e_{i,1}, w_1{\cdot}z_1^\infty, \ldots, w_n{\cdot}z_n^\infty, \beta_{n+1}\cdots\beta_t),$$

and we may further assume that each z_ℓ is not a proper power.

With u, a and b as before, we can expand the prefixes a, b of ν to longer prefixes and arrange that

$$\begin{aligned}&a^*\cdot\nu = b^*\cdot\nu = (e_{1,1}, z_1^\infty, \ldots, z_n^\infty, \beta'_{n+1}, \cdots, \beta'_t), \text{ and then}\\ &a = e_{i,1}(w_1{\cdot}z_1^{q_1})\cdots(w_n{\cdot}z_n^{q_n})w_{n+1}\cdots w_t,\\ &b = e_{i,1}(w_1{\cdot}z_1^{q'_1})\cdots(w_n{\cdot}z_n^{q'_n})w_{n+1}\cdots w_t,\end{aligned}$$

where $q_{[\![1\uparrow n]\!]}, q'_{[\![1\uparrow n]\!]} \in ([0{\uparrow}\infty[)^n$ and $\beta_\ell = w_\ell\cdot\beta'_\ell$, $\ell\in[(n+1){\uparrow}t]$; notice that irrationality implies that tails match up with unique prefixes, while the fact that z_ℓ is not a proper power implies that the tail z_ℓ^∞ matches up with a prefix that is unique up to right multiplication by a power of z_ℓ. Then

$$a\cdot b^* = e_{i,i}(w_1{\cdot}z_1^{q_1}{\cdot}z_1^{*q'_1}{\cdot}w_1^*)\cdots(w_n{\cdot}z_n^{q_n}{\cdot}z_n^{*q'_n}{\cdot}w_n^*). \tag{24}$$

Thus every element of $\mathrm{rep}(\nu)$ contains an element of the form (24). Conversely, every element of the form (24) lies in some element of $\mathrm{rep}(\nu)$.

It is now straightforward to show that $\mathrm{Germs}(\nu, tV_{r,m}) \cong \mathbb{Z}^n$, with elements represented by the expression (24) corresponding to $(q_1-q'_1, \ldots, q_n-q'_n) \in \mathbb{Z}^n$.

Let us now show that, for each $n \in [0{\uparrow}t]$, there exists some $\nu \in \mathcal{E}^{(t)}_{r,m}$ such that $\mathrm{Germs}(\nu, tV_{r,m}) \cong \mathbb{Z}^n$. Since there are only countably many rational right-infinite words, there exists some $\nu = (e_{1,1}, y_{1,1}^\infty, y_{2,1}^\infty, \ldots, y_{n,1}^\infty, \beta_{n+1}, \cdots, \beta_n) \in \mathcal{E}^{(t)}_{r,m}$, such that, for each $\ell \in [(n+1){\uparrow}t]$, β_ℓ is irrational. By the foregoing, $\mathrm{Germs}(\nu, tV_{r,m}) \cong \mathbb{Z}^n$.

We have now shown that the set of isomorphism classes of groups given by the set $\{\mathrm{Germs}(\nu; tV_{r,m}) : \nu \in \mathcal{E}^{(t)}_{r,m}\}$ equals the set of isomorphism classes of groups given by the set $\{\mathbb{Z}^n : n \in [0{\uparrow}t]\}$. It now follows from Theorem 7.2.1 that if $t'V_{r',m'} \cong tV_{r,m}$, then $t' = t$.

In fact, we can say more. The class of groups isomorphic to $\mathbb{Z}^n$ is closed under taking subgroups of finite index. Any (conjecturally rare) subgroup of finite index in $tV_{r,m}$ is a locally dense group of self-homeomorphisms of $\mathcal{E}^{(t)}_{r,m}$ and has the same $t+1$ types of germs as $tV_{r,m}$. Thus, if $t'V_{r',m'}$ and $tV_{r,m}$ are commensurable, then $t' = t$.

8 Summary

The following builds on work of Abrams, Ánh, Bleak, Brin, Higman, Lanoue, Pardo, and Thompson.

8.1 Theorem. *Let* $r_1,\, r_2 \in [2\uparrow\infty[\,,\ m_1,\, m_2,\, t_1,\, t_2 \in [1\uparrow\infty[\,.$ *The following are equivalent.*

(a). $r_1 = r_2$, $\gcd(m_1, r_1{-}1) = \gcd(m_2, r_2{-}1)$, *and* $t_1 = t_2$.

(b). $\mathrm{M}_{m_1}(\mathbb{L}_{r_1}^{\otimes t_1})$ *and* $\mathrm{M}_{m_2}(\mathbb{L}_{r_2}^{\otimes t_2})$ *are isomorphic as partially ordered rings with involution.*

(c). $t_1 V_{r_1,m_1} \cong t_2 V_{r_2,m_2}$.

Proof. (a) $\Rightarrow$ (b) by Theorem 4.3.

(b) $\Rightarrow$ (c). Suppose that $\mathrm{M}_{m_1}(\mathbb{L}_{r_1}^{\otimes t_1})$ and $\mathrm{M}_{m_2}(\mathbb{L}_{r_2}^{\otimes t_2})$ are isomorphic as partially ordered rings with involution. Then $\mathrm{PU}_{m_1}(\mathbb{L}_{r_1}^{\otimes t_1}) \cong \mathrm{PU}_{m_2}(\mathbb{L}_{r_2}^{\otimes t_2})$. Now by Theorem 5.4, $t_1 V_{r_1,m_1} \cong \mathrm{PU}_{m_1}(\mathbb{L}_{r_1}^{\otimes t_1}) \cong \mathrm{PU}_{m_2}(\mathbb{L}_{r_2}^{\otimes t_2}) \cong t_2 V_{r_2,m_2}$.

(c) $\Rightarrow$ (a). Suppose that $t_1 V_{r_1,m_1} \cong t_2 V_{r_2,m_2}$. By Conclusions 7.6, $t_1 = t_2$. By Conclusions 6.2, $r_1 = r_2$ and $\gcd(m_1, r_1{-}1) = \gcd(m_2, r_2{-}1)$. □

8.2 Remarks. Let $r \in [2\uparrow\infty[\,,\ m,\, t \in [1\uparrow\infty[\,,$ and let $R := \mathrm{M}_m(\mathbb{L}_r^{\otimes t})$.

(i). Pere Ara has shown that r and $\gcd(m, r{-}1)$ are invariants of the isomorphism class of R within the class of rings; we sketch his argument in (ii) below.

Also it follows from work of Jason Bell and George Bergman that t is an invariant of the isomorphism class of R within the class of rings; see (iii) below.

Hence the conditions in Theorem 8.1 are further equivalent to

(b′). $\mathrm{M}_{m_1}(\mathbb{L}_{r_1}^{\otimes t_1})$ *and* $\mathrm{M}_{m_2}(\mathbb{L}_{r_2}^{\otimes t_2})$ *are isomorphic as rings.*

Here, (b) $\Rightarrow$ (b′) is clear, while (b′) $\Rightarrow$ (a) is a consequence of the foregoing results of Ara, Bell and Bergman. Consequently, with r and t fixed, and m varying, the set of isomorphism classes of the rings $\mathrm{M}_m(\mathbb{L}_r^{\otimes t})$ is in bijective correspondence with the set of positive divisors of $r{-}1$.

(ii). Here we record the argument of Ara.

Let $i \in \mathbb{Z}$ and let A be any ring. We shall use the homotopy algebraic K-theory groups, $\mathrm{KH}_i(A)$, introduced by Weibel [21].

When we apply the Ara-Brustenga-Cortiñas result [2, Theorem 8.6] to the quiver E with one vertex and r loops, where $L_A(E) := \mathbb{L}_r \otimes_{\mathbb{Z}} A$, we obtain an exact sequenece

$$\mathrm{KH}_i(A) \xrightarrow{\text{mult. by } r-1} \mathrm{KH}_i(A) \xrightarrow{\text{natural}} \mathrm{KH}_i(\mathbb{L}_r \otimes_{\mathbb{Z}} A) \to \mathrm{KH}_{i-1}(A) \xrightarrow{\text{mult. by } r-1} \mathrm{KH}_{i-1}(A).$$

If $A = \mathbb{Z}$, then $\mathrm{KH}_i(A) = 0$ if $i < 0$, while $\mathrm{KH}_0(A) \cong \mathbb{Z}$, with the class of A in $\mathrm{KH}_0(A)$ corresponding to 1; see [21, Example 1.4]. It then follows by induction on t that if $A = \mathbb{L}_r^{\otimes t}$, then $\mathrm{KH}_i(A) = 0$ if $i < 0$, while $\mathrm{KH}_0(A) \cong \mathbb{Z}_{r-1}$ with the class of A in $\mathrm{KH}_0(A)$ corresponding to the class of 1 in $\mathbb{Z}_{r-1}$.

Recall that R denotes $\mathrm{M}_m(\mathbb{L}_r^{\otimes t})$. It now follows that $\mathrm{KH}_0(R) \cong \mathbb{Z}_{r-1}$ with the class of R corresponding to the class of m in $\mathbb{Z}_{r-1}$. Thus $\mathrm{KH}_0(R)$ is cyclic of order $r{-}1$ and the class of R in $\mathrm{KH}_0(R)$ has order $\frac{r-1}{\gcd(m,\, r-1)}$. Hence r and $\gcd(m, r{-}1)$ are invariants of the isomorphism class of the ring R, as desired.

(iii). Here we build on unpublished work of Bell and Bergman.

Let K be a commutative field, and let $\Gamma := K \otimes_{\mathbb{Z}} \mathbb{L}_r$.

Let Γ^{op} denote the opposite ring of Γ. Let $\Gamma^{\mathrm{e}} := \Gamma \otimes_K \Gamma^{\mathrm{op}}$. Where K is understood, the projective dimension of the left Γ^{e}-module Γ is denoted $\dim \Gamma$. Bergman-Dicks [3, (17) and (4)] showed that there exists an exact sequence of left Γ^{e}-modules $0 \to (\Gamma^{\mathrm{e}})^r \to \Gamma^{\mathrm{e}} \to \Gamma \to 0$. Thus $\dim \Gamma \leqslant 1$.

Straightforward normal-form arguments show that the element $x_1 - 1$ of Γ does not have a left inverse and is not a left zerodivisor; thus $\mathrm{w.\,gl.\,dim}\, \Gamma \geqslant 1$.

Since $\dim \Gamma \leqslant 1$ and $\mathrm{w.\,gl.\,dim}\, \Gamma \geqslant 1$, the Eilenberg-Rosenberg-Zelinsky result [9, Proposition 10(2)] implies that, for each K-algebra Λ, $\mathrm{l.\,gl.\,dim}(\Lambda \otimes_K \Gamma) = 1 + \mathrm{l.\,gl.\,dim}(\Lambda)$, that is, $\mathrm{l.\,gl.\,dim}(\Lambda \otimes_{\mathbb{Z}} \mathbb{L}_r) = 1 + \mathrm{l.\,gl.\,dim}(\Lambda)$.

Now $\mathrm{l.\,gl.\,dim}(K \otimes_{\mathbb{Z}} R) = \mathrm{l.\,gl.\,dim}(\mathrm{M}_m(K) \otimes_{\mathbb{Z}} \mathbb{L}_r^{\otimes t}) = t$, by induction on t. Thus t is an invariant of the isomorphism class of the ring R.

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