RIGIDITY OF CERTAIN HOLOMORPHIC FOLIATIONS

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Abstract. There is a well-known rigidity theorem of Y. Ilyashenko ([10]) for (singular) holomorphic foliations in $\mathbb{CP}^2$ and also the extension given in [7]. We present here a different generalization of the result of Ilyashenko: some cohomological and (generic) dynamical conditions on a foliation $\mathcal{F}$ on a fibred surface S imply the d-rigidity, i.e. any topologically trivial deformation of $\mathcal{F}$ is also analytically trivial. We particularize this result to the case of ruled surfaces. In this context, the foliations not verifying the cohomological hypothesis above were completely classified in a precedent work by X. Gómez-Mont ([8]). Hence we obtain a (generic) characterization of non d-rigid foliations in ruled surfaces. We point out that the widest class of them are Riccati foliations.

1. Introduction and main results

There is a well-known theorem due to Y. Ilyashenko (see [10]) assuring the topological rigidity of a wide class of foliations on $\mathbb{CP}^2$. Later, X. Gómez-Mont and L. Ortíz-Bobadilla generalize this result in [7] to many foliations on arbitrary algebraic surfaces admitting an invariant ample curve. We can also cite the results of [9], [13] and [14] on the existence and description of a versal space of equisingular unfoldings. Following the same ideas, it is possible to adapt the proof of Ilyashenko rigidity theorem to many foliations on fibred surfaces admitting an invariant fibre. More concretely, we present the main result:

Theorem A. *Let S be a fibred compact complex surface (i.e. admitting a submersion over a complex curve) and let $\mathcal{F}$ be a holomorphic foliation on S having reduced singularities and verifying the following conditions:*

(a) There is an invariant irreducible fibre F whose holonomy group is rigid and such that all the singularities of $\mathcal{F}$ over F are hyperbolic.

(b) $H^1(S, \mathcal{O}_S(T\mathcal{F})) = 0$, where $T\mathcal{F}$ is the tangent bundle of the foliation $\mathcal{F}$.

Then, $\mathcal{F}$ is d-rigid, i.e. any topologically trivial holomorphic deformation $\mathcal{F}_t$ of $\mathcal{F}$ is also analytically trivial.

Remark 1.1. The rigidity of the holonomy group $G \hookrightarrow \mathrm{Diff}(\mathbb{C}, 0)$ means that if $\psi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is a germ of homeomorphism such that for any $g \in G$ the composition $\psi_* g := \psi \circ g \circ \psi^{-1}$ is an element of $\mathrm{Diff}(\mathbb{C}, 0)$, then ψ is a conformal or anticonformal mapping. In particular, if ψ preserves the orientation then ψ is a biholomorphism. There are many situations assuring the rigidity of G, see [4, 10]. For instance, if G is not abelian and the linear parts of the elements in G form a dense subset of $\mathbb{C}^*$ then G is rigid. Another situation implying rigidity is when G

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is non solvable (see [15]). We point out that this hypothesis on the rigidity of G was also required in the precedent versions of that theorem [10, 7].

We will particularize the above result to the case of a ruled surface $\pi : S \to B$. A complete study of the coholomogy groups $H^1(S, \mathcal{O}_S(L))$ for any line bundle $L \to S$ over a ruled surface is done in [8]. It follows from it that if $\mathcal{F}$ is a holomorphic foliation on S with isolated singularities then $H^1(S, \mathcal{O}_S(T\mathcal{F})) = 0$ except in the following four exceptional cases:

(i) The foliation $\mathcal{F}$ is the fibration $\pi : S \to B$, which is unique unless for the product $S = \mathbb{CP}^1 \times \mathbb{CP}^1$.
(ii) The foliation $\mathcal{F}$ is transverse to a the fibration outside a finite number of invariant fibres, i.e. when $\mathcal{F}$ is a Riccati foliation.
(iii) The foliation $\mathcal{F}$ is a regular foliation (i.e. without singularities) over a ruled surface whose base B is an elliptic curve E. Moreover, if a such foliation $\mathcal{F}$ is not in the precedent two cases then S is a non-ramified finite quotient of the product $\mathbb{CP}^1 \times E$. This class includes the *turbulent* foliations studied by M. Brunella in [3] and [2].
(iv) The foliations on the first Hirzebruch surface obtained blowing-up a regular point of certain foliations on $\mathbb{CP}^2$ (see [8]).

From Theorem A and the precedent considerations we immediately deduce:

Corollary B. *Let S be a ruled surface and let $\mathcal{F}$ be a holomorphic foliation on S with reduced singularities. Assume that $\mathcal{F}$ possesses an invariant fibre F whose holonomy group is rigid and the singularities of $\mathcal{F}$ over F are hyperbolic. If $\mathcal{F}$ does not belong to the classes (i)-(iv) listed above then $\mathcal{F}$ is d-rigid.*

Thus, we can conclude saying that Riccati foliations form the widest class of non rigid foliations on ruled surfaces. Hence it is interesting to determine its analytic moduli space, see [12].

2. Deformations and unfoldings

In order to explain Theorem A, we begin recalling the difference between topologically trivial deformations and unfoldings and some well-known facts about them. In order to simplify the exposition we restrict ourselves to the case in that the parameter space is a disk $\Delta \ni 0$, o more precisely, the germ $(\Delta, 0)$.

Let S be a complex surface and let $\mathcal{F}$ be a (singular) holomorphic foliation on S. A deformation of $\mathcal{F}$ is a deformation $\pi : \mathcal{S} \to \Delta$ of S and a holomorphic foliation by curves $\mathcal{F}^\Delta$ on $\mathcal{S}$ such that:

- the singular set Σ of $\mathcal{F}^\Delta$ has codimension greater than 1 and none of its irreducible components is contained in a fibre of π;
- the leaves of $\mathcal{F}^\Delta$ are contained in the fibres of π, $\pi^{-1}(0) = S$ and $\mathcal{F}^\Delta_{|S} = \mathcal{F}$.

The deformation $\mathcal{F}^\Delta$ is said to be topologically trivial if there exists a homeomorphism $\Phi : \mathcal{S} \to S \times \Delta$ commuting with the projections over Δ such that $\Phi^*(\mathcal{F}^{\text{triv}}) = \mathcal{F}^\Delta$ (i.e. Φ send leaves of $\mathcal{F}^\Delta$ into leaves of the constant foliation $\mathcal{F}^{\text{triv}} = \mathcal{F} \times \Delta$ on $S \times \Delta$). An unfolding of $\mathcal{F}$ is a holomorphic foliation $\mathcal{G}$ of codimension one on the total space $\mathcal{S}$ of some deformation $\pi : \mathcal{S} \to \Delta$ such that the singular set of $\mathcal{G}$ has codimension greater than 1, none of its irreducible components is contained in a fibre of π, $\pi^{-1}(0) = S$ and $\mathcal{G}_{|S} = \mathcal{F}$.

Each unfolding $\mathcal{G}$ of $\mathcal{F}$ determines a deformation in the obvious way: its leaves are the intersections of the leaves of $\mathcal{G}$ with the fibres of π. Clearly, a priori, an unfolding has a more rich structure than a deformation because in the former we have prescribed how the one-dimensional leaves are put together in order to form the two-dimensional leaves of the unfolding. In section 4, we recall some known facts about unfoldings of holomorphic foliations. The hypothesis on the singularities and the cohomological assumptions in Theorem A will imply that any unfolding of $\mathcal{F}$ is analytically trivial. Hence, the key step in the proof of Theorem A is a partial converse of the above remark *unfolding implies deformation.* More concretely, we have the following result in the spirit of Ilyashenko's rigidity theorem, although the proof presented here is inspired by the exposition of [7].

Theorem 2.1. *Let $\mathcal{F}$ be a holomorphic foliation in a complex surface S admitting an invariant curve C such that*

(a) each irreducible component of C contains a hyperbolic singularity of $\mathcal{F}$ and the holonomy group of each connected component of $C^* = C \setminus \mathrm{Sing}\,(\mathcal{F})$ is rigid;

(b) each leaf L of $\mathcal{F}$ verifies $\overline{L} \cap C^* \neq \emptyset$, unless maybe a finite set of closed leaves.

Then every topologically trivial deformation of $\mathcal{F}$ is underlying to an unfolding.

Proof. Let $\mathcal{F}^\Delta$ be a topologically trivial deformation of $\mathcal{F}$ on $\pi : \mathcal{S} \to \Delta$. Let Σ be the singular set of $\mathcal{F}^\Delta$, which is an analytic set of $\mathcal{S}$ of dimension ≤ 1. Consider the open subset $\mathcal{S}^* = \mathcal{S} \setminus \Sigma$ of $\mathcal{S}$. Since $\mathcal{F}^\Delta$ is topologically trivial there exists a homeomorphism $\Phi : \mathcal{S} \to S \times \Delta$ commuting with the projection over Δ such that $\Phi^*(\mathcal{F}^{\mathrm{triv}}) = \mathcal{F}^\Delta$. Let us consider the topological foliation $\mathcal{G}^{\mathrm{top}} = \Phi^*(\mathcal{F} \times \Delta)$ on $\mathcal{S}^*$ of codimension one whose leaves are $\Phi^{-1}(L \times \Delta)$, where L is any leaf of $\mathcal{F}$. Notice that the deformation induced by the *topological unfolding* $\mathcal{G}^{\mathrm{top}}$ is just the holomorphic deformation $\mathcal{F}^\Delta$. We shall see that $\mathcal{G}^{\mathrm{top}}$ is a holomorphic foliation of codimension one on $\mathcal{S}^*$ which will extend holomorphically to the whole $\mathcal{S}$ thanks to Hartogs' theorem, concluding in this way the proof.

Let $\{\mathcal{U}_i, \varphi_i\}$ be a system of distinguished maps of $\mathcal{F}^\Delta$ restricted to $\mathcal{S}^*$, i.e. $\mathcal{U}_i$ are open set of $\mathcal{S}^*$ and $\varphi_i : \mathcal{U}_i \to V_i \subset \mathbb{C}^2$ are submersions defining $\mathcal{F}^\Delta_{|\mathcal{U}_i}$. Let us define the local (topological) foliation $\mathcal{G}_i = (\varphi_i)_*(\mathcal{G}^{\mathrm{top}})$ on V_i.

Since $\mathcal{F}^\Delta$ is topologically trivial, without loos of generality we can assume that $\mathcal{U}_i \cong U_i \times \Delta$, $V_i = V_i^0 \times \Delta$ and $\varphi_i(p,t) = (\varphi_i^t(p), t)$. On the other hand, the holomorphy of $\mathcal{G}^{\mathrm{top}}$ is a local property. Hence, it suffices to show that $\mathcal{G}_i$ is holomorphic because $\mathcal{G}^{\mathrm{top}}_{|\mathcal{U}_i} = \varphi_i^* \mathcal{G}_i$. In order to show this, we proceed in four steps:

(I) If $\mathcal{U}_i \cap C \neq \emptyset$ then the leaf C_i of $\mathcal{G}_i$ corresponding to $\Phi^{-1}(C \times \Delta)$ is holomorphic. To see this we will use the existence of a hyperbolic singularity of $\mathcal{F}$ on the corresponding connected component of C^*.

(II) Using the rigidity of the holonomy group G, we will see that $\mathcal{G}_i$ is holomorphic in a neighborhood of C_i.

(III) Hypothesis (b) implies that $\mathcal{G}_i$ is holomorphic outside a finite set of closed leaves.

(IV) Finally, using a variant of Riemann extension theorem we will conclude that $\mathcal{G}_i$ is holomorphic in the whole open set $V_i \subset \mathbb{C}^2$.

(I) Let $\Sigma_i = \Sigma_i^0 \times \Delta$ be a transverse section to $\mathcal{F}^\Delta$ in $\mathcal{U}_i$. In fact, we could identify Σ_i with V_i through the restriction to Σ_i of the map φ_i. Consider $p_i \in C \cap \Sigma_i$ and $\psi_i : (\Sigma_i, p_i) \to (\Sigma_i, p_i)$ the holonomy transformation associated to a loop turning around a hyperbolic singularity of $\mathcal{F}$ on the connected component of C^* containing

p_i. We can write $\psi_i(z,t) = (\psi_i^t(z), t)$ if $z \in \Sigma_i^0 \subset \mathbb{C}$ and $t \in \Delta$. The hyperbolicity of that singularity on C^* implies that

$$\left| \frac{\partial}{\partial z}\Big|_{z=0} \psi_i^t(z) \right| < 1 \tag{1}$$

holds for $t = 0$, where the coordinates of the point p_i are $(z,t) = (0,0)$. Hence, if Δ is small enough then (1) also holds for every $t \in \Delta$. Let Z_i be the fixed set of ψ_i. Thanks to (1), we can apply the Implicit Function Theorem to conclude that Z_i is a regular curve in V_i which is locally the graph of a holomorphic function $z_i : \Delta \to \Sigma_i^0 \subset \mathbb{C}$ with $z_i(0) = 0$. It is easy to see that $C_i = \varphi_i(Z_i) \subset V_i$ and consequently, the leaf C_i is holomorphic.

(II) Since $z_i : \Delta \to \Sigma_i^0$ is holomorphic, we can change the coordinates in order to have $z_i \equiv 0$ and therefore $C_i = \varphi_i(\{0\} \times \Delta)$. Let us consider the local homeomorphism $\bar{\phi}_i^t : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ which makes commutative the following diagram:

$$\begin{array}{ccc} \mathcal{U}_i \cap S_t & \xrightarrow{\phi_t} & U_i \\ {\scriptstyle \varphi_i^t}\downarrow & & \downarrow{\scriptstyle \varphi_i^0} \\ (\mathbb{C},0) & \xrightarrow[\bar{\phi}_i^t]{} & (\mathbb{C},0) \end{array}$$

where $S_t = \pi^{-1}(t)$ and $\phi_t = \Phi_{|S_t}$. By construction, $\bar{\phi}_i^t$ is a topological conjugation of the holonomies of $C_t = \phi_t^{-1}(C)$ and C with respect to the foliations by curves $\mathcal{F}_t = \mathcal{F}^{\Delta}_{|S_t}$ and $\mathcal{F}$. Hypothesis (b) implies that $\bar{\phi}_i^t(z)$ is holomorphic with respect to the coordinate z. We shall see that $\bar{\phi}_i^t$ is also holomorphic with respect to the coordinate $t \in \Delta$. To prove this, we consider an analytic family of contractive holonomy diffeomorphisms ψ_i^t conjugated to ψ_i^0 by $\bar{\phi}_i^t$. Thanks to the Schroeder linearization theorem for analytic families of diffeomorphism of $(\mathbb{C}, 0)$, cf. for instance [7], there is an analytic family of linearizing coordinates ζ_t such that $\bar{\phi}_i^t(\zeta_t) = \sigma(t)\zeta_0$ for some continuous function $\sigma : \Delta \to \mathbb{C}^*$. Since the holonomy group G is rigid and contains a contractive element, we deduce that G is non abelian. Therefore, there exists an element $g \in G$ such that $g(\zeta_0) = \sum a_k \zeta_0^k$ with $a_k \neq 0$ for some $k > 1$. Since the holonomy element

$$(\bar{\phi}_i^t)^{-1} \circ g \circ \bar{\phi}_i^t = \sum_{k=1}^{\infty} \frac{a_k}{\sigma(t)^{k-1}} \zeta_t^k$$

depends analytically on t, we conclude that σ is holomorphic and consequently $\bar{\phi}_i^t$ also is holomorphic with respect to t. It is easy to see that $\mathcal{G}_i$ is defined in a neighborhood of C_i by the level sets of the map $(z,t) \mapsto \bar{\phi}_i^t(z)$.

(III) Since the hypothesis (ii) is purely topological, it is also verified by $(S_t, \mathcal{F}_t, C_t)$. Therefore, for every j, the set V_j^* of points $\varphi_j(p) \in V_j$ such that the leaf of $\mathcal{F}^{\Delta}$ through p is adherent to $\mathcal{C} = \Phi^{-1}(C \times \Delta) = \bigcup_{t \in \Delta} C_t$ in regular points is an open set whose complementary is a finite union of plaques of $\mathcal{G}_j$. Point (II) implies that $\mathcal{C}$ is a holomorphic leaf of the topological foliation $\mathcal{G}^{\text{top}}$ having a neighborhood where it is holomorphic. Consider $(z,t) \in V_j^*$ and $p \in U_j$ such that $\varphi_j(p) = (z,t)$. Since the leaf L_t de $\mathcal{F}^{\Delta}$ through p accumulates C_t in regular points, there exists $q \in L_t \cap \mathcal{U}_i$ close enough to C_t such that $\mathcal{G}_i$ is holomorphic in a neighborhood of $\varphi_i(q) \in V_i$. Let γ be a path inside L_t joining the points p and q, and consider the holonomy transformation $\psi_\gamma : V_j \to V_i$ of the foliation $\mathcal{F}^{\Delta}$ associated to γ. It is easy to see that $\mathcal{G}_j = \psi_\gamma^* \mathcal{G}_i$ and therefore, $\mathcal{G}_j$ is holomorphic in a neighborhood of (z,t). Hence $\mathcal{G}_j$ is holomorphic in V_j^*.

(IV) Finally, we will prove that $\mathcal{G}_j$ is holomorphic in the whole V_j. As we have seen, $\mathcal{G}_j$ can be defined by a continuous function $f : V_j \to \mathbb{C}$ (sending (z,t) into $f(z,t) = \bar{\phi}_i^t(z)$) which is holomorphic in V_j^*. By construction, there exist positive constants $r_2 > r_1 > 0$ such that $K := \{z \in \Sigma_i^0 \,:\, r_1 < |z| < r_2\} \times \Delta \subset V_j^*$. For any $r \in (r_1, r_2)$, the function

$$\tilde{f}(z,t) = \frac{1}{2i\pi} \int_{|\zeta|=r} \frac{f(\zeta,t)}{\zeta - z} d\zeta$$

is holomorphic in z and coincides with f on V_j^*. On the other hand,

$$\frac{\partial \tilde{f}}{\partial \bar{t}}(z,t) = \frac{1}{2i\pi} \int_{|\zeta|=r} \frac{\frac{\partial f}{\partial \bar{t}}(\zeta,t)}{\zeta - z} d\zeta = 0,$$

because f is holomorphic in $K \subset V_j^*$. It follows from Osgood's lemma that $\tilde{f}$ is holomorphic in V_j. Since $\tilde{f} = f$ in V_j^* and f is continuous in V_j, it follows from the Riemann extension theorem that $f = \tilde{f}$ is holomorphic in the whole V_j. □

Remark 2.2. Thanks to the rigidity of the non-solvable dynamics in $\text{Diff}(\mathbb{C}, 0)$ and the density of contractive fixed points (see [15, 1]), we can substitute the hypothesis (a) in Theorem 2.1 by

(a') the holonomy group of each connected component of $C^* = C \setminus \text{Sing}(\mathcal{F})$ is non solvable.

3. Fibred surfaces

For a fibred complex surface S we mean a surface admitting an analytic locally trivial fibration $\pi : S \to B$ over a complex curve B. We will use the following results of R. Gérard and A. Sec in [5], which were originally inspired by the work of P. Painlevé (cf. [16]).

Definition 3.1 (Gerard-Sec)**.** *A foliation $\mathcal{F}$ on S is said to be simple with respect to a map $\pi : S \to B$ if and only if each point $p \in S$ has a $\mathcal{F}$-distinguished neighborhood U such that the plaque through p meets $\pi^{-1}(\pi(p))$ only in the point p.*

Remark 3.2. If S is a compact surface and $\pi : S \to B$ is a submersion it follows from Ehresmann's Lemma that π is a locally trivial fibration. In this situation, assume that $\mathcal{F}$ is a (singular) holomorphic foliation on S and let us denote by Σ the union of the singular set of $\mathcal{F}$ and the fibres of π invariant by $\mathcal{F}$. Then it is easy to see that $\mathcal{F}_{|S^*}$ is simple with respect to $\pi_{|S^*} : S^* \to B^*$, where $B^* = B \setminus \pi(\Sigma)$ and $S^* = \pi^{-1}(B^*)$.

Theorem 3.3 (Gerard-Sec)**.** *If $\pi : S \to B$ is an analytic locally trivial fibration and $\mathcal{F}$ is a simple foliation with respect to π then for each path $\gamma : [0,1] \to B$ and for each point $p \in \pi^{-1}(\gamma(0))$ there exists a lifting $\tilde{\gamma}$ of γ to the leaf of $\mathcal{F}$ passing through $p = \tilde{\gamma}(0)$. Moreover, the number of liftings is bounded by an integer depending only on the point p.*

We will also need a theorem of E. Ghys generalizing to the non algebraic case the well known result of J.-P. Jouanolou concerning the finiteness of closed leaves of holomorphic foliations, see [11, 6].

Theorem 3.4 (Ghys)**.** *If $\mathcal{F}$ is a codimension one (possibly singular) holomorphic foliation on a compact, connected complex manifold, then $\mathcal{F}$ has only a finite number of closed leaves except when $\mathcal{F}$ admits a meromorphic first integral, in which case all leaves are closed.*

Now, we proceed to continue with the following step of the proof of Theorem A.

Proposition 3.5. *Under the hypothesis of Theorem A, if we put $C = F$ then the hypothesis of Theorem 2.1 hold.*

Proof. First of all, we point out that if a leaf L of $\mathcal{F}$ meets all the fibres of π in a finite number of points then L is compact. Indeed, the restriction of $\pi : S \to B$ to L is finite and hence it is a proper map. Secondly, we note that the existence of a meromorphic first integral of $\mathcal{F}$ in S is not allowed because the hyperbolicity of the singularities of $\mathcal{F}$ on F. From Theorem 3.4 we deduce that there is a finite number of compact leaves of $\mathcal{F}$.

Let L be a non compact leaf of $\mathcal{F}$, i.e. $\overline{L} \setminus L \not\subset \mathrm{Sing}(\mathcal{F})$. Using the notations of Remark 3.2, we fix a point b_0 in B^* and a path $\gamma : [0,1) \to B^*$ such that $\lim_{t\to 1} \gamma(t)$ exists and is equal to $b_1 = \pi(F)$. For each $0 < t < 1$ and for each point p of $L \cap \pi^{-1}(b_0)$, we consider the lift $\tilde{\gamma}_t$ of γ restricted to $[0,t]$ to the leaf L such that $\tilde{\gamma}_t(0) = p$. The compactness of S implies that the set $A(\gamma) \subset F$ of accumulation points of $\tilde{\gamma}_t$, when t tends to 1 and p ranges $L \cap \pi^{-1}(b_0)$, is nonempty. The hyperbolicity of the singularities of $\mathcal{F}$ along F implies that $A(\gamma)$ contains necessarily some regular point of $\mathcal{F}$. □

4. Some facts about versal unfoldings

A holomorphic unfolding $\mathcal{G}$ of a singular foliation $\mathcal{F}$ (over a parameter space P of arbitrary dimension) on a complex surface S is an unfolding admitting a reduction of the singularities *with parameters*, see for instance [14] for a proper definition. An unfolding $\mathcal{G}^{\mathrm{ver}}$ of $\mathcal{F}$ over P^{ver} is called versal if it *contains* any other unfolding $\mathcal{G}$ of $\mathcal{F}$ over P. More precisely, if the there exist a map $\lambda_P : P \to P^{\mathrm{ver}}$ such that we can complete the pull-back diagram, i.e., $\mathcal{S} \cong \lambda_P^* \mathcal{S}^{\mathrm{ver}}$:

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\lambda} & \mathcal{S}^{\mathrm{ver}} \\ \downarrow & & \downarrow \\ P & \xrightarrow{\lambda_P} & P^{\mathrm{ver}} \end{array}$$

and $\mathcal{G} \cong \lambda^* \mathcal{G}^{\mathrm{ver}}$. In [14], a versal equisingular unfolding is constructed for every foliation $\mathcal{F}$, being P^{ver} a (possibly singular) analytic space. Under some cohomological assumptions ($H^1(S, \mathcal{O}_S(T\mathcal{F})) = H^2(S, \mathcal{O}_S(T\mathcal{F})) = 0$) there it is showed that P^{ver} is smooth and naturally identified with the product of local the parameter spaces P_i^{loc} corresponding to the local equisingular versal unfoldings of the germs of $\mathcal{F}$ at each singular point p_i of $\mathcal{F}$ (see also [13]).

It only remains to note that under the hypothesis of Theorem A, the parameter space of the versal equisingular unfolding of $\mathcal{F}$ is trivial. This is a consequence of the works of X. Gómez-Mont, cf. [9] and J.-F. Mattei and M. Nicolau, cf. [13, 14]. Indeed, if all the singularities of $\mathcal{F}$ are reduced then $P_i^{\mathrm{loc}} = 0$. This is also the case if all them have Milnor number equal to one and non vanishing trace, see [9]. If, in addition, $H^1(S, \mathcal{O}_S(T\mathcal{F})) = 0$ then we have $P^{\mathrm{ver}} = 0$, i.e. every equisingular unfolding is analytically trivial. To finish the proof of Theorem A, we need only point out that every unfolding of $\mathcal{F}$ is equisingular provided that all the singularities of $\mathcal{F}$ are reduced.

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