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UNFOLDING OF RESONANT SADDLES AND THE DULAC TIME

Pavao Mardešić

Institut de Mathématiques de Bourgogne, UFR Sciences et Techniques
Université de Bourgogne, UMR 5584 du CNRS, B.P. 47870, 21078 Dijon, France

David Marín

Departament de Matemàtiques, Facultat de Ciències
Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain

Jordi Villadelprat

Departament d'Enginyeria Informàtica i Matemàtiques, ETSE
Universitat Rovira i Virgili, 43007 Tarragona, Spain

Abstract. In this work we study unfoldings of planar vector fields in a neighbourhood of a resonant saddle. We give a $\mathcal{C}^k$ normal form for the unfolding with respect to the conjugacy relation. Using our normal form we determine an asymptotic development, uniform with respect to the parameters, of the Dulac time of a resonant saddle deformation. Conjugacy relation instead of weaker equivalence relation is necessary when studying the time function. The Dulac time of a resonant saddle can be seen as the basic building block of the total period function of an unfolding of a hyperbolic polycycle.

Introduction. In this work we study unfoldings of planar vector fields in a neighbourhood of a resonant saddle. We give a $\mathcal{C}^k$ normal form for the unfolding with respect to the conjugacy relation. This generalizes the known orbital normal form with respect to the equivalence relation [4] and [13].

Using our normal form we determine an asymptotic development, uniform with respect to the parameters, for the Dulac time of a resonant saddle. Our asymptotic development of the Dulac time is of a similar nature as the asymptotic expansion of the Dulac map given in [13]. It generalizes our previous work [7] dealing with the Dulac time of orbitally linearizable families, but without being as explicit on the coefficients.

Our initial motivation was the problem of finite "cyclicity" (i.e., existence of a local uniform bound) for the number of critical points of the period function of polynomial vector fields on hyperbolic or more general polycycles. The condition of non-criticality of the period appears for instance in the bifurcation theory of subharmonics. Under the non-criticality of the period, zeros of appropriate Melnikov functions guarantee the persistence of a subharmonic periodic orbit of a Hamiltonian under a periodic non-autonomous deformation (see Theorem 4.62 of [3]).

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We see our asymptotic development of the Dulac time as the basic building block in establishing an asymptotic development of the total period function (Poincaré time), which we hope to study in a subsequent work. In its turn, such a uniform asymptotic development should be the main ingredient in the proof of finite "cyclicity" for critical points of the period function on hyperbolic polycycles.

For a *fixed* vector field several results are known: An asymptotic development of the Poincaré time was obtained in [16, 17]. Non-accumulation of critical periods of a fixed polynomial vector field on hyperbolic polycycles has been recently proved in [9]. In [2] Chicone and Dumortier show that the Poincaré time of a fixed vector field on a polycycle is non-oscillating if the polycyle has at least one finite saddle point.

Hence, special attention must be payed to the study of polycycles whose all vertices are at infinity in the Poincaré disc. For that reason, in our study of *unfoldings* of saddle points (2) we permit polar factors. They can come from the line at infinity in a saddle at infinity or, more generally, appear in a divisor after desingularizing more general singular points at infinity in a polycycle. The case of lines of zeros in at least one of the separatrices is also allowed as it can appear after desingularizing a degenerate singular point at finite distance.

We think that our normal form is also of independent interest. Note that due to unfolding of resonances, one cannot hope for a C^∞ or analytic normal form in a neighbourhood of a resonant saddle. When studying unfoldings of polycycles of finite codimension a $\mathcal{C}^k$ normal form should be sufficient. For studying unfoldings of infinite codimension, analytic normal forms in some domains unfolding sectors should be developed in the spirit of the unfoldings of saddle-node in [14].

This paper consists of two parts. The first part is dedicated to establishing the normal form Theorem A of an unfolding of a resonant saddle with possibly polar factors in the axes. In the second part we apply this normal form to obtain an asymptotic development Theorem B for the Dulac time.

Part 1. Temporal normal form

This part is organized as follows. In Section 1 the theorem on normal form for conjugacy is formulated. In Section 2 tools necessary for its proof are collected. In Section 3 the normal form is proved modulo the tools. Finally Section 4 is devoted to prove these tools, the most important of them being the existence of solution of an adapted homological equation stated in Theorem 2.3.

1. Statement of Theorem A.

Let us consider a $\mathcal{C}^\infty$ unfolding $\{X_\mu\}_{\mu\in\mathcal{U}}$ of a saddle point at the origin. More precisely

$$X_\mu = a_\mu(x,y)x\partial_x + b_\mu(x,y)y\partial_y, \text{ with } a_\mu(0,0)=1 \text{ and } \lambda(\mu):=-b_\mu(0,0)>0, \quad (1)$$

where a_μ and b_μ are $\mathcal{C}^\infty$ functions at the origin and $\mathcal{U}$ is an open subset of $\mathbb{R}^{\mathfrak{m}}$. We also consider the collinear family

$$Y_\mu = \frac{1}{v}X_\mu, \quad \text{where} \quad v = x^m y^n \quad \text{and} \quad m,n\in\mathbb{Z}. \quad (2)$$

In what follows we shall say that two vector fields (or germs of vector fields) Z and W are *conjugated* if there exists a change of coordinates Φ transforming Z to W, i.e., $\Phi^\star Z = W$, where

$$\big(\Phi^\star Z\big)(p) = \big(D\Phi\big)_p^{-1}\big(Z\circ\Phi(p)\big).$$

We shall say that two germs of vector fields Z and W are *equivalent* at a point p_0, if they are conjugated up to a germ of a nonzero multiple: $\Phi^\star Z = fW$ with $f(p_0) \neq 0$. The two notions extend to germs of families of vector fields.

Definition 1.1. Given $\mu_0 \in \mathcal{U}$, let us denote $\lambda_0 := \lambda(\mu_0)$. The *orbital codimension* $\kappa \in \mathbb{N} \cup \{\infty\}$ of the saddle of the vector field X_{μ_0} is defined as follows. If $\lambda_0 \notin \mathbb{Q}$, then we set $\kappa := \infty$. If $\lambda_0 \in \mathbb{Q}$, then the infinite jet of X_{μ_0} at the origin is $\mathcal{C}^\infty$ equivalent to

$$x\partial_x + \Big(-p/q + \sum_{i\geqslant 0} \alpha_{i+1}(x^p y^q)^i\Big) y\partial_y, \quad \text{with} \quad \lambda_0 = p/q \quad \text{and} \quad \gcd(p,q) = 1. \tag{3}$$

In case that $\alpha_{i+1} \neq 0$ for some i we set $\kappa := \min\big\{i \in \mathbb{N} : \alpha_{i+1} \neq 0\big\}$ and, otherwise, $\kappa := \infty$.

Remark 1. The orbital codimension does not depend on the particular equivalence used to bring X_{μ_0} to a normal form (3) because the monomial $(x^p y^q)^\kappa$ can not be annihilated by means of a smooth coordinate transformation preserving the normal form.

The main theorem proved in the first part is the following.

Theorem A. Let $\{X_\mu\}_{\mu\in\mathcal{U}}$ be a $\mathcal{C}^\infty$ unfolding of a saddle point as in (1) and consider some $\mu_0 \in \mathcal{U}$. Then for any $k \in \mathbb{N}$ the family $\{Y_\mu\}_{\mu\in\mathcal{U}}$ is $\mathcal{C}^k$ conjugated by a diffeomorphism of the form $\Phi(x, y, \mu) = (\Phi_\mu(x, y), \mu)$ defined in a neighbourhood of $(0, 0, \mu_0) \in \mathbb{R}^2 \times \mathcal{U}$ to

$$Y_\mu^{NF} = \frac{1}{v + u^\ell Q_\mu(u)} \Big(x\partial_x + \big(-\lambda(\mu) + P_\mu(u)\big) y\partial_y \Big), \tag{4}$$

where

(a) if $\lambda_0 \notin \mathbb{Q}$, then $P_\mu \equiv Q_\mu \equiv 0$,

(b) if $\lambda_0 = p/q$ with $(p, q) = 1$, then P_μ and Q_μ are polynomials in the resonant monomial $u = x^p y^q$ and

$$\ell = \min\big\{\beta \in \mathbb{Z} : \beta(p, q) > (m, n)\big\}.$$

Moreover, in case that X_{μ_0} has orbital codimension $\kappa < \infty$ then we have that $\deg P_\mu \leqslant 2\kappa$ and $\deg Q_\mu \leqslant \kappa - \min(\ell, 1)$.

Remark 2. In the definition of ℓ above, the symbol $>$ stands for the partial order in $\mathbb{Z}^2$. Note that u^ℓ/v is regular at $(x, y) = (0, 0)$ and that if $m \geqslant 0$ or $n \geqslant 0$ then $\ell \geqslant 1$. The integer ℓ plays the role analogous to the orbital codimension in the bound of the degree of Q_μ. However, a priori the order of $Q_{\mu_0}(u)$ at $u = 0$ is a more natural notion of "temporal codimension", but it does not seem to have immediate applications.

2. **Tools.**

In this section we collect some tools used in the proof of Theorem A. They will be proved in Section 4 except for Theorem 2.1, for which we give only a sketch of proof. This theorem is part of folklore. It appears, as we state it here, in [13] but referring to [1] for the proof. However, [1] deals only with a related problem of normal forms for diffeomorphisms. A proof of Theorem 2.1 appears in [4] but there is a delicate point concerning the elimination of the remainder term which is not dealt with in that paper. Later on we point it out in the sketch of the proof of Theorem 2.1. The mentioned delicate point can be overcome by applying the results of Samovol in a very technical paper [15]. We do not give a complete proof as it can be done along the lines of the proof of our Theorem 2.3.

Theorem 2.1. *Let $\{X_\mu\}_{\mu\in\mathcal{U}}$ be a $\mathcal{C}^\infty$ unfolding of a saddle point as in (1) and consider some $\mu_0\in\mathcal{U}$. Then for any $s\in\mathbb{N}$ the family $\{X_\mu\}_{\mu\in\mathcal{U}}$ is $\mathcal{C}^s$ equivalent by a diffeomorphism of the form $\Phi(x,y,\mu)=(\Phi_\mu(x,y),\mu)$ defined in a neighbourhood of $(0,0,\mu_0)\in\mathbb{R}^2\times\mathcal{U}$ to*

$$X_\mu^{NF}=x\partial_x+\big(-\lambda(\mu)+P_\mu(u)\big)y\partial_y, \tag{5}$$

where

(a) *if $\lambda_0\notin\mathbb{Q}$, then $P_\mu\equiv 0$,*
(b) *if $\lambda_0=p/q$ with $(p,q)=1$, then P_μ is a polynomial in the resonant monomial $u=x^py^q$. Moreover, in case that X_{μ_0} has orbital codimension $\kappa<\infty$ then* $\deg P_\mu\leqslant 2\kappa$.

Lemma 2.2. *Let $\{Y_\mu\}_{\mu\in\mathcal{U}}$ be a family of vector fields as in (2) and let $\{f_\mu\}_{\mu\in\mathcal{U}}$ be a $\mathcal{C}^k$ family of functions with $f_\mu(0,0)=0$. Then, for each $\mu_0\in\mathcal{U}$, there exists a family of $\mathcal{C}^k$ diffeomorphisms $\{\Phi_\mu\}$ defined in a neighbourhood of $(0,0,\mu_0)\in\mathbb{R}^2\times\mathcal{U}$ such that, on $xy\neq 0$,*

$$(\Phi_\mu)^\star(Y_\mu)=\frac{X_\mu}{v+X_\mu(vf_\mu)}\,.$$

In fact $\Phi_\mu(x,y)=\varphi_\mu\big(F_\mu(x,y);x,y\big)$, where $\varphi_\mu(t;x,y)$ denotes the flow of X_μ passing through $(x,y)\in\mathbb{R}^2$ at $t=0$ and $\{F_\mu\}$ is a $\mathcal{C}^k$ family of functions with $F_\mu(0,0)=0$ which is defined implicitly by

$$vf_\mu(x,y)=\int_0^{F_\mu(x,y)}v\circ\varphi_\mu(\xi;x,y)\,d\xi.$$

Remark 3. For fixed $m,n\in\mathbb{Z}$, the diffeomorphism Φ_μ in Lemma 2.2 depends only on the initial data $\{Y_\mu\}$ and $\{f_\mu\}$. Since we shall apply it several times, changing both data (vector fields and functions), we introduce the notation $\Phi_\mu=\Phi[Y_\mu,f_\mu]$.

Let V be an open subset of $\mathbb{R}^n$ and consider a smooth function $f\colon V\longrightarrow\mathbb{R}$. We define

$$\|f\|_V=\sup\{|f(x)|\,:\,x\in V\}.$$

If $I=(i_1,\ldots,i_n)$ is a multi-index with $i_j\in\mathbb{N}\cup\{0\}$ then we use the notation $i=|I|=i_1+\cdots+i_n$ and

$$\partial_I^i=\frac{\partial^i}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}}.$$

Thus, dealing with partial derivatives, we shall use the convention that if J is a multi-index then the small letter j stands for $|J|$. Moreover, given $p\in V$, we denote by $\big(D^if\big)(p)$ the total differential of order i of f at p, which is defined as the symmetric i-linear form

$$\begin{array}{rcl}\big(D^if\big)(p):\mathbb{R}^n\times\cdots\times\mathbb{R}^n & \longrightarrow & \mathbb{R}\\ \big(x^{(1)},\ldots,x^{(n)}\big) & \longmapsto & \displaystyle\sum_{|I|=i}\big(\partial_I^if\big)(p)\,x_{i_1}^{(1)}\cdots x_{i_n}^{(n)},\end{array}$$

where the sum is taken over all the multi-index $I=(i_1,\ldots,i_n)$ with $|I|=i$ and $x^{(j)}=(x_1^{(j)},\ldots,x_n^{(j)})$ for $j=1,\ldots,n$. Finally, we define

$$\|D^if(p)\|=\max\{|\partial_I^if(p)|:|I|=i\}.$$

We extend these definitions to vector functions in the usual way. More concretely, if $f = (f_1, \dots, f_m)$ is a vector function from $V \subset \mathbb{R}^n$ to $\mathbb{R}^m$ then

$$\partial_I^i f = (\partial_I^i f_1, \dots, \partial_I^i f_m) \text{ and } \big(D^i f\big)(p) : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \longrightarrow \mathbb{R}^m.$$

Similarly,

$$\|f\|_V = \max\{\|f_j\|_V : j = 1, \dots, m\}$$

and

$$\|\big(D^i f\big)(p)\| = \max\{\|\big(D^i f_j\big)(p)\| : j = 1, \dots, m\}.$$

From now on we must distinguish between parameters $\mu \in \mathbb{R}^{\mathtt{m}}$ and phase variables $(x, y) \in \mathbb{R}^2$ when considering a smooth function $f : V \subset \mathbb{R}^{2+\mathtt{m}} \longrightarrow \mathbb{R}$. We say that such a function is N-flat with respect to (x, y) if it is $\mathcal{C}^{N+1}$ and verifies the estimates

$$\|\big(D^i f\big)(x, y, \mu)\| \leqslant C\|(x, y)\|^{N-i}, \qquad i = 0, \dots, N,$$

in some neighbourhood of $(0, 0, \mu_0) \in \mathbb{R}^{2+\mathtt{m}}$ and for some constant $C > 0$. The flatness with respect to x or y is defined analogously by replacing $\|(x, y)\|$ by $|x|$ or $|y|$ respectively.

Theorem 2.3. *Let $\{X_\mu\}_{\mu\in\mathcal{U}}$ be a family of vector fields as in (1) and consider some $\mu_0 \in \mathcal{U}$. Then for any $k \in \mathbb{N}$ there exists a natural number $N = N\big(k, \lambda_0, m, n\big)$ such that if $\{h_\mu\}$ is a $\mathcal{C}^N$ family of N-flat functions, then the homological equation*

$$X_\mu(vf_\mu) = vh_\mu \tag{6}$$

has a $\mathcal{C}^k$ family of solutions $\{f_\mu\}$ defined in a neighbourhood of $(0, 0, \mu_0) \in \mathbb{R}^2 \times \mathcal{U}$. More precisely, we can take

$$N(k, \lambda_0, m, n) := 2\left[\max\{(\nu_0 + 1)k - m + \lambda_0 n, (\nu_0/\lambda_0 + 1)k + m/\lambda_0 - n\} + 1\right],$$

where $\nu_0 = \max\{1, \lambda_0\}$ and $[\,\cdot\,]$ denotes the integer part.

Remark 4. Analogously to the definition of ℓ in Theorem A, the natural number $N\big(k, \lambda_0, m, n\big)$ can be written as

$$2\min\Big\{M \in \mathbb{N} \,:\, M \cdot \Big(1, \lambda_0\Big) > \Big((\nu_0 + 1)k - m + \lambda_0 n, (\nu_0 + \lambda_0)k + m - \lambda_0 n\Big)\Big\}.$$

Note that the above formula is not symmetric with respect to m and n. This is so because in the proof we first show that it suffices to consider a vector field in normal form and we choose one containing all the resonant monomials in the ∂_y direction. It is important to mention that N depends only on the linear part of X_{μ_0}.

Before proving our main theorem in which we give a normal form for conjugacy, we sketch the proof of Theorem 2.1 that deals with orbital normal form in order to see which kind of ideas are involved in this type of results. Theorem 2.1 is part of folklore (see [4, 13]) but we want to point out a delicate point, which we think did not receive the required attention in the literature.

One uses first the Takens normal form theorem [18] (see also [3]). Let H^h be the space of polynomial vector field families in the (x, y) plane depending on the parameter μ and homogeneous of degree h in (x, y). Let $L = L(\mu) = x\partial_x - \lambda(\mu)y\partial_y$ be the linear part of X_μ and for each h, consider the action of the Lie bracket $[L, -] : H^h \to H^h$. For fixed μ and any h, the mapping $[L, -]$ is linear on H^h. Denote by B^h the image of H^h by $[L, -]$ and let G^h be some complementary space so that $H^h = B^h \oplus G^h$. Then, for any N, there exists a polynomial change of coordinates transforming the vector field family X_μ to the form

$$x\partial_x - \lambda(\mu)y\partial_y + g_2 + \cdots + g_N + R(x,y), \tag{7}$$

where g_h is a homogeneous vector field family belonging to $G^h \subset H^h$, for $h = 1, \ldots, N$ and the remainder term $R(x,y)$ is a vector field $R(x,y) = o(|(x,y)|^N)$. Moreover,

$$[L, x^i y^j \partial_x] = \big(1 - i + j\lambda(\mu)\big)x^i y^j \partial_x \text{ and } [L, x^i y^j \partial_y] = \big(-i + (j-1)\lambda(\mu)\big)x^i y^j \partial_y$$

so the action of $[L, -]$ on H^h is diagonal, and G^h can be taken as the kernel of $[L, -]$. That is, if λ_0 is irrational, then for λ sufficiently close to λ_0, the family is linearizable up to an N-flat term for any N. If $\lambda_0 = p/q$, with p, q positive, relatively prime integers, then for $\lambda(\mu)$ sufficiently close to λ_0 up to an N-flat term, all monomials can be eliminated except for the resonant monomials: $u^k x\partial_x$ and $u^k y\partial_y$. When working with the equivalence and not conjugacy relation, it is legitimate to divide (7) by the component of $x\partial_x$. Hence, for any N there exists a polynomial change of coordinates transforming orbitally the vector field family X_μ to

$$X^{NF} + R(x,y)$$

with X^{NF} as in (3) and $R(x,y)$, N-flat, with respect to $|(x,y)|$.

One next applies the second step in the normalization process, eliminating the N-flat term R by means of a $\mathcal{C}^k$ diffeomorphism. We use here the homotopic method (see for instance [4, 12]). As the dependence with respect to the parameter μ is inessential, we omit mentioning it. In general, the homotopic method says that vector fields X and $X + R$ are $\mathcal{C}^k$ smoothly conjugate if the homological equation

$$[X + tR, Z_t] = R \tag{8}$$

has a $\mathcal{C}^k$ solution Z_t. The time-one flow of the vector field Z_t realizes the conjugation (if it exists). In [12] it is proved that for X hyperbolic and R infinitely flat, the homological equation (8) has a solution in the class C^∞. The proof is done first in the semihyperbolic case. That is, one decomposes the remainder $R = R_1 + R_2$ where R_1 is flat with respect to the y variable and R_2 flat with respect to the x variable. One uses first the contractibility of the flow of X in the y direction for solving the equation

$$[X + tR_1, Z_t] = R_1 \tag{9}$$

and hence proving that X is conjugated to $X + R_1$. Next, one proves that $X + R_1$ and $X + R_1 + R_2$ are conjugated by solving the equation

$$[X + R_1 + tR_2, Z_t] = R_2$$

using the contractibility of $X + R_1 + tR_2$ for negative time. The two equations being of the same type, we comment only on (9). In order to solve it one globalizes first the vector field. That is, one modifies the ∂_y component of X in a complement of a small neighbourhood of the origin in such a way that the flow of the modified vector field is well defined for positive time and all solutions tend to the x axes as $t \to +\infty$. By abuse, we keep the same notation for the modified vector field X. A solution of (9) is given by

$$Z_t(x,y) = -\int_0^\infty (D(X + tR_1))^{-1} \circ R_1(\phi(\tau,(x,y)))d\tau, \tag{10}$$

where $D(X + tR_1)$ is the solution of the first variational equation of the modified vector field $X + tR_1$ and ϕ is its flow (see [4]). Using the flow-box theorem for the vector field $X+tR_1$, it is easy to see that if the integral (10) is uniformly convergent, then it verifies (9). Further dominated convergence estimates are needed to assure the differentiability of Z_t. In [12], these estimations are given in the C^∞ smoothness case. It is easy, following this proof, to see that a solution Z_t of class $\mathcal{C}^k$ of (9) exists provided that R_1 is sufficiently flat with respect to the y variable. The difficulty is that the required flatness as it appears in the proof of Proposition 2.2.11 in [12] depends on the norm of X. In our application, the vector field X appears as a result of Takens normal form procedure. It could happen that when obtaining higher flatness of R_1 as a result of applying Takens normal form procedure, the norm of X grows and an even higher flatness of R_1 would be required. It is sufficient to show that the required flatness $N(k)$ of R_1 assuring the existence of a $\mathcal{C}^k$ solution of (9) depends only on the linear part of the vector field X (which in not modified by the Takens normal form procedure). This is proved by Samovol in [15] where he proves that the required flatness $N(k)$ in the homological equation (9) depends only on the linear part of the vector field (but only the case of vector field without poles is considered). An explicit estimates of $N(k)$ appears also in [10] in the case of linear X, but the proof is very sketchy. In general, the independence of $N(k)$ on higher order terms of X can be proved along the lines of our proof of Theorem 2.3. Yakovenko informed us that equation (9) can be reduced to (6) with $v=1$. A detailed proof of an analogous problem for diffeomorphisms appears in [1].

Finally, in the finite orbital codimension case, the polynomial normal form can be improved using the Weierstrass preparation theorem (see [4]). We perform a similar construction concerning the temporal part in the next section.

3. **Proof of Theorem A.**

Proof of Theorem A. Fix $k\in\mathbb{N}$ and $\mu_0\in\mathcal{U}$, and let $N=N(k,\lambda_0,m,n)$ be the integer given by Theorem 2.3. Take any $s>N$. By Theorem 2.1, there exists a $\mathcal{C}^s$ change of coordinates Φ_μ^0 such that

$$\big(\Phi_\mu^0\big)^\star(Y_\mu)=V_\mu^s:=\frac{1}{v}\,\frac{X_\mu^s}{1+R_\mu^s(x,y)}\,,$$

where $X_\mu^s=x\partial_x+\big(-\lambda(\mu)+P_\mu^s(u)\big)y\partial_y$ and R_μ^s is a $\mathcal{C}^s$ function vanishing at the origin. (Here Φ_μ^0 is the equivalence between X_μ and X_μ^s that provides Theorem 2.1 and we took into account that it is tangent to the identity and preserving the axes.)

Next we shall "simplify" the function R_μ^s by means of a conjugation and to this end we apply Lemma 2.2. Thus, recall Remark 3, the idea is to take the diffeomorphism $\Phi_\mu^1:=\Phi\left[V_\mu^s,f_\mu\right]$, where $\{f_\mu\}$ is to be chosen appropriately. Notice that

$$\big(\Phi_\mu^1\big)^\star(V_\mu^s)=\frac{X_\mu^s}{v\big(1+R_\mu^s(x,y)\big)+X_\mu^s(vf_\mu)}.$$

The vector field X_μ^s acts linearly on the vector space $(v)\mathbb{R}[x,y]$ and note that its image contains all the monomials of $(v)\mathbb{R}[x,y]$ which are not inside $\mathbb{R}(u)$ because

$$X_\mu^s(x^ay^b)=\big(a-\lambda(\mu)b\big)x^ay^b+bP_\mu^s(u)x^ay^b.$$

In other words, $u^\ell\mathbb{R}[u]\subset(v)\mathbb{R}[x,y]$ is a supplementary subspace of the image of X_μ^s acting on $(v)\mathbb{R}[x,y]$. Hence we can choose $f_\mu(x,y)$ as a polynomial so that

$$vR_\mu^s(x,y)+X_\mu^s(vf_\mu)=u^\ell Q_\mu^s(u)+vh_\mu^s(x,y),$$

where $h^s_\mu(x,y)$ is a s-flat function and $Q^s_\mu \equiv 0$, in case that $\lambda_0 \notin \mathbb{Q}$, or $Q^s_\mu(u)$ polynomial in $u = x^p y^q$, in case that $\lambda_0 = p/q$ with $(p,q) = 1$. Accordingly $\big(\Phi^1_\mu\big)^\star(V^s_\mu) = Z^s_\mu$, where

$$Z^s_\mu := \frac{X^s_\mu}{v + u^\ell Q^s_\mu(u) + v h^s_\mu(x,y)} = \frac{1}{v}\,\frac{X^s_\mu}{1 + u^\ell Q^s_\mu(u)/v + h^s_\mu(x,y)}.$$

We point out that the vector field Z^s_μ can be written as in (2), i.e., it is of the form $1/v$ times a smooth vector field at the origin because u^ℓ/v has the same property. Therefore we can apply Lemma 2.2 and consider the coordinate transformation $\Phi^2_\mu := \Phi\left[Z^s_\mu, g_\mu\right]$, which verifies

$$\big(\Phi^2_\mu\big)^\star(Z^s_\mu) = \frac{X^s_\mu}{v + u^\ell Q^s_\mu(u) + v h^s_\mu(x,y) + X^s_\mu(v g_\mu)}.$$

Our goal is to annihilate $v h^s_\mu$ by choosing an appropriate g_μ. The problem reduces to solving the homological equation $X^s_\mu(v g_\mu) = -v h^s_\mu$. Since h^s_μ is a s-flat function with $s > N$, by applying Theorem 2.3 we can assert that there exists a $\mathcal{C}^k$ function g_μ verifying the aforementioned homological equation. In short we have that

$$\big(\Phi^2_\mu \circ \Phi^1_\mu \circ \Phi^0_\mu\big)^\star(Y_\mu) = W^s_\mu := \frac{X^s_\mu}{v + u^\ell Q^s_\mu(u)}$$

with $Q^s_\mu(u)$ polynomial. It is important to mention that N in Theorem 2.3 depends only on the linear part of the vector field X^s_μ, which is independent of s. This enables us to fix in advance the required flatness of $h^s_\mu(x,y)$ in order to get the $\mathcal{C}^k$ conjugacy Φ^2_μ that annihilates it. This constitutes the key point in all the process because X^s_μ does depends on s.

Assume finally that the original vector field X_{μ_0} has orbital codimension $\kappa < \infty$. In this case, by applying Theorem 2.1 we have that $X^s_\mu = x\partial_x + \big(-\lambda(\mu) + P^s_\mu(u)\big) y\partial_y$, where P^s_μ is a polynomial in u with $\deg(P^s_\mu) \leqslant 2\kappa$ for $\mu \approx \mu_0$ and such that $P^s_{\mu_0}$ has order κ at $u = 0$. Again, on account of the definition of ℓ, W^s_μ can be written as in (2) because

$$W^s_\mu = \frac{1}{v}\,\frac{X^s_\mu}{1 + u^\ell Q^s_\mu(u)/v}.$$

As before we consider $\Phi^3_\mu := \Phi\left[W^s_\mu, \widehat{\tau}_\mu\right]$ where $\widehat{\tau}_\mu$ is a smooth function to be determined. However now we want it of the form $\widehat{\tau}_\mu(x,y) = \tau_\mu(u)/v$. The reason for this will be clear in a moment but note that if $u^\ell | \tau_\mu(u)$ then, by the definition of ℓ, $\widehat{\tau}_\mu$ will be regular at $(x,y) = (0,0)$. By Lemma 2.2 we can assert that

$$\big(\Phi^3_\mu\big)^\star(W^s_\mu) = \frac{X^s_\mu}{v + u^\ell Q^s_\mu(u) + X^s_\mu(v\widehat{\tau}_\mu)}.$$

Then, since $v\widehat{\tau}_\mu = \tau_\mu$ depends only on u, the above denominator becomes $v + u^\ell Q^s_\mu(u) + \tau'_\mu(u) X^s_\mu(u)$ and an easy computation shows that $X^s_\mu(u) = u\big(p - \lambda(\mu)q + P^s_\mu(u)\big)$. Thus, since $\lambda_0 = p/q$ and $P^s_{\mu_0}(u)$ has order κ at $u = 0$, by applying the Weierstrass Preparation Theorem, we have that $X^s_\mu(u) = u A^s_\mu(u) B^s_\mu(u)$ where $B^s_\mu(u)$ is a polynomial of degree κ in u for $\mu \approx \mu_0$ and $A^s_{\mu_0}(0) \neq 0$. Accordingly

$$\big(\Phi^3_\mu\big)^\star(W^s_\mu) = \frac{X^s_\mu}{v + u^\ell Q^s_\mu(u) + u\tau'(u) A^s_\mu(u) B^s_\mu(u)}$$

and so we seek for a function τ_μ such that $u^\ell Q^s_\mu(u)+u\tau'_\mu(u)A^s_\mu(u)B^s_\mu(u)$ has few monomials. This "simplification" depends on weather ℓ is positive or negative. Setting $u^\ell Q^s_\mu(u)=\sum_{i=\ell}^r a_iu^i$ with $r>0$ we decompose $u^\ell Q^s_\mu(u)=S^1_\mu(u)+S^2_\mu(u)$, where $S^1_\mu(u)=\sum_{i=\ell}^{-1}a_iu^i$ and $S^2_\mu(u)=\sum_{i=0}^r a_iu^i$ in case that $\ell<0$, and $S^1_\mu(u)\equiv 0$ and $S^2_\mu(u)=u^\ell Q^s_\mu(u)$ in case that $\ell\geqslant 0$. (Here we can assume that $r>0$ taking some $a_i=0$ if necessary.) With this decomposition we perform the (polynomial) division of $S^2_\mu(u)$ by $u^\nu B^s_\mu(u)$, where $\nu:=\max(\ell,1)$, i.e.,

$$S^2_\mu(u)=C^s_\mu(u)u^\nu B^s_\mu(u)+R^s_\mu(u) \tag{11}$$

and thus $\deg(R^s_\mu)\leqslant\nu+\kappa-1$. Finally, τ_μ is to be chosen so that

$$u^\ell Q^s_\mu(u)+u\tau'_\mu(u)A^s_\mu(u)B^s_\mu(u)=S^1_\mu(u)+R^s_\mu(u),$$

which, due to $u^\ell Q^s_\mu(u)=S^1_\mu(u)+S^2_\mu(u)$, yields

$$\tau'_\mu(u)=\frac{R^s_\mu(u)-S^2_\mu(u)}{uA^s_\mu(u)B^s_\mu(u)}=-u^{\nu-1}\frac{C^s_\mu(u)}{A^s_\mu(u)}.$$

(The last equality follows from taking (11) into account.) That is,

$$\tau_\mu(u):=-\int_0^u\xi^{\nu-1}\frac{C^s_\mu(\xi)}{A^s_\mu(\xi)}\,d\xi,$$

which is a smooth function for $(u,\mu)\approx(0,\mu_0)$ because $A^s_{\mu_0}(0)\neq 0$ and $\nu\geqslant 1$. Moreover it verifies $u^\ell|\tau_\mu(u)$ as desired due to $\nu\geqslant\ell$. In short, the choice of $\widehat{\tau}_\mu(x,y)=\tau_\mu(u)/v$ for $\Phi^3_\mu=\Phi\left[W^s_\mu,\widehat{\tau}_\mu\right]$ leads to

$$\big(\Phi^3_\mu\circ\Phi^2_\mu\circ\Phi^1_\mu\circ\Phi^0_\mu\big)^\star(Y_\mu)=\frac{X^s_\mu}{v+S^1_\mu(u)+R^s_\mu(u)}.$$

It remains only to check that $S^1_\mu(u)+R^s_\mu(u)=u^\ell Q_\mu(u)$ for some polynomial Q_μ of degree $\kappa-\min(\ell,1)$. In the case that $\ell\geqslant 0$ this is simple because then $S^1_\mu\equiv 0$ and $S^2_\mu(u)=u^\ell Q^s_\mu(u)$. Consequently, from (11) we have that $R^s_\mu(u)=u^\ell\big(Q^s_\mu(u)-u^{\nu-\ell}C^s_\mu(u)B^s_\mu(u)\big)$ and so $S^1_\mu(u)+R^s_\mu(u)=R^s_\mu(u)=u^\ell Q_\mu(u)$ with

$$\deg(Q_\mu)=\nu+\kappa-1-\ell=\kappa-\min(\ell,1).$$

(In the second equality above we took $\nu=\max(\ell,1)$ and $\ell\geqslant 0$ into account.) Finally in the case that $\ell<0$, then $S^1_\mu(u)=\sum_{i=\ell}^{-1}a_iu^i=u^\ell\sum_{i=1}^{-\ell}a_{i+\ell-1}u^{i-1}$ and $R^s_\mu(u)=\sum_{i=0}^{\nu+\kappa-1}b_iu^i=u^\ell\sum_{i=0}^{\nu+\kappa-1}b_iu^{i-\ell}$. Therefore $S^1_\mu(u)+R^s_\mu(u)=u^\ell Q_\mu(u)$, where Q_μ is a polynomial with

$$\deg(Q_\mu)=\max\big(-\ell-1,\nu+\kappa-1-\ell\big)=\max\big(-\ell-1,\kappa-\ell\big)=\kappa-\ell=\kappa-\min(\ell,1).$$

Here we used that $\ell<0$, $\kappa>0$ and $\nu=\max(\ell,1)=1$. This completes the proof of the theorem. □

4. **The homological equation.**

This section is dedicated to showing the two main results that we used in the proof of Theorem A, namely, Lemma 2.2 and Theorem 2.3.

Proof of Lemma 2.2. Since the origin is a hyperbolic saddle for X_μ with both separatrices in the axes, it is clear that $v\circ\varphi_\mu(t;x,y)=v\,\chi_\mu(t,x,y)$ for some $\mathcal{C}^\infty$ function

χ_μ with $\chi_\mu(0,0,0) \neq 0$. Thus, given $\{f_\mu\}$ as in the statement, we have to find $\{F_\mu\}$ verifying

$$f_\mu(x,y) = \int_0^{F_\mu(x,y)} \chi_\mu(\xi,x,y)\,d\xi.$$

Note that the $\mathcal{C}^k$ function

$$R(x,y,\mu,\tau) := f_\mu(x,y) - \int_0^{\tau} \chi_\mu(\xi,x,y)\,d\xi$$

satisfies $R(0,0,\mu_0,0) = 0$ and $\frac{\partial}{\partial\tau}R(0,0,\mu_0,0) = -\chi_{\mu_0}(0,0,0) \neq 0$. Therefore, by the Implicit Function Theorem, there exits a $\mathcal{C}^k$ family of functions $\{F_\mu\}_\mu$ with $R\big(x,y,\mu,F_\mu(x,y)\big) = 0$ for $(x,y,\mu) \approx (0,0,\mu_0)$ and $F_{\mu_0}(0,0) = 0$. The fact that $F_\mu(0,0) = 0$ for all μ follows easily using that, by assumption, $f_\mu(0,0) = 0$. It is clear then that $\Phi_\mu(x,y) := \varphi_\mu\big(F_\mu(x,y);x,y\big)$ is a local diffeomorphism with $\Phi_\mu(0,0) = (0,0)$ for all μ. Moreover, from [5], we have that

$$(\Phi_\mu)^\star\big(X_\mu\big) = \frac{X_\mu}{1 + X_\mu(F_\mu)}\,. \tag{12}$$

Since $Y_\mu = \frac{1}{v}\,X_\mu$, we have that

$$(\Phi_\mu)^\star\big(Y_\mu\big) = \frac{1}{v \circ \Phi_\mu}\,(\Phi_\mu)^\star\big(X_\mu\big).$$

Consequently, on account of (12), the result will follow once we prove that

$$(v \circ \Phi_\mu)\big(1 + X_\mu(F_\mu)\big) = v + X_\mu(vf_\mu).$$

To see this note that some easy manipulations yield

$$\begin{aligned}
X_\mu\big(vf_\mu\big)(x,y) &= \frac{d}{ds}\Big(\big(vf_\mu\big)\circ\varphi_\mu(s;x,y)\Big)\Big|_{s=0} \\
&= \frac{d}{ds}\left(\int_0^{F_\mu(\varphi_\mu(s;x,y))} v\circ\varphi_\mu\big(s+\xi;x,y\big)\,d\xi\right)\Bigg|_{s=0} \\
&= v\circ\varphi_\mu\Big(s + F_\mu\big(\varphi_\mu(s;x,y)\big);x,y\Big)\Big(1 + \frac{d}{ds}F_\mu\big(\varphi_\mu(s;x,y)\big)\Big|_{s=0}\Big) \\
&\quad - v\circ\varphi_\mu(s;x,y)\Big|_{s=0} \\
&= \big(v\circ\Phi_\mu\big)(x,y)X_\mu\big(F_\mu\big) + \big(v\circ\Phi_\mu\big)(x,y) - v\circ(x,y).
\end{aligned}$$

In the first equality above we use the definition of the derivative of a function with respect to a vector field, and in the last one we took $\Phi_\mu(x,y) = \varphi_\mu\big(F_\mu(x,y);x,y\big)$ into account. This proves the result. □

Since the proof of Theorem 2.3 is very technical, we begin by giving first its idea omitting the dependence on μ to simplify the exposition. Let $\varphi_t : (x,y) \longmapsto \varphi(t;x,y)$ be flow at time t of a given vector field X and consider also a given function H. In this case, if

$$F(x,y) = \int_{\pm\infty}^0 H\circ\varphi_t(x,y)\,dt$$

is a well-defined smooth function then it is a solution of the *homological equation* $X(F) = H$. Indeed, by making the change of variables $\tau = t + s$ we obtain

$$X(F) = \frac{d}{ds}\int_{\pm\infty}^{0} H \circ \varphi_t \circ \varphi_s \, dt\Big|_{s=0} = \frac{d}{ds}\int_{\pm\infty}^{s} H \circ \varphi_\tau \, d\tau\Big|_{s=0} = H.$$

Our goal is to solve the homological equation (6), where recall that $v = x^m y^n$ with $m, n \in \mathbb{Z}$. Note that it coincides with the above one taking $H = vh$ and $F = vf$. The strategy consists in modifying conveniently X and h in order to make F well-defined and $f = \frac{F}{v}$ to be of class $\mathcal{C}^k$. Taking this into account, let us introduce the functions that will appear in the proof of Theorem 2.3.

So let us consider the homological equation $X(vf) = vh$. Since h is N-flat, denoting by M the integer part of $N/2$, we can decompose it as a sum $h = h_1 + h_2$, with h_1 and h_2 being M-flat with respect to x and y respectively (see [15]). The first step in the proof will be to show that there is no loss of generality in assuming that the homological equation is $X^{NF}(vf) = vh$, where X^{NF} is the vector field in normal form provided by Theorem 2.1. Accordingly we consider

$$F(x,y) = \int_{-\infty}^{0} (vh_1) \circ \varphi_t(x,y)\, dt + \int_{+\infty}^{0} (vh_2) \circ \varphi_t(x,y)\, dt,$$

where φ_t is the flow of X_μ^{NF}. In order to study F we must control the function $v \circ \varphi_t$, which satisfies the differential equation

$$\frac{d}{dt}(v \circ \varphi_t) = X(v) \circ \varphi_t = \big(v(m - \lambda n + n\,P)\big) \circ \varphi_t = (v \circ \varphi_t)\,(m - \lambda n + n\,P \circ \varphi_t)\,.$$

Consequently

$$\frac{v \circ \varphi_t}{v} = e^{(m-\lambda n)t} \exp\left(n \int_0^t P \circ \varphi_s \, ds\right),$$

and therefore $f = \frac{F}{v}$ is given by

$$f(x,y) = \int_{-\infty}^{0} \mathcal{I}_1(x,y,t)\, dt - \int_0^{\infty} \mathcal{I}_2(x,y,t)\, dt,$$

where

$$\mathcal{I}_i(x,y,t) = e^{(m-\lambda n)t}\big(h_i \circ \varphi_t(x,y)\big) \exp\left(n \int_0^t P \circ \varphi_s(x,y)\, ds\right).$$

In order to prove that f is a well-defined $\mathcal{C}^k$ function we must bound the derivatives of $\mathcal{I}_i$ and, in particular, the derivatives of the flow φ_t with respect to $(x, y, \mu) \in \mathbb{R}^{2+\mathfrak{m}}$. To this end some technical lemmas are needed.

From now on, if g is a symmetric l-linear form on $\mathbb{R}^n$ and $v_1, \ldots, v_l \in \mathbb{R}^n$, then we shall write $g(v_1, \ldots, v_l) \in \mathbb{R}$ as $gv_1 \cdots v_l$. The following result provides an expression for the chain rule of higher order. (Its proof, being straightforward, is omitted for the sake of shortness.)

Lemma 4.1. *Let $h\colon \mathbb{R}^n \longrightarrow \mathbb{R}$, $\chi\colon \mathbb{R}^m \longrightarrow \mathbb{R}$ and $\varphi\colon \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be differentiable functions. Then, for each $I = (i_1, \ldots, i_m)$ with $|I| = i \geqslant 1$, we have*

(a) $\partial_I^i (h \circ \varphi) = \sum_{l=1}^{i} \sum_J C_J^I \,(D^l h \circ \varphi)\, \partial_{J_1}^{j_1}\varphi \cdots \partial_{J_l}^{j_l}\varphi,$

(b) $\partial_I^i e^{\chi} = e^{\chi} \sum_{l=1}^{i} \sum_J C_J^I \,\partial_{J_1}^{j_1}\chi \cdots \partial_{J_l}^{j_l}\chi.$

Here $J=(J_1,\dots,J_l)$ *is any* l*-tuple of vectors in* $(\mathbb{N}\cup\{0\})^m$ *verifying* $J_1+\cdots+J_l=I$ *and* $\{C_J^I\}$ *is a collection of constants with* $C_I^I=1$.

We shall also use the well-known Gronwall's Lemma (see for instance [19]).

Lemma 4.2 (Gronwall). *Let* $u,k,g:[a,b]\longmapsto\mathbb{R}$ *be continuous functions and assume that* $k\geqslant 0$.

(*a*) *If*

$$u(t)\leqslant g(t)+\int_a^t k(s)u(s)ds \textit{ for all } t\in[a,b],$$

then

$$u(t)\leqslant g(t)+\int_a^t g(s)k(s)\exp\left(\int_s^t k(r)dr\right)ds \textit{ for all } t\in[a,b].$$

(*b*) *If*

$$u(t)\leqslant g(t)+\int_t^b k(s)u(s)ds \textit{ for all } t\in[a,b],$$

then

$$u(t)\leqslant g(t)+\int_t^b g(s)k(s)\exp\left(\int_t^s k(r)dr\right)ds \textit{ for all } t\in[a,b].$$

Lemma 4.3. *Let* X *be a complete vector field in some open set* V *of* $\mathbb{R}^n$ *such that* $\|DX\|_V\leqslant\nu$. *Then, for each* $i\geqslant 1$, *there exists a constant* $K_i>0$ *such that the total* i*-differential of the flow* φ_t *of* X *verifies* $\|D^i\varphi_t\|_V\leqslant K_ie^{i\nu|t|}$ *for all* $t\in\mathbb{R}$.

Proof. We proceed by induction on i. Due to $\|D^i\varphi_t\|_V=\max\big\{\|\partial_I^i\varphi_t\|_V : |I|=i\big\}$, it is clear that it suffices to prove the inequality for any partial derivative of order i.

Let us prove the result for $i=1$. To this end let $\big\{I_1,I_2,\dots,I_n\big\}$ be the canonical basis of $\mathbb{R}^n$ and consider some $\partial_{I_j}\varphi_t$. This partial derivative verifies the first variational equation, namely

$$\frac{d}{dt}\partial_{I_j}\varphi_t=\big(DX\circ\varphi_t\big)\partial_{I_j}\varphi_t \text{ with initial condition } \partial_{I_j}\varphi_t\big|_{t=0}=I_j.$$

Accordingly

$$\partial_{I_j}\varphi_t=I_j+\int_0^t(DX\circ\varphi_s)\partial_{I_j}\varphi_s\,ds.$$

Consequently, since $\|DX\|_V\leqslant\nu$ by assumption, the function $u_1(t)=\|\partial_{I_j}\varphi_t\|_V$ satisfies

$$u_1(t)\leqslant 1+\int_0^t\nu u_1(s)\,ds,\ \text{ if } t\geqslant 0, \text{ and } u_1(t)\leqslant 1+\int_t^0\nu u_1(s)\,ds,\ \text{ if } t\leqslant 0.$$

Applying Gronwall's Lemma to each inequality we obtain respectively

$$u_1(t)\leqslant 1+\int_0^t\nu e^{\nu(t-s)}\,ds=e^{\nu t} \text{ for } t\geqslant 0$$

and

$$u_1(t)\leqslant 1+\int_t^0\nu e^{\nu(s-t)}\,ds=e^{-\nu t} \text{ for } t\leqslant 0.$$

Hence $u_1(t)\leqslant e^{\nu|t|}$ and so this proves the result for $i=1$.

Assume now that the result is true for $j < i$ and fix some multi-index I with $|I| = i$. Since φ_t is the flow of the vector field X, we have that

$$\frac{d}{dt}\,\partial_I^i\varphi_t = \partial_I^i(X\circ\varphi_t) \text{ with } \partial_I^i\varphi_t\big|_{t=0} = 0.$$

We expand the right hand side of the above equality by applying (a) in Lemma 4.1 to the each component and, after integration, we obtain

$$\partial_I^i\varphi_t = \sum_{l=1}^{i}\sum_J C_J^I\int_0^t (D^lX\circ\varphi_s)\,\partial_{J_1}^{j_1}\varphi_s\cdots\partial_{J_l}^{j_l}\varphi_s\,ds.$$

Note that the second summation above is taken over all the l-tuples $J = (J_1,\dots,J_l)$ with $J_1+\cdots+J_l = I$. Therefore we can split it up as

$$\partial_I^i\varphi_t = C_I^I\int_0^t (DX\circ\varphi_s)\,\partial_I^i\varphi_s\,ds + \sum_{l=2}^{i}\sum_J C_J^I\int_0^t (D^lX\circ\varphi_s)\,\partial_{J_1}^{j_1}\varphi_s\cdots\partial_{J_l}^{j_l}\varphi_s\,ds.$$

Then, denoting $u_i(t) = \|\partial_I^i\varphi_t\|_V$ and taking $C_I^I = 1$ into account, by using the inductive hypothesis we obtain

$$u_i(t)\leqslant K_I\int_0^t e^{i\nu s}\,ds + \int_0^t \nu u_i(s)\,ds \leqslant \frac{K_I}{i\nu}e^{i\nu t} + \int_0^t \nu u_i(s)\,ds,\ \text{ if } t\geqslant 0,$$

and

$$u_i(t)\leqslant K_I\int_t^0 e^{-i\nu s}\,ds + \int_t^0 \nu u_i(s)\,ds \leqslant \frac{K_I}{i\nu}e^{-i\nu t} + \int_t^0 \nu u_i(s)\,ds,\ \text{ if } t\leqslant 0.$$

Here the positive constant

$$K_I = \sum_{l=2}^{i}\sum_J C_J^I\|D^lX\|_V K_{j_1}\cdots K_{j_l}$$

depends continuously on $\|D^jX\|_V$, $j = 2,\dots,i$. Finally, by applying Gronwall's Lemma, it follows that

$$u_i(t)\leqslant \frac{K_I}{i\nu}e^{i\nu t} + \int_0^t \frac{K_I}{i\nu}e^{i\nu s}\nu e^{\nu(t-s)}\,ds \leqslant \frac{K_I}{i(i-1)\nu}e^{i\nu t}\leqslant K_i e^{i\nu t},\ \text{ if } t\geqslant 0,$$

and

$$u_i(t)\leqslant \frac{K_I}{i\nu}e^{i\nu t} + \int_t^0 \frac{K_I}{r\nu}e^{-i\nu s}\nu e^{\nu(s-t)}\,ds \leqslant \frac{K_I}{i(i-1)\nu}e^{-i\nu t}\leqslant K_i e^{-i\nu t},\ \text{ if } t\leqslant 0,$$

where we take $K_i = \max\big\{\frac{K_I}{i(i-1)\nu} : |I| = i\big\}$. This completes the proof of the result. □

The following result will be used to bound the derivatives of $\mathcal{I}_k(x,y,\mu,t)$ with respect to x, y and μ. Note that it refers to the functions $\mathcal{J}_k(x,y,\mu,t)$ such that $\mathcal{I}_k = e^{(m-\lambda n)t}\mathcal{J}_k$.

Lemma 4.4. *Let $X(x,y,\mu) = x\partial_x + \big(-\lambda_0 + P(x,y,\mu)\big)y\partial_y$ be a complete vector field in the open subset $V_\delta = \mathbb{R}^2\times\big\{\|\mu-\mu_0\| < \delta\big\}\subset\mathbb{R}^2\times\mathcal{U}$ such that $\|P\|_{V_\delta}\leqslant\eta$ and $\|DX\|_{V_\delta}\leqslant\nu$. Let h_1 and h_2 be M-flat functions on $\mathbb{R}^2\times\mathcal{U}$ with respect to x and y respectively. In addition, for $k = 1,2$, define*

$$\mathcal{J}_k(x,y,\mu,t) = \big(h_k\circ\varphi_t(x,y,\mu)\big)\exp\left(n\int_0^t P\circ\varphi_s(x,y,\mu)\,ds\right), \tag{13}$$

where φ_t is the flow of X. Then, for each $i = 0, \ldots, M$, we have

$$|\partial_I^i \mathcal{J}_1(x,y,\mu,t)| \le K|x|^{M-i} e^{(M-(\nu+1)i-|n|\eta)t} \qquad \text{if } t \in (-\infty, 0),$$

$$|\partial_I^i \mathcal{J}_2(x,y,\mu,t)| \le K|y|^{M-i} e^{(-\lambda_0 M+(\nu+\lambda_0)i+(|n|-M+i)\eta)t} \qquad \text{if } t \in (0, +\infty),$$

for all $(x,y,\mu) \in V_\delta$ and some positive constant K (independent of x, y, μ and t).

Proof. The flatness assumption on h_1 and h_2 means that, for $0 \leqslant r \leqslant M$,

$$\|D^r h_1(x,y,\mu)\| \leqslant C|x|^{M-r} \text{ and } \|D^r h_2(x,y,\mu)\| \leqslant C|y|^{M-r} \tag{14}$$

for all $(x,y,\mu) \in \mathbb{R}^2 \times \mathcal{U}$. It is easy to show that the first two components of the flow φ_t are given by

$$\varphi_t^1(x,y,\mu) = xe^t \text{ and } \varphi_t^2(x,y,\mu) = ye^{-\lambda_0 t} e^{\chi(x,y,\mu,t)}$$

for all $t \in \mathbb{R}$, where

$$\chi(x,y,\mu,t) = \int_0^t P \circ \varphi_s(x,y,\mu)\, ds.$$

Moreover, due to $\|P\|_{V_\delta} \leqslant \eta$, we have that $|\chi(x,y,\mu,t)| \leqslant |t|\eta$ for all $(x,y,\mu,t) \in V_\delta \times \mathbb{R}$. Accordingly, if $t \leqslant 0$ then $|\varphi_t^2(x,y,\mu)| \leqslant |y|e^{-(\lambda_0-\eta)t}$ for all $(x,y,\mu) \in V_\delta$. The combination of this with (14) yields

$$\begin{aligned} &\|D^r h_1 \circ \varphi_t(x,y,\mu)\| \leqslant C|x|^{M-r} e^{(M-r)t} && \text{if } t \leqslant 0,\\ &\|D^r h_2 \circ \varphi_t(x,y,\mu)\| \leqslant C|y|^{M-r} e^{-(M-r)(\lambda_0-\eta)t} && \text{if } t \geqslant 0, \end{aligned} \tag{15}$$

for each $r = 0, \ldots, M$. The case $i = 0$ follows easily from the above inequalities with $r = 0$ and the bound for χ. On the other hand, from (a) in Lemma 4.1, if $j \geqslant 1$ then

$$\partial_J^j \chi = \int_0^t \partial_J^j (P \circ \varphi_s)\, ds = \sum_{\ell=1}^{j} \sum_{L=(L_1,\ldots,L_\ell)} C_L^J \int_0^t (D^\ell P \circ \varphi_s) \partial_{L_1}^{l_1} \varphi_s \cdots \partial_{L_\ell}^{l_\ell} \varphi_s\, ds.$$

It is important to note that the second summation above is taken over all the multi-indices $L_1, \ldots, L_\ell$ such that $L_1 + \cdots + L_\ell = J$. In order to avoid cumbersome notations, when there is no risk of confusion we use a "universal" positive constant K (meaning that it is something independent from x, y, μ and t). Taking this into account, by using Lemma 4.3 we get

$$|\partial_J^j \chi(x,y,\mu,t)| \leqslant K \int_0^{|t|} e^{j\nu s}\, ds \leqslant K\, e^{j\nu|t|} \text{ for all } t \in \mathbb{R}.$$

Hence, from (b) in Lemma 4.1, it follows that

$$\left|\partial_J^j e^{n\chi(x,y,\mu,t)}\right| \leqslant K e^{(|n|\eta+j\nu)|t|} \text{ for all } t \in \mathbb{R}.$$

Exactly in the same way as we bound $|\partial_J^j \chi|$, the combination of Lemmas 4.1 and 4.3 shows that

$$\left|\partial_J^j \big(h_k \circ \varphi_t(x,y,\mu)\big)\right| \leqslant K e^{j\nu|t|} \sum_{l=1}^{j} \|D^l h_k \circ \varphi_t(x,y,\mu)\| \text{ for all } t \in \mathbb{R}.$$

Now, the two last inequalities and the well-known formula

$$\partial_I^i (a\, b) = \sum_{J+L=I} C_{J,L} \partial_J^j a\, \partial_L^l b, \tag{16}$$

imply that if $i \geq 1$ then

$$|\partial_I^i \mathcal{J}_k(x,y,\mu,t)| \leqslant K e^{(|n|\eta+i\nu)|t|} \sum_{l=1}^{i} \|D^l h_k \circ \varphi_t(x,y,\mu)\| \text{ for all } t \in \mathbb{R}.$$

Finally, thanks to (15), we obtain the desired inequalities. □

Proof of Theorem 2.3. Given $\{X_\mu\}$ as in (1), we must prove that if $\{h_\mu\}$ is a family of N-flat functions with $N \geqslant N(k,\lambda_0,m,n)$, then the homological equation $X_\mu(vf_\mu) = vh_\mu$ has a solution $\{f_\mu\}$ of class $\mathcal{C}^k$. We claim that it suffices to prove this taking the normal form family $\{X_\mu^{NF}\}$ that appears in (5) instead of the original $\{X_\mu\}$. Indeed, thanks to Theorem 2.1, there exists a family of diffeomorphisms $\{\Phi_\mu\}$ such that $\Phi_\mu^\star X_\mu = \kappa_\mu X_\mu^{NF}$, where κ_μ is a function verifying that $\kappa_\mu(0,0) \neq 0$. Since Φ_μ preserves the axes, we have that $\Phi_\mu^\star v := v \circ \Phi_\mu = v\chi_\mu$, where χ_μ is a function with $\chi_\mu(0,0) \neq 0$. Define $h_\mu^{NF} = \frac{\chi_\mu}{\kappa_\mu}\Phi_\mu^\star h_\mu$, which clearly is also a family of N-flat functions. Now, if the corresponding homological equation

$$X_\mu^{NF}\big(vf_\mu^{NF}\big) = vh_\mu^{NF}$$

has a C^k solution $\{f_\mu^{NF}\}$ then, using the equality $\Phi^\star(X(F)) = (\Phi^\star X)(\Phi^\star F)$, one can easily check that

$$f_\mu = \tfrac{1}{v}\big(\Phi_\mu^{-1}\big)^\star\big(vf_\mu^{NF}\big)$$

is a $\mathcal{C}^k$ solution of the original homological equation, i.e., it verifies $X_\mu(vf_\mu) = vh_\mu$.

From now on and, as we have just shown, without loss of generality, we study the equation

$$X_\mu\big(vf_\mu\big) = vh_\mu \text{ with } X_\mu = x\partial_x + \big(-\lambda(\mu) + P_\mu(u)\big)y\partial_y. \tag{17}$$

In order to construct a solution of the homological equation it is convenient that the flow of X_μ is defined for all $t \in \mathbb{R}$. This can be achieved by a "globalization process" using a suitable family of bump functions. More precisely, we consider a family of $\mathcal{C}^\infty$ bump functions $\{\psi_\varepsilon\}$ such that

$$\psi_\varepsilon(x,y) = \begin{cases} 1 & \text{if } \|(x,y)\| \leqslant \varepsilon/2, \\ 0 & \text{if } \|(x,y)\| \geqslant \varepsilon, \end{cases}$$

and verifying moreover $\|D\psi_\varepsilon\| < \frac{c}{\varepsilon}$ for a fixed $c > 2$. Then, setting $h_\mu^\varepsilon = h_\mu\psi_\varepsilon$ and

$X_\mu^\varepsilon = x\partial_x + \big(-\lambda_0 + P_\mu^\varepsilon(x,y)\big)y\partial_y$, where $P_\mu^\varepsilon(x,y) = \big(\lambda_0 - \lambda(\mu) + P_\mu(u)\big)\psi_\varepsilon(x,y)$,

we consider the homological equation

$$X_\mu^\varepsilon(vf_\mu^\varepsilon) = vh_\mu^\varepsilon, \tag{18}$$

which coincides with (17) on the $\varepsilon/2$-disk centered at the origin. Now, as we explained before, taking $M = \left[\frac{N}{2}\right]$, we write $h_\mu^\varepsilon = h_{\mu,1}^\varepsilon + h_{\mu,2}^\varepsilon$ with $h_{\mu,1}^\varepsilon$ and $h_{\mu,2}^\varepsilon$ being M-flat with respect to x and y respectively. Then we define

$$f_\mu^\varepsilon(x,y) = \int_{-\infty}^{0} \mathcal{I}_1^\varepsilon(x,y,\mu,t)\,dt - \int_0^{+\infty} \mathcal{I}_2^\varepsilon(x,y,\mu,t)\,dt, \tag{19}$$

where

$$\mathcal{I}_l^\varepsilon(x,y,\mu,t) = e^{(m-\lambda(\mu)n)t}\big(h_{\mu,l}^\varepsilon \circ \varphi_t(x,y,\mu)\big)\exp\left(n\int_0^t P_\mu^\varepsilon \circ \varphi_s(x,y,\mu)\,ds\right)$$

and $\varphi_t(x,y,\mu)$ is the flow of X_μ^ε. It is important to mention that this flow is defined for all $t \in \mathbb{R}$ because X_μ^ε is linear outside a compact set. Now the key point is to prove that (19) is a well defined $\mathcal{C}^k$ function because then, as we showed before, it is straightforward to verify that it is a solution of (18). The rest of the proof is dedicated to showing that this is the case provided that ε and $\|\mu-\mu_0\|$ are small enough.

For each positive ε and δ we consider the subsets

$$V_{\varepsilon,\delta}=\big\{(x,y,\mu)\in\mathbb{R}^{2+\mathtt{m}}:\|(x,y)\|<\varepsilon,\ \|\mu-\mu_0\|<\delta\big\}\ \text{ and }\ V_\delta=\mathbb{R}^2\times\big\{\|\mu-\mu_0\|<\delta\big\}.$$

Then we have the estimate

$$\|P_\mu^\varepsilon\|_{V_\delta}\leqslant\sup_{\|\mu-\mu_0\|<\delta}\{|\lambda(\mu)-\lambda_0|\}+\|P_\mu(x,y)\|_{V_{\varepsilon,\delta}}=:\eta(\varepsilon,\delta),$$

where note that $\eta(\varepsilon,\delta)$ is a continuous function tending to 0 as $(\varepsilon,\delta)\to(0,0)$. Define $\nu_0=\max\{1,\lambda_0\}$. Then, since $\|yD\psi_\varepsilon\|_{V_{\varepsilon,\delta}}\leqslant c$, it follows that

$$\|DX_\mu^\varepsilon\|_{V_\delta}\leqslant\nu_0+\|yDP_\mu\|_{V_{\varepsilon,\delta}}+\|yD\lambda(\mu)\|_{V_{\varepsilon,\delta}}+c\eta(\varepsilon,\delta)=:\nu(\varepsilon,\delta),$$

where $\nu(\varepsilon,\delta)$ is a continuous function tending to ν_0 as $(\varepsilon,\delta)\to(0,0)$. We can hence apply Lemma 4.4 to bound the partial derivatives of

$$\mathcal{J}_l^\varepsilon(x,y,\mu,t):=\big(h_{\mu,l}^\varepsilon\circ\varphi_t(x,y,\mu)\big)\exp\left(n\int_0^t P_\mu^\varepsilon\circ\varphi_s(x,y,\mu)\,ds\right)\text{ for }l=1,2.$$

Our goal is to choose $\bar\varepsilon$ and $\bar\delta$ small enough so that the bounds of the partial derivatives of $\mathcal{I}_l^{\bar\varepsilon}$ are integrable functions with respect to t. To this end note that if

$$N\geqslant N(k,\lambda_0,m,n)=2\big[\max\{(\nu_0+1)k-m+\lambda_0n,(\nu_0/\lambda_0+1)k+m/\lambda_0-n\}+1\big],$$

then $M=\left[\frac{N}{2}\right]$ satisfies the inequalities

$$M>(\nu_0+1)k-m+\lambda_0n\ \text{ and }\ M>(\nu_0/\lambda_0+1)k+m/\lambda_0-n.$$

By continuity of $\eta(\varepsilon,\delta)$ and $\nu(\varepsilon,\delta)$, there exist $\bar\varepsilon,\bar\delta>0$ such that $\eta=\eta(\bar\varepsilon,\bar\delta)$ and $\nu=\nu(\bar\varepsilon,\bar\delta)$ are close enough to 0 and ν_0 respectively, in order that the inequalities

$$\alpha_1:=M-(\nu+1)k-|n|\eta+m-\lambda(\mu)n>0$$

$$\alpha_2:=-\lambda_0M+(\nu+\lambda_0)k+(|n|-M+k)\eta+m-\lambda(\mu)n<0$$

hold for $\|\mu-\mu_0\|<\bar\delta$. Thus, by applying Lemma 4.4, we can assert that the inequalities

$$|\partial_I^i\mathcal{I}_1^{\bar\varepsilon}(x,y,\mu,t)|\leqslant Ke^{(m-\lambda(\mu)n)t}\sum_{1\leqslant|J|\leqslant i}|\partial_J^j\mathcal{J}_1^{\bar\varepsilon}(x,y,\mu,t)|\leqslant K|x|^{M-k}e^{\alpha_1t}$$

and

$$|\partial_I^i\mathcal{I}_2^{\bar\varepsilon}(x,y,\mu,t)|\leqslant Ke^{(m-\lambda(\mu)n)t}\sum_{1\leqslant|J|\leqslant i}|\partial_J^j\mathcal{J}_2^{\bar\varepsilon}(x,y,\mu,t)|\leqslant K|y|^{M-k}e^{\alpha_2t}$$

are verified for $0\leqslant i\leqslant k$. (To see this we also used the formula in (16) for the derivation of a product.) Therefore, since $\alpha_1>0$ and $\alpha_2<0$ by construction, the functions $\partial_I^i\mathcal{I}_1^{\bar\varepsilon}$ and $\partial_I^i\mathcal{I}_1^{\bar\varepsilon}$ are integrable with respect to t on $(-\infty,0)$ and $(0,\infty)$ respectively. Thus, $f_\mu^{\bar\varepsilon}$ is a well-defined $\mathcal{C}^k$ function and, accordingly, it is a solution of the homological equation (17) for $\|(x,y)\|<\bar\varepsilon/2$ and $\|\mu-\mu_0\|<\bar\delta$. $\square$

Part 2. Asymptotic expansion of the Dulac time

5. **Statement and proof of Theorem B.** In this section we give an asymptotic development of the Dulac time (time of passing around a corner) of a family of vector fields unfolding a saddle point with possibly polar factors in the coordinate axes. We see this result as a basic building block for studying the Poincaré time (time associated to the Poincaré map) near a polycycle. Critical periods of the Poincaré time are particulary important since the condition of non-criticality of the period appears for instance in the bifurcation theory of subharmonics. Under the non-criticality of the period, zeros of appropriate Melnikov functions guarantee the persistence of a subharmonic periodic orbit of a Hamiltonian under a periodic non-autonomous deformation (see Theorem 4.6.2 of [3]). Moreover, the problem of existence of a uniform bound for the number of critical points of the period function on a family of polynomial (or analytic) vector fields can be seen as a problem analogous to the second part of 16th Hilbert problem on limit cycles. We see our work as a contribution to establishing a finite "cyclicity" result in finite codimension (i.e., existence of a local uniform bound) for the number of critical points of the period function of polynomial vector fields on hyperbolic or more general polycycles.

Let $\mathcal{U}$ be an open set of $\mathbb{R}^{\mathtt{m}}$ and let $\{X_\mu,\, \mu\in\mathcal{U}\}$ be a $\mathcal{C}^\infty$ family of vector fields defined in some open set U of $\mathbb{R}^2$. Assume that the vector field X_μ has a hyperbolic saddle p_μ as unique critical point inside U. In this situation it is well know that there exists exactly two smooth transverse invariant curves $\mathcal{S}_\mu$ and $\mathcal{T}_\mu$ through p_μ (depending also smoothly on μ). We also consider a family Y_μ proportional to X_μ but having poles along $\mathcal{S}_\mu$ and $\mathcal{T}_\mu$ of order m and n respectively. We make the convention that if m (respectively, n) is a negative integer then Y_μ vanishes along the invariant curve $\mathcal{S}_\mu$ (respectively, $\mathcal{T}_\mu$) with multiplicity $-m$ (respectively, $-n$). We can take a coordinate system (x,y,μ) on $U\times\mathcal{U}\subset\mathbb{R}^{2+\mathtt{m}}$ such that $p_\mu=(0,0,\mu)$, $\mathcal{S}_\mu=\big\{(x,y,\mu):x=0\big\}$ and $\mathcal{T}_\mu=\big\{(x,y,\mu):y=0\big\}$.

In the coordinates mentioned above X_μ and Y_μ can be written as in (1) and (2) respectively. Our goal is to study these two families in a neighbourhood of a parameter $\mu_0\in\mathcal{U}$ such that

$$\lambda(\mu_0)=\frac{p}{q} \text{ with with } (p,q)=1.$$

By applying Theorem A, in a neighbourhood of $(0,0,\mu_0)\in\mathbb{R}^{2+\mathtt{m}}$ there exists a $\mathcal{C}^k$ diffeomorphism Φ such that

$$\big(\Phi^\star Y_\mu\big)=Y_\mu^{NF}:=\frac{1}{v+u^\ell Q_\mu(u)}\Big(x\partial_x+\big(-\lambda(\mu)+P_\mu(u)\big)y\partial_y\Big).$$

Here recall that $v=x^my^n$, P_μ and Q_μ are polynomials in the resonant monomial $u=x^py^q$ and

$$\ell=\min\big\{\beta\in\mathbb{Z}:\beta(p,q)>(m,n)\big\}.$$

Composing Φ with suitable homotheties we can assume that Φ is defined on $\{|x|<2,\, |y|<2\}$. Let Σ_1^N and Σ_2^N be two *normalized* transverse sections to the separatrices $x=0$ and $y=0$ respectively. To be more precise, Σ_1^N and Σ_2^N are parameterized by $\sigma_1(s):=\Phi(s,1)$ and $\sigma_2(s):=\Phi(1,s)$ respectively, so that $\Sigma_1^N=\Phi\big(\{y=1\}\big)$ and $\Sigma_2^N=\Phi\big(\{x=1\}\big)$. We denote the Dulac map and the time function associated to the transverse sections Σ_1^N and Σ_2^N by D and T respectively. More precisely, if

$\varphi(t,(x_0,y_0);\mu)$ is the solution of Y_μ passing through (x_0,y_0) at $t=0$, for each $s>0$ we define $D(s;\mu)$ and $T(s;\mu)$ by means of the relation

$$\varphi\big(T(s;\mu),\sigma_1(s);\mu\big)=\sigma_2\big(D(s;\mu)\big).$$

Theorem B is the main result of this part of the paper. It gives an asymptotic development of $T(s;\mu)$ near $s=0$, uniform with respect to μ, assuming that m and n are not both negative. (This assumption implies that $\ell\geqslant 1$.) After exchanging coordinates if necessary, we assume that $n\geqslant 0$. In order to state the result we must introduce the so called *Roussarie-Écalle compensator*, namely,

$$\omega(s;\alpha)=\begin{cases}\frac{s^{-\alpha}-1}{\alpha} & \text{if } \alpha\neq 0,\\ -\log s & \text{if } \alpha=0.\end{cases}$$

We also define $\alpha(\mu):=p-\lambda(\mu)q$.

Theorem B. With the above notation and assumptions, for each $K\in\mathbb{N}$, we have that

$$T(s^q;\mu)=a_0\log s+s^{mq}A_\mu(s^{pq})+B_\mu\big(s^p,s^p\omega(s,\alpha(\mu))\big)+\Psi_K(s;\mu),$$

where, for $\mu\approx\mu_0$, $\Psi_K(s;\mu)$ is a K-flat function at $s=0$ uniformly on μ. Moreover

(a) $a_0=-q$ in case that $(m,n)=(0,0)$ and zero otherwise.
(b) $A_\mu(z)$ and $B_\mu(z,w)$ are polynomials in z and w and their coefficients are rational functions in the coefficients of P_μ and Q_μ in (4) without poles at $\mu=\mu_0$.
(c) The order of $B_\mu(z,w)$ at $(0,0)$ is $\geqslant\min(n,q\ell)$ and, if $mq-pn\neq 0$, then $A_\mu(0)=\frac{1}{\lambda(\mu)n-m}$.

Let us clarify that the above expression of $T(s^q;\mu)$ provides, after the substitution of s by $s^{1/q}$, the asymptotic development of $T(s;\mu)$ at $s=0$ for $\mu\approx\mu_0$. We prefer to state it in this way for the sake of shortness and simplicity in the proof. In order to prove Theorem B let us first note that, by construction, if $\big(x_t(s),y_t(s)\big)$ is the solution of $X_\mu^{NF}=x\partial_x+\big(-\lambda(\mu)+P_\mu(u)\big)y\partial_y$ with initial condition $(x_0,y_0)=(s,1)$, then

$$T(s;\mu)=\int_0^{-\log s}\big(v+u^\ell Q_\mu(u)\big)\Big|_{(x_t(s),y_t(s))}dt,$$

where recall that $v=x^my^n$ and $u=x^py^q$. Thus $T(s;\mu)$ is a finite linear combination of terms

$$T_{ij}(s)=\int_0^{-\log s}x_t(s)^iy_t(s)^j\,dt$$

with $(i,j)\in\mathcal{I}:=\big\{(m,n)\big\}\cup\big\{\nu\,(p,q):\nu=\ell,\dots,\ell+\deg Q_\mu\big\}$. Here and in what follows, in order to avoid long formulae we omit the dependence on μ when there is no risk of ambiguity. Clearly $T_{00}(s)=-\log s$ and, in case that $i\neq 0$, $T_{i0}(s)=\frac{s^i-1}{i}$. So it suffices to study $T_{ij}(s)$ for $j\neq 0$ and to this end we take advantage of some results of Roussarie in [13, Chapter 5]. For the sake of clarity we collect them in the following lemma:

Lemma 5.1. *For each $t\geqslant 0$, $u_t(s)=x_t(s)^py_t(s)^q$ can be expanded as a series in s as*

$$u_t(s)=\sum_{k=1}^{\infty}g_k(t)s^{pk},$$

where $g_1(t) = e^{\alpha t}$ *and* $g_k(0) = 0$ *for* $k \geqslant 2$. *In addition, for each* $r \geqslant 0$, *we have that*

$$|\partial^r g_k(t)| \leqslant C_r C^k e^{tk/3} \text{ for all } t \geqslant 0 \text{ and } \mu \approx \mu_0$$

for some constants C *and* C_r *(independent of* t, μ *and* k*). Finally,* $g_k(t) = e^{\alpha t}\bar{g}_{k-1}(t)$ *with* $\bar{g}_{k-1}(t)$ *being a polynomial of degree* $\leqslant k-1$ *in* $\Omega(t,\alpha) := \frac{e^{\alpha t}-1}{\alpha}$.

It is to be noted that the upper bound of $\partial^r g_k$ in Lemma 5.1 is slightly different from the one in [13] because there the exponential factor is $e^{tk/2}$ instead of $e^{tk/3}$. This is only a technicality. Indeed, one can easily verify that if μ is close enough to μ_0 so that $|\alpha(\mu)| < 1/3$ then we can replace $1/2$ by $1/3$ in the exponent. Now, with the notation introduced in Lemma 5.1, it follows that

$$y_t(s) = e^{-\lambda t}\left(\sum_{k=0}^{\infty} \bar{g}_k(t) s^{kp}\right)^{1/q} \quad \text{for } t \in [0, -\log s].$$

Since $(1+z)^{j/q} = \sum_{l=0}^{\infty} \binom{j/q}{l} z^l$ for $|z| < 1$, we get

$$y_t(s)^j = e^{-\lambda j t}\sum_{k=0}^{\infty} \bar{g}_{jk}(t) s^{kp},$$

with

$$\bar{g}_{jk} := \sum_{l=1}^{k} \sum_{i_1+\cdots+i_l=k} \binom{j/q}{l} \bar{g}_{i_1}\cdots\bar{g}_{i_l}. \tag{20}$$

Note that there are as many summands above as the number $p(k)$ of partitions of k and it is easy to see that $p(k) \leqslant \binom{2k-1}{k} \leqslant 2^{2k-1} \leqslant 4^k$. On the other hand, if $l \leqslant k$ then $\binom{j/q}{l} \leqslant |j|^l \leqslant |j|^k$. Thus, using the inequality in Lemma 5.1 for $r = 0$, it is easy to check that

$$|\bar{g}_{jk}| \leqslant \big(4|j|C_0\big)^k e^{(2/3+|\alpha|)kt}$$

for some positive constant C_0. Consequently, if $s \approx 0$ and $\alpha \approx 0$, then

$$T_{ij}(s) = \int_0^{-\log s} \sum_{k=0}^{\infty} s^{pk+i} e^{(i-\lambda j)t}\bar{g}_{jk}(t)\,dt = \sum_{k=0}^{\infty} s^{pk+i}\int_0^{-\log s} e^{(i-\lambda j)t}\bar{g}_{jk}(t)\,dt,$$

since the right-hand side of

$$\left| s^{pk+i}\int_0^{-\log s} e^{(i-\lambda j)t}\bar{g}_{jk}(t)\,dt\right| \leqslant s^{\lambda j}\left(4|j|C_0 s^{(p-2/3-|\alpha|)}\right)^k \tag{21}$$

is the general term of a convergent series in k provided that s and α are small enough. In short, we have shown that

$$T_{ij}(s) = \sum_{k=0}^{\infty} s^{pk+i} T_{ijk}(s) \text{ with } T_{ijk}(s) := \int_0^{-\log s} e^{(i-\lambda j)t}\bar{g}_{jk}(t)\,dt. \tag{22}$$

Our next goal is to bound the derivatives of the k-th term in the above series. More concretely, we prove the following result:

Lemma 5.2. *For each* $r \geqslant 0$ *there exits a positive constant* C_r *such that*

$$|\partial^r\big(s^{pk+i}T_{ijk}(s)\big)| \leqslant k^r\big(4|j|C_r\big)^k s^{(p-2/3-|\alpha|)k-r+\lambda j}.$$

Proof. The case $r=0$ follows directly from (21). To study the case $r \geqslant 1$ let us introduce the function

$$\bar{h}_{jk}(s)=\bar{g}_{jk}(-\log s),$$

so that $\partial T_{ijk}(s)=-\bar{h}_{jk}(s)s^{\lambda j-i-1}$. By (a) in Lemma 4.1, we have that

$$\partial^r\big(g_k\circ(-\log s)\big)=\sum_{l=1}^{r}\sum_{i_1+\cdots+i_l=r}C_i^r\big((\partial^l g_k)\circ(-\log s)\big)\partial^{i_1}(-\log s)\cdots\partial^{i_l}(-\log s)$$

for some collection of constants $\{C_i^r\}_{i,r}$. Accordingly, by applying Lemma 5.1, there exist positive constants C and C_r such that

$$|\partial^r\big(g_k\circ(-\log s)\big)|\leqslant C_rC^ks^{-k/3-r}.$$

Here C is the same as in Lemma 5.1 whereas C_r is not. Since $\bar{g}_k(-\log s)=s^{\alpha}g_{k+1}(-\log s)$, on account of (16) we get

$$\partial^r\big(\bar{g}_k\circ(-\log s)\big)=\sum_{h=0}^{r}\binom{r}{h}\partial^h(s^{\alpha})\,\partial^{r-h}\big(g_{k+1}(-\log s)\big)$$

and consequently

$$|\partial^r\big(\bar{g}_k\circ(-\log s)\big)|\leqslant C_rC^{k+1}s^{-(k+1)/3-r-|\alpha|}.$$

Now, by using the above estimates in the r-th derivative of (20),

$$\begin{aligned}\partial^r\bar{h}_{jk}(s)&=\\&=\sum_{l=1}^{k}\sum_{i_1+\cdots+i_l=k}\binom{j/q}{l}\sum_{j_1+\cdots+j_l=r}C_{j_1,\ldots,j_l}\partial^{j_1}\big(\bar{g}_{i_1}(-\log s)\big)\cdots\partial^{j_l}\big(\bar{g}_{i_l}(-\log s)\big),\end{aligned}$$

we obtain that

$$\begin{aligned}|\partial^r\bar{h}_{jk}(s)|&\leqslant\sum_{l=1}^{k}|j|^l\sum_{i_1+\cdots+i_l=k}\sum_{j_1+\cdots+j_l=r}(C_rC)^lC^ks^{-(k+l)/3-r-l|\alpha|}\\&\leqslant\big(4|j|C_r\big)^ks^{-(2/3-|\alpha|)k-r}.\end{aligned}$$

(In the two inequalities above, and in what follows, for the sake of simplicity C_r stands for a "universal" constant not depending on k.) Hence, due to $\partial T_{ijk}(s)=-\bar{h}_{jk}(s)s^{\lambda j-i-1}$, from (16) we conclude that

$$|\partial^rT_{ijk}(s)|\leqslant(4|j|C_r)^ks^{-(2/3+|\alpha|)k-r+\lambda j-i}.$$

Finally, the inequality in the statement follows by applying the derivation formula in (16) once again. ◻

Proposition 1. *Fix $(i,j)\in\mathcal{I}$ with $j\neq 0$. Then, for each $K>0$, there exists $M(K)>0$ such that*

$$\Psi_{ij}^K(s)=\sum_{k=M(K)}^{\infty}s^{pk+i}T_{ijk}(s)$$

is a K-flat function at $s=0$ for $\mu\approx\mu_0$.

Proof. Since $\alpha(\mu_0)=0$, there exists a constant β such that

$$0<\beta<p-2/3-|\alpha| \text{ for } \mu\approx\mu_0.$$

If s is small enough then $\sum_{k=M(K)}^{\infty} k^r\big(4|j|C_r s^\beta\big)^k$ is a convergent series for each $r=0,\ldots,K$. Denote by $\widehat{C}$ the maximum of their values for $r=0,\ldots,K$. Then, by Lemma 5.2 and taking $M(K)>\frac{K-\lambda j}{p-2/3-|\alpha|-\beta}$, it follows that

$$\sum_{k=M(K)}^{\infty}|\partial^r(s^{pk+i}T_{ijk}(s))|\leqslant s^{K-r}\sum_{k=M(K)}^{\infty}k^r\big(4|j|C_r s^\beta\big)^k s^{(p-2/3-|\alpha|-\beta)k-K+\lambda j}$$
$$\leqslant \widehat{C}s^{K-r}.$$

This proves the result. □

Proof of Theorem B. Set $\Psi_{i0}^K\equiv 0$. For $j\neq 0$, consider the functions Ψ_{ij}^K given by Proposition 1 and define $T_{ij}^K=T_{ij}-\Psi_{ij}^K$. Then it follows that

$$T(s)=\sum_{(i,j)\in\mathcal{I}}T_{ij}(s)=T^K(s)+\Psi^K(s),$$

where $T^K:=\sum_{(i,j)\in\mathcal{I}}T_{ij}^K$ and Ψ^K is K-flat at $s=0$. Here recall that

$$\mathcal{I}=\big\{(m,n)\big\}\cup\big\{\nu(p,q):\nu=\ell,\ldots,\ell+\deg Q_\mu\big\}.$$

Note moreover that $T_{00}(s)=-\log s$ and, in case that $i\neq 0$, $T_{i0}(s)=\frac{s^i-1}{i}$. So it suffices to study T_{ij}^K with $(i,j)\in\mathcal{I}$ and $j>0$. To this end notice that

$$T_{ij}^K(s)=\sum_{k=0}^{M(K)-1}s^{pk+i}T_{ijk}(s),$$

where T_{ijk} are the functions introduced in (22). We claim that the following is verified:

(*a*) If $(i,j)=(m,n)$ with $mq-np\neq 0$ and $n>0$, then

$$s^{q(pk+m)}T_{mnk}(s^q)=b_0 s^{(m+pk)q}+B_0\big(s^p,s^p\omega(s,\alpha)\big),$$

where $B_0(z,w)$ is a polynomial of order $\geqslant n$ at $(0,0)$.

(*b*) If $(i,j)=\nu(p,q)$ with $\nu>0$, then $s^{p(k+\nu)}T_{\nu p,\nu q,k}(s)=B_\nu\big(s^p,s^p\omega(s,\alpha)\big)$, where $B_\nu(z,w)$ is a polynomial of order $\geqslant k+\nu$ at $(0,0)$.

We will show in addition that, for each $\nu\geqslant 0$, the coefficients of B_ν are rational functions in the coefficients of P_μ in (5) without poles at $\mu=\mu_0$.

In order to prove (*a*) note first that, taking (20) into account and applying Lemma 5.1, it follows that $\bar{g}_{jk}(t)=R_{jk}\big(\Omega(t,\alpha)\big)$ for some polynomial R_{jk} of degree $\leqslant k$. Then

$$T_{mnk}(s)=\int_0^{-\log s}e^{(m-\lambda n)t}R_{nk}\big(\Omega(t,\alpha)\big)dt=\int_0^{\omega(s,\alpha)}(1+\alpha w)^{\frac{m-\lambda n}{\alpha}-1}R_{nk}(w)dw,$$

where to obtain the second equality we perform the change of variables $w = \Omega(t, \alpha)$. Then, after integrating by parts k times, we get

$$T_{mnk}(s) = \left[\frac{(1+\alpha w)^{\frac{m-\lambda n}{\alpha}}}{m-\lambda n} \left(R_{nk}(w) - \frac{R'_{nk}(w)(1+\alpha w)}{m-\lambda n+\alpha} \right.\right.$$

$$\left.\left. + \frac{R''_{nk}(w)(1+\alpha w)^2}{(m-\lambda n+\alpha)(m-\lambda n+2\alpha)} + \cdots + \frac{(-1)^k R^{(k)}_{nk}(w)(1+\alpha w)^k}{(m-\lambda n+\alpha)\cdots(m-\lambda n+k\alpha)} \right) \right]_0^{\omega(s,\alpha)}.$$

Note that the denominators in the above expression are different from zero for $\mu \approx \mu_0$ because $\alpha(\mu_0) = 0$ and, due to $mq - np \neq 0$, $m - \lambda(\mu_0)n \neq 0$. Accordingly

$$T_{mnk}(s) = -\tau_k(0) + \big(1 + \alpha\omega(s,\alpha)\big)^{\frac{m-\lambda n}{\alpha}} \tau_k\big(\omega(s,\alpha)\big),$$

where τ_k is a polynomial of degree k with $\tau_0(0) = \frac{1}{m-\lambda n} \neq 0$. Then, using that $1 + \alpha\omega(s,\alpha) = s^{-\alpha}$ and $\alpha = p - \lambda q$, some easy manipulations show that

$$\begin{aligned}\big(1+\alpha\omega(s,\alpha)\big)^{\frac{m-\lambda n}{\alpha}} &= \big(1+\alpha\omega(s,\alpha)\big)^{\frac{qm-np}{q\alpha}} \big(1+\alpha\omega(s,\alpha)\big)^{\frac{n}{q}} \\ &= s^{\frac{p}{q}n-m}\big(1+\alpha\omega(s,\alpha)\big)^{\frac{n}{q}} = s^{-m}\left(s^{\frac{p}{q}} + \tfrac{\alpha}{q} s^{\frac{p}{q}} \omega\big(s^{\frac{1}{q}}, \alpha\big)\right)^n.\end{aligned}$$

Therefore

$$s^{pk+m} T_{mnk}(s) = -\tau_k(0) s^{pk+m} + s^{pk} \tau_k\big(\omega(s,\alpha)\big) \left(s^{\frac{p}{q}} + \tfrac{\alpha}{q} s^{\frac{p}{q}} \omega\big(s^{\frac{1}{q}}, \alpha\big)\right)^n. \tag{23}$$

Note at this point that $\omega(s,\alpha) = J_q\left(\omega\big(s^{1/q},\alpha\big)\right)$ for some polynomial J_q of degree q. Indeed, this is so because

$$\frac{s^{-\alpha}-1}{\alpha} = \frac{(s^{-\alpha/q})^q - 1}{\alpha} = \frac{s^{-\alpha/q}-1}{\alpha}\left(1 + s^{-\alpha/q} + \cdots + s^{-\alpha(q-1)/q}\right)$$

and $s^{-\alpha/q} = 1 + \alpha\omega\big(s^{1/q},\alpha\big)$. This shows that

$$s^p \omega(s,\alpha) = \big(s^{p/q}\big)^q J_q\big(\omega(s^{1/q},\alpha)\big) \tag{24}$$

is a polynomial in $s^{p/q}$ and $s^{p/q}\omega(s^{1/q},\alpha)$. Consequently, so it is $s^{pk}\tau_k\big(\omega(s,\alpha)\big)$ since it is a finite sum of terms of the form

$$s^{pk}\omega(s,\alpha)^i = s^{p(k-i)}\big(s^p\omega(s,\alpha)\big)^i = \big(s^{p/q}\big)^{q(k-i)}\big(s^p\omega(s,\alpha)\big)^i$$

with $0 \leqslant i \leqslant k$. This fact, on account of (23), proves (a).

Next let us prove part (b) of the claim. The same change of variables as before gives

$$T_{\nu p,\nu q,k}(s) = \int_0^{-\log s} e^{\alpha\nu t} R_{\nu q,k}\big(\Omega(t,\alpha)\big)dt = \int_0^{\omega(s,\alpha)} (1+\alpha w)^{\nu-1} R_{\nu q,k}(w)dw,$$

which is a polynomial of degree $\leqslant k+\nu$ in $\omega(s,\alpha)$. Exactly the same way as before this shows that $s^{p(k+\nu)}T_{\nu p,\nu q,k}(s) = B_\nu\big(s^p, s^p\omega(s,\alpha)\big)$ for some polynomial B_ν of order $\geqslant k+\nu$ at $(0,0)$, proving the validity of (b).

In view of (a) it is clear that to prove the result it suffices to study those terms arising from $(i,j) = \nu(p,q)$ with $\nu > 0$. However this is easy because, once again from (24), we can write $s^{pq(k+\nu)}T_{\nu p,\nu q,k}(s^q)$ as a polynomial in s^p and $s^p\omega\big(s,\alpha(\mu)\big)$, which contributes to the terms of $B_\mu\big(s^p, s^p\omega\big(s,\alpha(\mu)\big)\big)$ in $T(s;\mu)$. This concludes the proof of the result. □

6. **Perspectives.** In this section we give some perspectives for future work. The principal motivation for this work was the study of asymptotic properties of the period function near a hyperbolic polycycle. We give an asymptotic development of the Dulac time near a hyperbolic singular point. It is important to note that our Dulac time T in Theorem B is measured between *normalized* transverse sections which are constructed using the diffeomorphism that brings to the normal form (4). In order to have a result on the Dulac time between *arbitrary* transverse sections one must add to the local Dulac time T in Theorem B the two times necessary to go from given transverse sections to the normalized ones. The times must be calculated in the coordinate on the source transversal. This leads to a composition problem. We postpone the solution of this problem to the general paper dealing with hyperbolic polycycles to which we hope to come in a near future. In any case we must study then the composition problem in detail.

Note that the monomials $\log s$, s^{mq+kpq}, $s^{jp}\omega^j$ appearing in the asymptotic development permit a process of derivation division generating a Chebyshev system (see [6, 11]). One can hope that this can be generalized to the total period of a hyperbolic polycycle and that hence in finite codimension one can prove non-accumulation of critical periods on hyperbolic polycycles.

It would be useful to know the structure of the coefficients in the Dulac time and divide the Dulac time similarly as for the Dulac map in [11]. It seems out of reach for the moment.

Note that our study covers all the cases of the polar factors of the vector field (2), except for the case $m, n < 0$. This last case occurs when both separatrices are lines of zeros of the vector field Y. For studying the accumulation of critical periods we don't have to study this case, since in this case the derivative of the Dulac time tends to infinity uniformly on the parameters and hence no critical periods can appear. Nevertheless, in this case very interesting resonances between the order of poles (m, n) and the eigen-values (p, q) seem to appear, leading to higher order compensators. We hope to return to this problem later.

Some parts of the present study apply also to the saddle node case. However, in the treatment of the remainder term via Theorem 2.3 only the part corresponding to the strong variable can be eliminated.

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E-mail address: Pavao.Mardesic@u-bourgogne.fr
E-mail address: davidmp@mat.uab.es
E-mail address: Jordi.Villadelprat@urv.cat