The period function of quadratic centers

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Abstract. In this paper we investigate the bifurcation diagram of the period function associated to a family of quadratic centers, namely the dehomogenized Loud's systems. The local bifurcation diagram of the period function at the center is fully understood using the results of Chicone and Jacobs [4]. Most of the present paper deals with the local bifurcation diagram at the polycycle that bounds the period annulus of the center. The techniques that we use here are different from the ones in [4] because, while the period function extends analytically at the center, it has no smooth extension to the polycycle. At best one can hope that it has some asymptotic expansion. Another major difficulty is that in order to prove that a parameter is not a bifurcation value it is necessary that the asymptotic development is uniform with respect to the parameters. We study also the bifurcations in the interior of the period annulus and we show that there exist three germs of curves in the parameter space that correspond to this type of bifurcation. Moreover we determine some regions in the parameter space for which the corresponding period function has at least one or two critical periods. Finally we propose a complete conjectural bifurcation diagram of the period function of the dehomogenized Loud's systems. Our results can also be viewed as a contribution to the proof of Chicone's conjecture [2].

1 Introduction and main results

In this work we study the bifurcation diagram of the period function associated to a family of quadratic centers. Chicone [2] has conjectured that if a quadratic system has a center with a period function which is not monotonic then, by an affine transformation and a constant rescaling of time, it can be brought to the Loud normal form

$$\begin{cases} \dot{x} = -y + Bxy, \\ \dot{y} = x + Dx^2 + Fy^2, \end{cases} \tag{1}$$

and that the period function of these centers has at most two critical periods. In fact, there is much analytic evidence that the conjecture is true (see [5, 14, 18] for instance). On the other hand, it is proved in [7] that if $B = 0$ then the period function of the center at the origin of system (1) is monotonous. So, from the point of view of the study of the period function, the most interesting stratum of quadratic centers is the family (1) with $B \neq 0$, which can be brought to $B = 1$ by means of a rescaling, i.e., to

$$\begin{cases} \dot{x} = -y + xy, \\ \dot{y} = x + Dx^2 + Fy^2. \end{cases} \tag{2}$$

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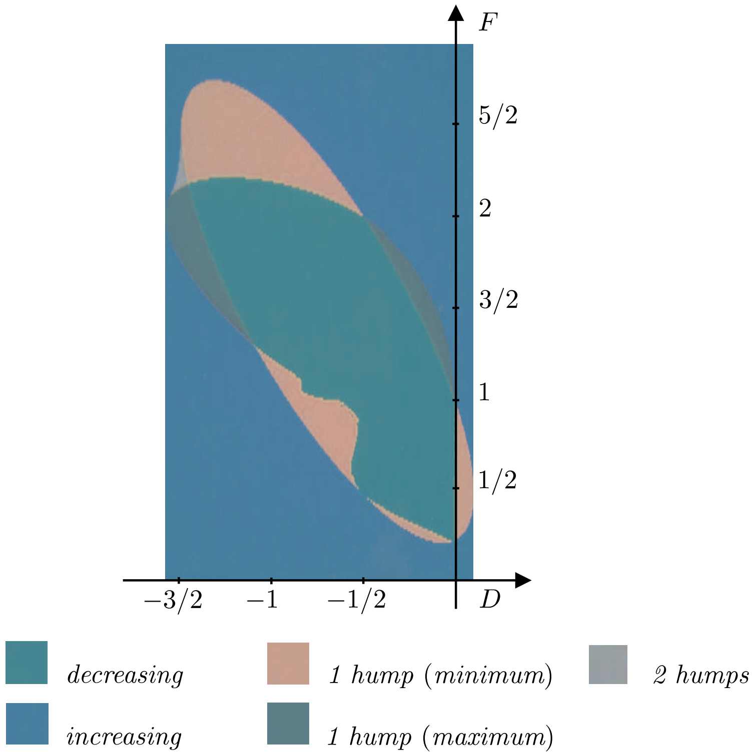

Figure 1: Numerical bifurcation diagram by Chicone and Jacobs.

This is precisely the family of quadratic centers that we study in this paper and, following the terminology in [4], we call them dehomogenized Loud's systems. Compactifying $\mathbb{R}^2$ to the Poincaré disc, the boundary of the period annulus of the center has two connected components, the center itself and a polycycle. We call them respectively the inner and outer boundary of the period annulus. It follows (see Lemma 2.7) that the bifurcation diagram of the period function consists of three parts:

(a) Bifurcations of the period function at the inner boundary (i.e., the center).

(b) Bifurcations of the period function at the outer boundary (i.e., the polycycle).

(c) Bifurcations of the period function in the interior of the period annulus.

For the precise definitions see section 2. The local bifurcation diagram of the period function at the inner boundary is fully understood for the quadratic centers using the results of Chicone and Jacobs [4]. They determined the parameter values from which the maximal number (two) of critical periods bifurcate from the inner boundary and showed that at most one critical period bifurcates from the isochronous centers. Around 1990, Chicone and Jacobs made public a numerical computation of the complete bifurcation diagram (see Figure 1). It presents a strange ellipse-like figure corresponding to the bifurcation parameters of the period function at the outer boundary. The major part of this paper is devoted to the precise determination of this set.

Consider the dotted curve Γ_U and the one in bold Γ_B represented in Figure 2 (here the subscripts B and U stand for bifurcation and unspecified respectively). The curve Γ_U corresponds, except for the segment $(-1,-1/2)\times\{1/2\}$, to bifurcations of the phase portrait that affect the outer boundary of the period annulus (see section 3.1). The curve Γ_B is the union of some explicit straight segments and a curve that joins the

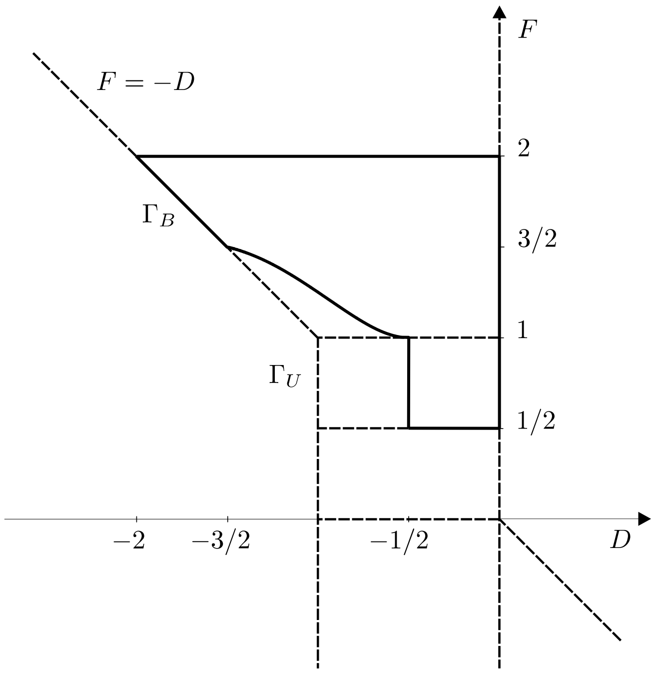

Figure 2: Bifurcation diagram of the period function at the outer boundary

points $(-3/2, 3/2)$ and $(-1/2, 1)$. To be more precise, let us advance that this curve is defined as the zero level set of an explicit function that we introduce in section 3.2.1. To draw it in Figure 2 we have computed it numerically. Analytically, among other properties that are gathered in Proposition 3.11, we have proved that it is the graphic of an analytic function $D = \mathcal{G}(F)$. From Proposition 3.11 it follows in particular that Γ_B is a Jordan curve. We can consider therefore the bounded and unbounded components of $\mathbb{R}^2 \setminus \Gamma_B$, which we denote by $\mathcal{D}_B$ and $\mathcal{I}_B$ (for decreasing and increasing) respectively. With this notation we can now state our main result:

Theorem A. *Denoting $\mu = (D, F)$, let $\{X_\mu,\, \mu \in \mathbb{R}^2\}$ be the family of vector fields in* (2) *and consider the period function of the center at the origin. Then the open set $\mathbb{R}^2 \setminus \{\Gamma_B \cup \Gamma_U\}$ corresponds to local regular values of the period function at the outer boundary of the period annulus. In addition,*

(*a*) *If $\mu_0 \in \mathcal{I}_B \setminus \Gamma_U$ then the period function of X_{μ_0} is monotonous increasing near the outer boundary.*

(*b*) *If $\mu_0 \in \mathcal{D}_B \setminus \Gamma_U$ then the period function of X_{μ_0} is monotonous decreasing near the outer boundary.*

Finally, the parameters in Γ_B are local bifurcation values of the period function at the outer boundary of the period annulus.

We have not determined the character of the parameters in Γ_U. We conjecture that they are not bifurcation values at the outer boundary except for the the segment $\{0\} \times [0, 1/2]$. The numerical picture of Chicone and Jacobs fits relatively well with the bifurcation curve Γ_B. There are however some striking differences. In particular, unlike the numerical picture, most parts of the bifurcation curve are straight segments.

One encounters two major difficulties in the study of the bifurcation diagram of the period function at the outer boundary of the period annulus. The first one is that, contrary to the situation in the inner boundary, the period function does not extend smoothly on the outer boundary. At best one can hope that it has some asymptotic development. The second one is that in order to prove that a parameter is not a bifurcation value one needs an asymptotic development which is uniform with respect to the parameters. This is not easily achieved because the shape of the polycycle in the outer boundary changes as the parameters vary.

The paper is organized in the following way. In section 2 we introduce the precise definitions that we shall use. Section 3 is devoted to the proof of Theorem A. In all the cases that we study (see Figure 3), the polycycle in the outer boundary of the period annulus has one or two singular points, which are saddles. In these cases, the symmetry of the Loud's systems allows to split up the period function and to consider only the time function associated to the passage around one saddle. The most complicated situations are those in which the period annulus is unbounded because then the saddle is at infinity and one has to consider meromorphic vector fields. In order to obtain the asymptotic development mentioned above we use a result proved in [11], which provides the first terms in the expansion of this type of time function (see Proposition 3.9). Theorem 3.3 deals with this situation and so it is the most difficult result to prove. In section 4 we study the bifurcations of the period function in the interior of the period annulus and we show that there exist three germs of curves with this type of bifurcation values. Next, in section 5, we determine some regions in the parameter space for which the corresponding period function has at least one or two critical periods. Finally in section 6 we propose a complete conjectural bifurcation diagram of the period function of the dehomogenized Loud's systems. We also pose some precise open questions remaining to prove its validity. In particular, for certain values of the parameters, the polycycle in the outer boundary of the period annulus has singular points at infinity that are resonant saddles or saddle-nodes. In these cases, tools analogous to Proposition 3.9 still have to be developed. The global study of the bifurcation values of the period function in the interior of the period annulus seems out of reach for the moment.

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2 Basic definitions

We say that a critical point p of a planar differential system is a *center* if it has a punctured neighbourhood that consists entirely of periodic orbits surrounding p. The largest punctured neighbourhood with this property is called the *period annulus* of the center and it will be denoted by $\mathcal{P}$. Compactifying $\mathbb{R}^2$ to the Poincaré disc, the boundary of $\mathcal{P}$ has two connected components, the center itself and a polycycle. We call them respectively the *inner* and *outer boundary* of $\mathcal{P}$.

Definition 2.1 Let Λ be an open subset of $\mathbb{R}^m$ and consider a continuous family of analytic planar vector fields $\{X_\mu,\ \mu \in \Lambda\}$. Suppose that, for each $\mu \in \Lambda$, X_μ has a center at $p_\mu \in \mathbb{R}^2$. We say that the family of corresponding period annuli *varies continuously* if there exists a continuous family of analytic functions $\{\xi_\mu : \mu \in \Lambda\}$ such that, for each $\mu \in \Lambda$, $\xi_\mu \colon [0,1] \longrightarrow \mathbb{RP}^2$ verifies:

(a) $\xi_\mu(0) = p_\mu$ and $\xi_\mu(1)$ belongs to the outer boundary of $\mathcal{P}_\mu$,

(b) $\xi_\mu(s) \in \mathcal{P}_\mu$ for all $s \in (0,1)$,

(c) $\xi'_\mu(s)$ is transverse to $X_\mu\big(\xi_\mu(s)\big)$ for all $s \in (0,1)$.

□

Note that ξ_μ is the parametrization of a transverse section for X_μ in $\mathcal{P}_\mu$. In general, for each fixed $\mu \in \Lambda$, it is always possible to take such a transverse section. Definition 2.1 requires the existence of one that varies continuously with the parameter. As we will see in section 3.1, the period annuli of the family that we study vary continuously. Next remark shows however that this does not always occur.

Remark 2.2 The period annuli of the center at the origin of the 1-parameter family of potential systems

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = x + ax^3 + x^5, \end{cases}$$

do not vary continuously. Indeed, it is easy to show that, for $a < 2$, the period annulus $\mathcal{P}_a$ is the whole plane, while for $a \geq 2$ there exists a positive constant r (not depending on a) such that $\mathcal{P}_a$ is inside a disk of radius r. □

Let $\{X_\mu,\, \mu \in \Lambda\}$ be a continuous family of analytic vector fields with a center p_μ. Assume that the corresponding period annuli vary continuously and consider the family of transverse sections parametrized by $\{\xi_\mu,\, \mu \in \Lambda\}$. For each $(s;\mu) \in (0,1) \times \Lambda$, we denote the period of the periodic orbit of X_μ passing through the point $\xi_\mu(s)$ by $P_\mu(s)$. We say then that P_μ is a *parametrization of the period function* of X_μ. Note that P_μ is an analytic function on $(0,1)$. In order to study the qualitative properties of the period function we consider $Z_\mu(s) = P'_\mu(s)$, which is a function defined on $(0,1)$ for all $\mu \in \Lambda$. The following definition deals with a slightly more general situation, but it will be clear in a moment the convenience for this.

Definition 2.3 Let $\{I_\mu,\, \mu \in \Lambda\}$ be a continuous family of intervals in $\mathbb{R}$ and consider a continuous family of functions $\{Z_\mu\colon I_\mu \longrightarrow \mathbb{R},\, \mu \in \Lambda\}$. We say that $\mu_0 \in \Lambda$ is a *regular value of the family* $\{Z_\mu,\, \mu \in \Lambda\}$ if there exist a neighbourhood U of μ_0 and an isotopy $\{h_\mu\colon I_\mu \longrightarrow I_{\mu_0},\, \mu \in U\}$, with $h_{\mu_0} = id$, such that

$$sgn\Big(Z_\mu(s)\Big) = sgn\Big(Z_{\mu_0}\big(h_\mu(s)\big)\Big) \tag{3}$$

for all $s \in I_\mu$ and $\mu \in U$. A parameter μ_0 which is not regular is called a *bifurcation value.* □

Note that the domain of definition of Z_μ depends on μ. To be more precise, by a continuous family of functions we mean with respect to the induced topology on $\cup_{\mu\in\Lambda} I_\mu \times \{\mu\}$ as a subset of $\mathbb{R} \times \Lambda$.

Definition 2.4 Let $\{X_\mu,\, \mu \in \Lambda\}$ be a continuous family of analytic vector fields with a center p_μ and assume that the corresponding period annuli vary continuously.

(a) We say that $\mu_0 \in \Lambda$ is a *regular* (respectively, *bifurcation*) *value of the period function* if for some parametrization of the period function P_μ we have that μ_0 is a regular (respectively, bifurcation) value of the family $\{P'_\mu\colon (0,1) \longrightarrow \mathbb{R},\, \mu \in \Lambda\}$.

(b) We say that $\mu_0 \in \Lambda$ is a *local regular value of the period function in the interior* if there exists some parametrization of the period function P_μ such that for any $c \in (0,1)$ there exists a continuously varying neighbourhood $I_\mu(c)$ of c in $(0,1)$ such that μ_0 is a regular value of the family $\{P'_\mu\colon I_\mu(c) \longrightarrow \mathbb{R},\, \mu \in \Lambda\}$. A parameter which is not a local regular value in the interior is called a *local bifurcation value in the interior.*

(c) We say that $\mu_0 \in \Lambda$ is a *local regular value of the period function at the inner* (respectively, *outer*) *boundary* if for some parametrization of the period function P_μ there exists a continuously varying neighbourhood $I_\mu(c)$ of $c = 0$ (respectively, $c = 1$) such that μ_0 is a regular value of the family $\{P'_\mu\colon I_\mu(c) \cap (0,1) \longrightarrow \mathbb{R},\, \mu \in \Lambda\}$. A parameter which is not a local regular value at the inner (respectively, outer) boundary is called a *local bifurcation value at the inner* (respectively, *outer*) *boundary.*

(d) We say that the period function of X_{μ_0} is *monotonous increasing* (respectively, *decreasing*) *at the inner boundary* if for some parametrization of the period function P_μ there exists $\varepsilon > 0$ such that $P'_{\mu_0}(s) > 0$ (respectively, $P'_{\mu_0}(s) < 0$) for all $s \in (0,\varepsilon)$. The monotonicity in the outer boundary is defined exactly the same way using $(1-\varepsilon, 1)$ instead of $(0,\varepsilon)$.

$\square$

Remark 2.5 In the above definitions one can replace "some parametrization" by "any parametrization". Indeed, assume for instance that $\mu_0 \in \Lambda$ is a regular value using P_μ and consider another parametrization, say $\widetilde{P}_\mu$. Then, following the notation of Definition 2.3, take $\widetilde{h}_\mu := \tau_{\mu_0} \circ h_\mu \circ \tau_\mu^{-1}$ where τ_μ is the Poincaré mapping from the transverse section given by ξ_μ to the one given by $\widetilde{\xi}_\mu$. Now, taking $P_\mu(s) = \widetilde{P}_\mu\big(\tau_\mu(s)\big)$ and $\tau_\mu'(s) > 0$ into account, it is easy to verify that μ_0 is a regular value using $\widetilde{P}_\mu$. $\square$

Remark 2.6 There are two situations in which it is very easy to decide whether a parameter μ_0 is a local regular value or not:

(a) If any neighbourhood of μ_0 contains two parameters μ_+ and μ_- such that X_{μ_+} and X_{μ_-} have different monotonicity at the inner (respectively, outer) boundary, then μ_0 is a local bifurcation value at the inner (respectively, outer) boundary.

(b) If for some parametrization of the period function P_μ there exists a neighbourhood U of μ_0 and $\varepsilon > 0$ such that $P_\mu'(s) \neq 0$ for all $\mu \in U$ and $s \in (0, \varepsilon)$ (respectively, $s \in (1-\varepsilon, 1)$), then μ_0 is a local regular value in the inner (respectively, outer) boundary.

$\square$

Lemma 2.7. *Let $\{X_\mu,\, \mu \in \Lambda\}$ be a continuous family of analytic vector fields with a center p_μ and assume that the corresponding period annuli vary continuously. Then the bifurcation diagram of the period function is the union of the local bifurcation diagrams at the inner and outer boundary and in the interior.*

Proof. It is obvious that a regular value is a local regular at the inner and outer boundary and in the interior. Let us prove the converse. Let $\mu_0 \in \Lambda$ be a local regular value at the inner boundary, the outer boundary and the interior. Note that by Remark 2.5 we can assume that we use the same parametrization, say P_μ, of the period function. By the local regularity at the inner and outer boundary, there is a neighbourhood U of μ_0 and continuously varying neighbourhoods $I_\mu(0)$ and $I_\mu(1)$, of the inner and outer boundary respectively, on which an isotopy h_μ as in the Definition 2.3 exists for $Z_\mu = P_\mu'$. By analyticity, $P_{\mu_0}'(s)$ has at most a finite number of zeros, say $c_1, \ldots, c_k$, in an open neighbourhood J of $(0,1) \setminus \big(I_\mu(0) \cup I_\mu(1)\big)$. Using the local regularity of μ_0 in the interior, for each $i = 1, 2, \ldots, k$, there exists a continuously varying closed interval $I_\mu(c_i)$ containing c_i and an isotopy h_μ such that the equality (3) holds for all $s \in I_\mu(c_i)$ and $\mu \in U$. Reducing U and each $I_\mu(c_i)$ if necessary, we can assume in addition that $I_\mu(c_1), \ldots, I_\mu(c_k)$ are pairwise disjoint and that

$$P_\mu'(s) \neq 0 \;\text{ for } s \in J \setminus \big(\bigcup_{i=1}^{k} I_\mu(c_i)\big) \text{ and } \mu \in U.$$

On the other hand, reducing also $I_\mu(0)$ and $I_\mu(1)$ if necessary, we can assume that $I_\mu(c_1), \ldots, I_\mu(c_k)$ do not intersect $I_\mu(0)$ and $I_\mu(1)$ neither. It remains therefore to define the isotopy in a finite disjoint union of open intervals. In each of these intervals we define it as an affine map whose values at the endpoints are already defined. $\blacksquare$

The above result shows that if $\mathcal{P}_\mu$ varies continuously, then in order to obtain the bifurcation diagram it is enough to study the three possible types of local bifurcations given in (b) and (c) of Definition 2.4. However, dealing with a family of centers such that the period annuli do not vary continuously, it may occur that some bifurcation does not correspond to any of these three types. In fact this is the case of the period function of the centers in Remark 2.2 (see [12] for details).

As we already mention, the local bifurcation diagram at the inner boundary is fully understood for the quadratic centers (see section 4) thanks to the results of Chicone and Jacobs [4]. Let us point out that their definition of bifurcation value at the inner boundary is not equivalent to ours. Their definition allows to describe better the bifurcation, but its usefulness is strongly based on the fact that the period function of a nondegenerate center can be extended analytically to the inner boundary. In general this is not possible in the outer boundary, which is the case that we study. We want, on the other hand, a unified definition for both boundaries because otherwise a result as Lemma 2.7 is very difficult to obtain. This is the reason why we use here a different definition. We point out however that, for the quadratic centers, the bifurcation values at the inner boundary are the same with both definitions (see Remark 4.2).

3 Bifurcation at the outer boundary

This section is devoted to the proof of Theorem A and it is divided into four subsections. In the first one we study the phase portrait of the dehomogenized Loud's systems and we focus on the shape of the period annulus of the center at the origin. In brief, we show that, apart from a parameter subset which consists of some straight lines, there are four different types of period annuli. We turn then to the study of the period function in each situation. We consider the two cases in which the period annulus is unbounded in section 3.2, and the two cases in which it is bounded in section 3.3. Finally in section 3.4 we prove Theorem A.

3.1 Study of the phase portrait

In the sequel, setting $\mu=(D,F)$, we shall denote by $\{X_\mu,\ \mu\in\mathbb{R}^2\}$ the family of vector fields corresponding to the dehomogenized Loud's systems, i.e.,

$$X_\mu=y(1-x)\partial_x+(x+Dx^2+Fy^2)\partial_y.$$

For each value of μ, the vector field X_μ has a center at the origin, whose period function is our object of study. In order to do this, we need to determine the period annulus $\mathcal{P}_\mu$ of X_μ as well as its outer boundary, which is a polycycle in some compactification of $\mathbb{R}^2$. Usually, one takes the Poincaré disk but, for the sake of simplicity in the computations, we will use instead the real projective plane $\mathbb{RP}^2$. We consider $\mathbb{RP}^2$ covered by the charts (x,y), $(u,v)=\left(\frac{1}{1-x},\frac{y}{1-x}\right)$ and $(\zeta,\omega)=\left(\frac{1-x}{y},\frac{1}{y}\right)$. The expressions of X_μ in (u,v) and (ζ,ω) coordinates are given respectively by

$$\begin{aligned}X_\mu(u,v)=&\frac{1}{u}\Big(-uv\partial_u+\big(-u+u^2+D(u-1)^2+(F-1)v^2\big)\partial_v\Big)\\ X_\mu(\zeta,\omega)=&\frac{1}{\omega}\Big(\big((1-F)\zeta+D\zeta^3+(D+1)\zeta\omega^2-(2D+1)\omega\zeta^2\big)\partial_\zeta\\ &+\omega\big(-F-(\omega-\zeta)((D+1)\omega-D\zeta)\big)\partial\omega\Big).\end{aligned}$$

It is easy to check (see [17] for instance) that if $F\notin\{0,1,\frac{1}{2}\}$ then the vector field X_μ has a Darboux type first integral given by

$$H_\mu(x,y)=(1-x)^{-2F}\left(\tfrac{1}{2}\,y^2-q_\mu(x)\right), \tag{4}$$

where $q_\mu(x)=a(\mu)x^2+b(\mu)x+c(\mu)$ with

$$a(\mu):=\frac{D}{2(1-F)},\ b(\mu):=\frac{D-F+1}{(1-F)(1-2F)}\ \text{and}\ c(\mu):=\frac{F-D-1}{2F(1-F)(1-2F)}. \tag{5}$$

The line at infinity L_∞ (with respect to the (x,y)-coordinates), the conic $\mathcal{C}_\mu:=\{\frac{1}{2}\,y^2-q_\mu(x)=0\}$ and the line $L_1:=\{x=1\}$ are invariant curves of X_μ. The determinant associated to the conic $\mathcal{C}=\mathcal{C}_\mu$, which coincides with the discriminant of $q_\mu(x)$, is given by

$$\Delta(\mu) := \frac{(D+F)(D+1-F)}{(1-2F)^2(1-F)F}.$$

Thus, we can see that $\mathcal{C}$ degenerates into two lines when $(D+F)(D+1-F)=0$. Indeed, it is easy to check that the conic $\mathcal{C}$ splits into two real lines when $F=-D$ and $D \notin [-1,0]$. On the other hand, if $F=D+1$ (respectively, $F=-D \in (0,1)$), then the conic $\mathcal{C}$ becomes two complex conjugated lines having the center $(x,y)=(0,0)$ (respectively, $(x,y)=(-1/D,0)$) as the unique real common point. In the other cases the affine type of $\mathcal{C}$ can be determined by the sign of Δ and a in the following way:

- If $a<0$ and $\Delta<0$ then the conic $\mathcal{C}$ has no real points.
- If $a<0$ and $\Delta>0$ then the conic $\mathcal{C}$ is an ellipse.
- If $a>0$ then the conic $\mathcal{C}$ is a hyperbola and we have two subcases depending on the sign of Δ. If $\Delta>0$ then $\mathcal{C}$ cuts the x-axis in two points which will be denoted in the sequel by p_1 and p_2 with $p_1<p_2$. If $\Delta<0$ then the hyperbola $\mathcal{C}$ has no common point with $\{y=0\}$.
- If $a=0$ then the conic $\mathcal{C}$ is a parabola (this only occurs when $D=0$).

It is well-known that every quadratic system has seven singularities (in the projective complex domain and counting multiplicities). Taking the pairwise intersections of the invariant curves L_1, L_∞ and $\mathcal{C}$ we obtain five singular points. Moreover, apart from the center at the origin $(x,y)=(0,0)$, we have the singular point $[-1,0,D]$, which in general does not lye on none of the invariant curves L_1, L_∞ or $\mathcal{C}$. Now we proceed to study in some detail each singular point of X_μ:

- The two points $L_1 \cap \mathcal{C} = \{[1, \pm\sqrt{\frac{-(D+1)}{F}}, 1]\}$ are real when $(D+1)F<0$. The linear part of X_μ at these points has eigenvalues $\lambda_1 = \pm 2F\sqrt{\frac{-(D+1)}{F}}$ and $\lambda_2 = \pm\sqrt{\frac{-(D+1)}{F}}$.
- The two points $L_\infty \cap \mathcal{C} = \{[1, \pm\sqrt{\frac{D}{1-F}}, 0]\}$ are real when $(1-F)D>0$. Working in (u,v)-coordinates, the linear part of uX_μ at these points has eigenvalues $\lambda_1 = \mp 2D\sqrt{\frac{1-F}{D}}$ and $\lambda_2 = \mp\sqrt{\frac{D}{1-F}}$.
- At the point $L_1 \cap L_\infty = [0,1,0]$ the linear part of ωX_μ has eigenvalues $\lambda_1=-F$ and $\lambda_2 = 1-F$.
- At the point $[-1,0,D]$, with $D \neq 0$, the linear part of X_μ has eigenvalues $\lambda_i = \pm\sqrt{1+\frac{1}{D}}$.

Figure 3 shows the bifurcation diagram of the phase portrait of the dehomogenized Loud's systems. It is important to note that in each phase portrait we place the center $(0,0)$ on the left of the centered invariant line $L_1=\{x=1\}$. In addition the conic $\mathcal{C}$ appears in boldface type when it is relevant. We will describe next in brief all the bifurcations occurring in this diagram:

- Along $D=-1$ there is a collapse of the three singularities $L_1 \cap \mathcal{C}$ and $[-1,0,D]$. We point out that if $F>1$ then this bifurcation does not affect the period annulus.
- The bifurcation at $D=0$ occurs when the three singularities $L_\infty \cap \mathcal{C}$ and $[-1,0,D]$ collapse. This bifurcation always affects the period annulus.
- The bifurcations at $F=0$ and $F=1$ can also be easily described. Indeed, the three singular points $L_1 \cap L_\infty$ and $L_1 \cap \mathcal{C}$ collapse giving raise to a saddle-node at infinity, whose strong separatrix is on L_∞ when $F=0$ and on L_1 when $F=1$. Note that these bifurcations only affect the period annulus when $D \in [-1,0]$.

- Along $F+D=0$ the conic $\mathcal{C}$ degenerates, but this does not affect the outer boundary of the period annulus if $F \in (0,1)$.

- Along $F=D+1$ the conic $\mathcal{C}$ also degenerates, but this never affects the period annulus.

- Finally the bifurcation at $F=1/2$ is more subtle because there is no confluence of singularities. The position of the conic depends on $F>1/2$ or $F<1/2$ and it "explodes" to the limit set $L_1 \cup L_\infty$ as F tends to $1/2$. Note in addition that the singular point $L_1 \cap L_\infty$ is a saddle for $0<F<1$. In fact it can be shown that this saddle is orbitally linearizable for $F \neq 1/2$. In contrast, if $F=1/2$ then the singular point $L_1 \cap L_\infty$ (which belongs to the outer boundary of $\mathcal{P}_\mu$ when $D \in [-1,0]$) becomes a resonant saddle with hyperbolicity ratio equal to one. Notice however that this bifurcation never affects the structure of the outer boundary of the period annulus.

Remark 3.1 The discussion above shows that the bifurcations (in the structure) of the outer boundary of the period annulus of the center at the origin occur only on the dotted curve in Figure 3. Let us point out that we shall not study the period function corresponding to these parameters. □

Lemma 3.2. *The family of period annuli of the center at the origin of the dehomogenized Loud's systems* $\{X_\mu,\, \mu \in \mathbb{R}^2\}$ *varies continuously.*

Proof. Let $\big(b(\mu),0\big)$ be the intersection point of the outer boundary of $\mathcal{P}_\mu$ with the positive x-axis (see Figure 3). This provides us a natural parametrization for $\mathcal{P}_\mu$. Indeed, notice that $b(\mu)$ depends continuously on μ and that X_μ is transverse to the segment $\{(x,0): 0<x<b(\mu)\}$ for all $\mu \in \mathbb{R}^2$. It suffices therefore to consider $\xi_\mu : [0,1] \longrightarrow \mathbb{R}^2$ defined by means of $\xi_\mu(s) := \big(b(\mu)s,0\big)$. ■

As we already mentioned, in order to study the behaviour of the period function near the outer boundary of the period annulus we must treat separately the four different types of polycycle that bound it. We gather them in two sections according to wether the period annulus (considered as a subset of $\mathbb{R}^2$) is bounded or not. In section 3.2 we deal with the unbounded case, which is divided in two subcases: subsection 3.2.1 corresponds to period annuli with the outer boundary contained in $\mathcal{C} \cup L_\infty$ and subsection 3.2.2 to period annuli with the outer boundary contained in $L_1 \cup L_\infty$. As a matter of fact this last case was already treated in [11] and here, for the sake of completeness, we only recall the result that we obtained. Finally, section 3.3 deals with the cases in which the period annulus is bounded. From Figure 3 it follows that there are two possibilities for the polycycle in the outer boundary, namely, a saddle loop or a bicycle.

3.2 Unbounded period annulus

3.2.1 The case $F>1$, $F+D>0$ and $D<0$.

In this subsection we study the period function of the center at the origin of X_μ in case that the parameter μ belongs to

$$U := \big\{(D,F) \in \mathbb{R}^2 : F>1,\; F+D>0 \text{ and } D<0\big\}.$$

We shall prove the existence of a curve Γ_1 such that setting $\Gamma_2 = U \cap \{F=2\}$ then the following holds:

Theorem 3.3. *Let* $\{X_\mu,\, \mu \in \mathbb{R}^2\}$ *be the family of vector fields in* (2) *and consider the period function of the center at the origin. Then the open set* $U \setminus \{\Gamma_1 \cup \Gamma_2\}$ *corresponds to local regular values of the period function at the outer boundary of the period annulus. Moreover, for these parameters, the period function is monotonous near the outer boundary and the corresponding character is shown in Figure 4.*

To be more precise, let us advance that Γ_1 is the zero level set of an explicit function which is given in (28), see page 22. In order to draw Γ_1 in Figure 4 we have computed it numerically. Analytically, among

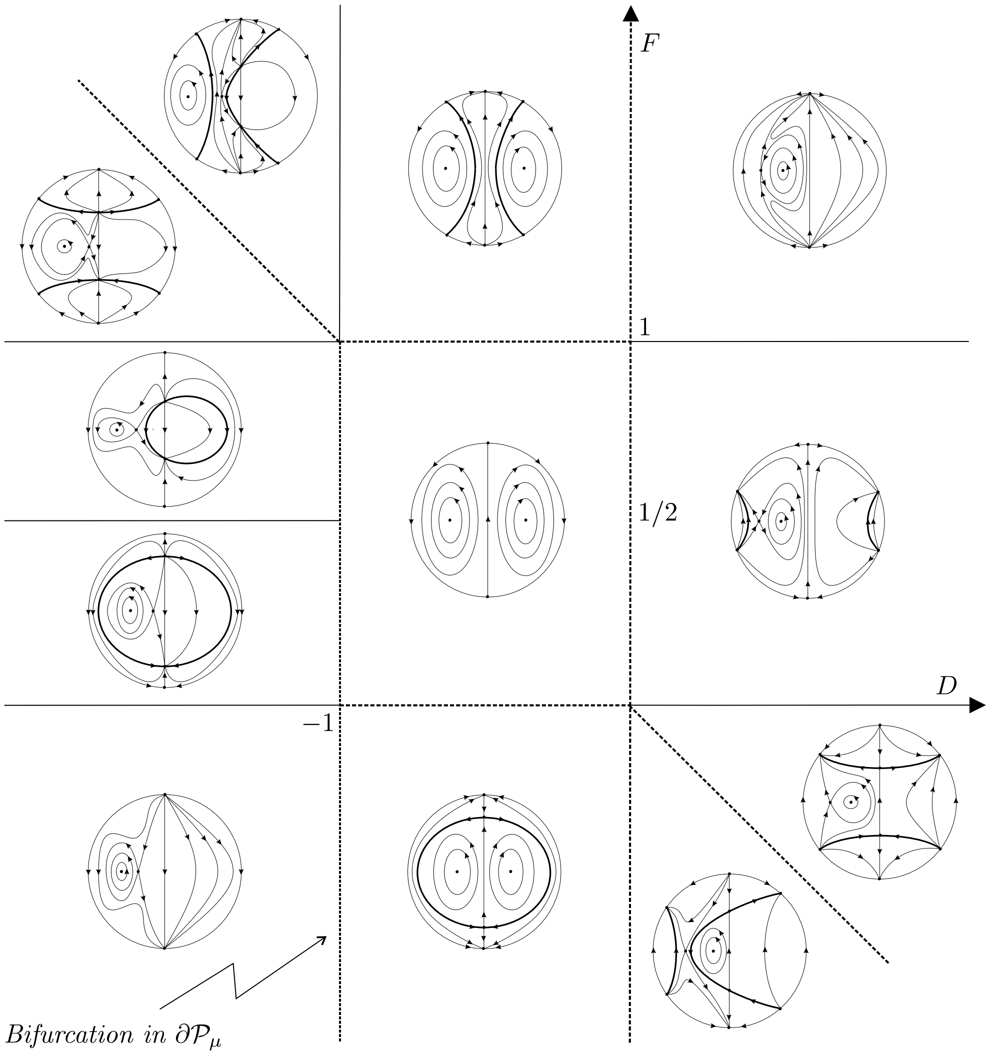

Figure 3: Phase portraits of the dehomogenized Loud's systems.

other properties that are gathered in Proposition 3.11, we have proved that Γ_1 is the graphic of an analytic function $D = \mathcal{G}(F)$.

In order to prove Theorem 3.3 we shall study the asymptotic development of the period function near the outer boundary of $\mathcal{P}_\mu$. For the parameter values under consideration (see Figure 3), recall that the outer boundary of the period annulus of the center is made up of the line at infinity and a branch of the conic $\mathcal{C}_\mu = \{\frac{1}{2}\,y^2 - q_\mu(x) = 0\}$, where $q_\mu(x) = ax^2 + bx + c$ with the coefficients a, b, c defined in (5). For $\mu \in U$, the conic has two different intersection points with $y = 0$, namely

$$p_1 := \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad p_2 := \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

which one can verify that $0 < p_1 < p_2$ and $p_1 < 1$. Notice in particular that $(p_1, 0)$ belongs to the outer

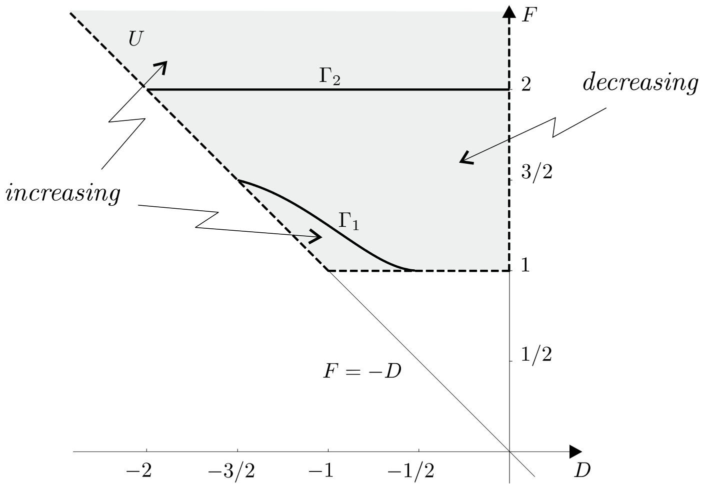

Figure 4: Monotonicity of the period function at the outer boundary of $\mathcal{P}_\mu$.

boundary of the period annulus. Since one can check that X_μ is transverse to $\{(x,0) : 0 < x < p_1\}$, we have a global parametrization of the set of periodic orbits in $\mathcal{P}_\mu$. Thus, for $(s,\mu) \in (0,p_1)\times U$, we denote by $P(s;\mu)$ the period of the periodic orbit of X_μ passing through the point $(p_1 - s, 0)$.

Notice that one can easily normalize $P(s;\mu)$ to obtain a parametrization of the period function defined for $s \in (0,1)$ and so that the inner and outer boundary correspond to $s \approx 0$ and $s \approx 1$ respectively. However for convenience in the computations we prefer to use the previous one instead, for which we stress that the outer boundary corresponds to $s \approx 0$.

Theorem 3.3 follows almost directly from Theorem 3.6, which gives the first nontrivial term of the asymptotic development of $P_s(s;\mu)$ at $s = 0$. In its statement we use the following definitions:

Definition 3.4 Let W be an open subset of $\mathbb{R}^m$. We denote by $\mathcal{I}(W)$ the set of germs of analytic functions $h(s;\mu)$ defined on $(0,\varepsilon)\times W$ for some $\varepsilon > 0$ such that

$$\lim_{s\to 0} h(s;\mu) = 0 \text{ and } \lim_{s\to 0} s\,\frac{\partial h(s;\mu)}{\partial s} = 0$$

uniformly (on μ) on every compact subset of W.

Let us also denote by $\mathcal{I}_0(W)$ the set of germs of analytic functions $h(s;\mu)$ defined on $(-\varepsilon,\varepsilon)\times W$ for some $\varepsilon > 0$ such that $h(0;\mu) \equiv 0$. Note therefore that $\mathcal{I}_0(W) \subset \mathcal{I}(W)$. □

Definition 3.5 The function defined for $s > 0$ and $\alpha \in \mathbb{R}$ by means of

$$\omega(s;\alpha) = \begin{cases} \frac{s^{\alpha-1}-1}{\alpha-1} & \text{if } \alpha \neq 1,\\ \log s & \text{if } \alpha = 1,\end{cases}$$

is called the *Roussarie-Ecalle compensator.* □

Let us define in addition

$$\lambda(\mu) := \frac{1}{2(F-1)}$$

and introduce the covering of the parameter space U given by the open subsets

$$U_1 := \{\mu \in U : F < 3/2\},\ U_2 := \{\mu \in U : F > 3/2\} \text{ and } U_3 := \{\mu \in U : 5/4 < F < 2\}, \tag{6}$$

which one can verify that correspond respectively to $\lambda(\mu) > 1$, $\lambda(\mu) < 1$ and $1/2 < \lambda(\mu) < 2$.

Now, with the definitions and notation introduced above, we prove the following:

Theorem 3.6. *Denote*

$$\Delta_0(\mu) = \frac{2\sqrt{2}}{\sqrt{a+b+c}}\, arctanh\left(\frac{2\,a+b-\sqrt{b^2-4\,ac}}{2\sqrt{a(a+b+c)}}\right).$$

Then the following holds:

(a) If $\mu \in U_1$ then $P(s;\mu) = \Delta_0(\mu) + \Delta_1(\mu)s + sf_1(s;\mu)$, where $f_1 \in \mathcal{I}(U_1)$ and

$$\Delta_1(\mu) = \frac{-1/\sqrt{2a}}{(p_2-p_1)(1-p_1)}\left\{2 - \int_0^1 \left(u^{-\frac{1}{\lambda}}\left(\frac{1-p_2}{1-p_1}(u-1)+1\right)^{1+\frac{1}{\lambda}} - 1\right)\frac{du}{(1-u)^{3/2}}\right\}.$$

(b) If $\mu \in U_2$ then $P(s;\mu) = \Delta_0(\mu) + \Delta_2(\mu)s^{\lambda} + s^{\lambda}f_2(s;\mu)$, where $f_2 \in \mathcal{I}(U_2)$ and

$$\Delta_2(\mu) = \sqrt{\frac{2\pi}{a}}\,\frac{\lambda\,(p_2-p_1)^{\lambda}}{(1-p_1)^{2\lambda+1}}\,\frac{\Gamma\left(\frac{1}{2(1-F)}\right)}{\Gamma\left(\frac{F-2}{2(F-1)}\right)}.$$

(c) If $\mu \in U_3$ then $P(s;\mu) = \Delta_0(\mu)+\Delta_3(\mu)s\omega(s;\lambda)+\Delta_4(\mu)s+sf_3(s;\mu)$, where $f_3 \in \mathcal{I}(U_3)$ and the functions $\Delta_3(\mu)$ and $\Delta_4(\mu)$ are analytic on U_3. Furthermore, if $\lambda(\mu_0) = 1$ then

$$\Delta_3(\mu_0) = -\frac{p_2-p_1}{\sqrt{2a}\,(1-p_1)^3}.$$

Notice that, in the Poincaré disc, the outer boundary of the $\mathcal{P}_\mu$ is a polycycle with two hyperbolic saddles located at infinity (see Figure 3 in page 10). In addition, taking advantage of the symmetry of the Loud's family with respect to the x-axes, in order to prove Theorem 3.6 it is enough to study half of the period. Consequently we must only study the time function associated to the passage through one of these saddles. To this end we shall use a result which appears in [11]. In that paper, given an analytic family of vector fields in $\mathbb{R}^2$ having a saddle point, we studied the asymptotic development of the time function along the union of two separatrices. Next, for the sake of completeness, we state this result (see Proposition 3.9) and we explain the related definitions.

Let W be an open set of $\mathbb{R}^m$ and let $\{\widetilde{X}_\mu,\, \mu \in W\}$ be an analytic family of vector fields defined on some open set V of $\mathbb{R}^2$. Assume that each vector field $\widetilde{X}_\mu$ has a hyperbolic saddle p_μ as the unique critical point inside V. In this situation it is well known that there exist exactly two analytic transverse invariant curves $\mathcal{S}_\mu$ and $\mathcal{T}_\mu$, the stable and unstable manifolds, passing through p_μ (depending also analytically on μ). We consider an analytic family of *meromorphic* vector fields X_μ proportional to $\widetilde{X}_\mu$ with a pole of order $n > 0$ along $\mathcal{T}_\mu$. We can take a coordinate system (u,v,μ) on $V \times W \subset \mathbb{R}^{2+m}$ such that $p_\mu = (0,0,\mu)$, $\mathcal{S}_\mu = \{(u,v,\mu) : u = 0\}$ and $\mathcal{T}_\mu = \{(u,v,\mu) : v = 0\}$. In these coordinates the family $\{X_\mu,\, \mu \in W\}$ can be written as

$$X_\mu(u,v) = \frac{1}{v^n}\big(uP(u,v;\mu)\partial_u + vQ(u,v;\mu)\partial_v\big), \tag{7}$$

where P and Q are analytic functions such that $P(u,0;\mu) > 0$ and $Q(0,v;\mu) < 0$ for any $(0,v,\mu) \in \mathcal{S}_\mu$ and $(u,0,\mu) \in \mathcal{T}_\mu$. Moreover, by hypothesis, we have that

$$\lambda(\mu) := -\frac{Q(0,0;\mu)}{P(0,0;\mu)} > 0.$$

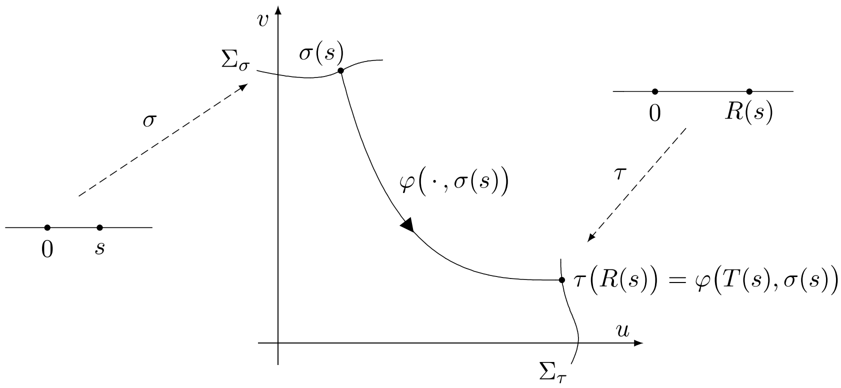

Figure 5: Definition of T and R in Proposition 3.9.

The family $\{X_\mu,\, \mu \in W\}$ can be thought of as a single vector field X defined on $V \times W \subset \mathbb{R}^{2+m}$ whose trajectories are contained inside the submanifolds $\{\mu = \text{const}\}$. Let $\sigma : I \times W \longrightarrow \Sigma_\sigma$ and $\tau : I \times W \longrightarrow \Sigma_\tau$ be two analytic transverse sections to X defined by

$$\sigma(s;\mu) = \big(\sigma_1(s;\mu), \sigma_2(s;\mu); \mu\big) \;\text{ and }\; \tau(s;\mu) = \big(\tau_1(s;\mu), \tau_2(s;\mu); \mu\big)$$

such that $\sigma(0;\mu) \in \mathcal{S}_\mu$ and $\tau(0;\mu) \in \mathcal{T}_\mu$. Here I denotes a small interval of $\mathbb{R}$ containing 0.

We denote the Dulac and time mappings between the transverse sections Σ_σ and Σ_τ by R and T respectively. More precisely (see Figure 5), if $\varphi\big(t,(u_0,v_0);\mu\big)$ is the solution of X_μ passing through (u_0,v_0) at $t=0$, for each $s>0$ we define $R(s;\mu)$ and $T(s;\mu)$ by means of the relation

$$\varphi\big(T(s;\mu), \sigma(s);\mu\big) = \tau\big(R(s;\mu)\big). \tag{8}$$

Definition 3.7 We will say that $\{X_\mu,\, \mu \in W\}$ verifies the *family linearization property* (FLP in short) if there exist an open set $U \subset \mathbb{R}^2$ containing the origin and an analytic *local* diffeomorphism $\Phi : U \times W \to V \times W$ of the form $\Phi(x,y;\mu) = \big(x + \text{h.o.t.}, y + \text{h.o.t.}, \mu\big)$ such that

$$X_\mu = \Phi_*\left(\frac{1}{f(x,y;\mu)}\big(x\partial_x - \lambda(\mu) y \partial_y\big)\right),$$

where f is an analytic function on $U \times W$. □

Remark 3.8 It is easy to show that the family of meromorphic vector fields $\{X_\mu,\, \mu \in W\}$ defined in (7) verifies FLP if it has a Darboux first integral

$$H_\mu(x,y) = f_1(x,y;\mu)^{\beta_1(\mu)} \cdots f_k(x,y;\mu)^{\beta_k(\mu)},$$

where f_j and β_j are analytic functions on $V \times W$ and W respectively.

Recall that $H_\mu(x,y) = (1-x)^{-2F}\big(\frac{1}{2}\,y^2 - q_\mu(x)\big)$ is a Darboux first integral for X_μ if $F(F-1)(2F-1) \neq 0$, so the FLP is verified in these cases. □

In order to simplify the expressions that appear in the statement of the next result we introduce the functions

$$L(u;\mu) := \exp\left(\int_{\sigma_2(0)}^{u}\left(\frac{P(0,y)}{Q(0,y)} + \frac{1}{\lambda}\right)\frac{dy}{y}\right),$$
$$M(u;\mu) := \exp\left(\int_0^{u}\left(\frac{Q(x,0)}{P(x,0)} + \lambda\right)\frac{dx}{x}\right),$$

and the covering of the parameter space W given by the open subsets

$$W_1 := \left\{\mu \in W : \lambda > \frac{1}{n}\right\},\ W_2 := \left\{\mu \in W : \lambda < \frac{1}{n}\right\} \text{ and } W_3 := \left\{\mu \in W : \frac{1}{n+1} < \lambda < \frac{2}{n}\right\}.$$

Proposition 3.9. *Let $\{X_\mu,\, \mu \in W\}$ be the family of vector fields defined in* (7) *and assume that it verifies FLP. Let R and T be respectively the Dulac map and the time function associated to the transverse sections Σ_σ and Σ_τ as introduced in* (8). *Denote*

$$\rho(\mu) = \frac{\sigma_1'(0)^\lambda \sigma_2(0)}{\tau_2'(0)\tau_1(0)^\lambda} L(0)^\lambda M\big(\tau_1(0)\big) \text{ and } \Delta_0(\mu) = \int_{\sigma_2(0)}^{0} \frac{v^{n-1}}{Q(0,v)}\,dv.$$

Then $R(s;\mu) = \rho(\mu)s^\lambda + s^\lambda f_0(s;\mu)$ with $f_0 \in \mathcal{I}(W)$. In addition, the time function $T(s;\mu)$ verifies the following:

(*a*) *If $\mu \in W_1$ then $T(s;\mu) = \Delta_0(\mu) + \Delta_1(\mu)s + sf_1(s;\mu)$, where $f_1 \in \mathcal{I}(W_1)$ and*

$$\Delta_1(\mu) = -\frac{\sigma_2'(0)\,\sigma_2(0)^{n-1}}{Q(0,\sigma_2(0))} + \sigma_1'(0)\sigma_2(0)^{1/\lambda} \int_0^{\sigma_2(0)} \frac{Q_u(0,v)L(v)v^{n-1/\lambda}}{Q(0,v)^2}\frac{dv}{v}.$$

(*b*) *If $\mu \in W_2$ then $T(s;\mu) = \Delta_0(\mu) + \Delta_2(\mu)s^{\lambda n} + s^{\lambda n} f_2(s;\mu)$, where $f_2 \in \mathcal{I}(W_2)$ and*

$$\Delta_2(\mu) = \sigma_1'(0)^{\lambda n}\sigma_2(0)^n L(0)^{\lambda n} \left\{ \frac{\tau_1(0)^{-\lambda n}}{nQ(0,0)} + \int_0^{\tau_1(0)} \left(\frac{M(u)^n}{P(u,0)} - \frac{M(0)^n}{P(0,0)}\right)\frac{du}{u^{\lambda n+1}} \right\}.$$

(*c*) *If $\mu \in W_3$ then $T(s;\mu) = \Delta_0(\mu) + \Delta_3(\mu)s\omega(s;\lambda n) + \Delta_4(\mu)s + sf_3(s;\mu)$, where $f_3 \in \mathcal{I}(W_3)$ and the functions $\Delta_3(\mu)$ and $\Delta_4(\mu)$ are analytic on W_3. Furthermore, if $\lambda(\mu_0) = 1/n$ then*

$$\Delta_3(\mu_0) = -n\sigma_1'(0)\sigma_2(0)^n L(0)\frac{Q_u(0,0)}{P(0,0)^2}.$$

Proposition 3.9 constitutes the main ingredient in the proof of Theorem 3.6. However, in order to apply it we must first perform a change of coordinates that sends each separatrix of the saddle at infinity to a straight line. This will raise some technical complications because the coordinate transformation that we use is *singular* and it creates a line of critical points. To bypass this problem we will have to split up the time function and to introduce an additional parameter associated to the new transverse sections. This makes the proof more complicated than one could expect. In particular we shall need the following result to study the remainder terms. Its proof can be found in [11] and, for the sake of brevity, in the statement we denote $\mathcal{I}(W)$ and $\mathcal{I}_0(W)$ by $\mathcal{I}$ and $\mathcal{I}_0$ respectively (see Definition 3.4).

Lemma 3.10. *Assume that $a(\mu), k(\mu)$ and $r(\mu)$ are positive analytic functions.*

(*a*) *If $g(s;\mu)$ and $f(s;\mu)$ belong to $\mathcal{I}_0$ and $\mathcal{I}$ respectively then $g\circ f \in \mathcal{I}$.*

(*b*) *If $f(s;\mu)$ belongs to $\mathcal{I}$ (resp. $\mathcal{I}_0$) and $\varphi := s^r(a+f)$ then $s^k \circ \varphi - a^k s^{kr}$ belongs to $s^{kr}\mathcal{I}$ (resp. $s^{kr}\mathcal{I}_0$).*

(*c*) *If $f(s;\mu)$ and $g(s;\mu)$ belong to $\mathcal{I}$ and $\varphi := s^r(a+f)$ then $(s^k g)\circ\varphi$ belongs to $s^{kr}\mathcal{I}$.*

(*d*) *If $g(s;\mu)$ belongs to $\mathcal{I}_0$ then $g\omega(s;r) \in \mathcal{I}$.*

(*e*) *If $g(s;\mu)$ belongs to $\mathcal{I}_0$ then $\big(s\omega(s;r)\big)\circ\big(s(a+g)\big) = a^r s\omega(s;r) + a\omega(a;r)s + \Psi$ with $\Psi \in s\mathcal{I}$.*

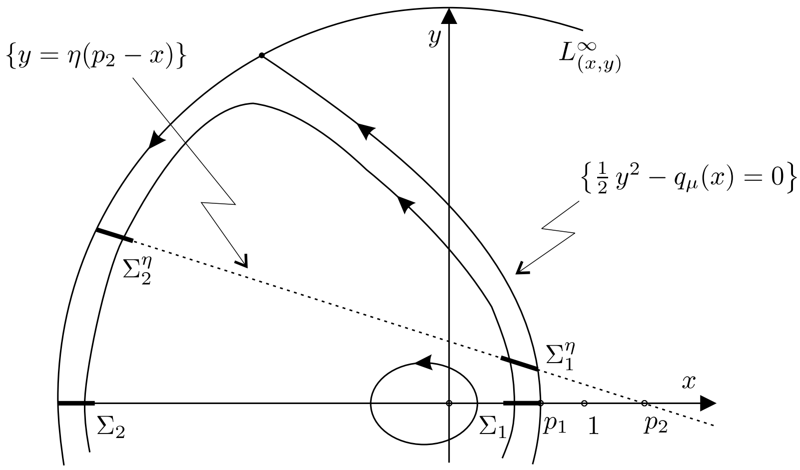

Figure 6: Auxiliary transverse sections.

Proof of Theorem 3.6 Note first that, since the transformation $(x, y, t) \longmapsto (x, -y, -t)$ preserves the Loud normal form, it is enough to study half of the period. More concretely, denoting the solution of X_μ passing through (x_0, y_0) at $t = 0$ by $\varphi\big(t, (x_0, y_0); \mu\big)$, for each $s \in (0, p_1)$ we define $T(s; \mu)$ as the minimum positive number so that $\varphi_2\big(T(s; \mu), (p_1 - s, 0); \mu\big) = 0$.

Thus we only need to obtain the coefficients of the asymptotic development of $T(s; \mu)$ at $s = 0$, which involves only one passage through a saddle at infinity. Clearly the coefficients of $P(s; \mu)$ at $s = 0$ will follow then using that $P(s; \mu) = 2T(s; \mu)$.

Notice now (see Figure 6) that $T(s; \mu)$ is the time function associated to the transverse sections Σ_1 and Σ_2, which are given respectively by $\alpha^1(s) = (p_1 - s, 0)$ and $\alpha^2(s) = (-1/s, 0)$. In order to study $T(s; \mu)$ we introduce two auxiliary transverse sections, say Σ_1^η and Σ_2^η, on the straight line $y = \eta(p_2 - x)$, where $\eta \in (0, \varepsilon)$. To this end, let $\big(x_\eta, \eta(p_2 - x_\eta)\big)$ be the intersection point between this straight line and the hyperbola $\{\frac{1}{2}\,y^2 - q_\mu(x) = 0\}$. We parametrize Σ_1^η and Σ_2^η by

$$\alpha_\eta^1(s) = \big(x_\eta - s\,, \eta\,(p_2 - x_\eta + s)\big) \text{ and } \alpha_\eta^2(s) = \big(-1/s\,, \eta\,(p_2 + 1/s)\big)$$

respectively. Let us denote the time function between Σ_1 and Σ_1^η by $T_1(s; \mu, \eta)$, the one between Σ_1^η and Σ_2^η by $T_2(s; \mu, \eta)$, and the one between Σ_2^η and Σ_2 by $T_3(s; \mu, \eta)$. Then

$$T(s; \mu) = T_1\big(s; \mu, \eta\big) + T_2\big(R_1(s; \mu, \eta); \mu, \eta\big) + T_3\big(R_2(s; \mu, \eta); \mu, \eta\big),$$

where $R_1(s; \mu, \eta)$ is the Poincaré mapping between Σ_1 and Σ_1^η and $R_2(s; \mu, \eta)$ is the one between Σ_1 and Σ_2^η.

It is well known that T_1, T_3 and R_1 can be extended analytically to $s = 0$, and it is also clear that they are analytical for $(\mu, \eta) \in U \times (-\varepsilon, \varepsilon)$. Hence

(9) $$T_1(s; \mu, \eta) = \Delta_0^1(\mu, \eta) + \Delta_1^1(\mu, \eta)s + sf_1(s; \mu, \eta) \text{ with } f_1 \in \mathcal{I}_0\big(U \times (-\varepsilon, \varepsilon)\big),$$

(10) $$T_3(s; \mu, \eta) = \Delta_1^3(\mu, \eta)s + sf_3(s; \mu, \eta) \text{ with } f_3 \in \mathcal{I}_0\big(U \times (-\varepsilon, \varepsilon)\big),$$

(11) $$R_1(s; \mu, \eta) = \rho_1(\mu, \eta)s + sg_1(s; \mu, \eta) \text{ with } g_1 \in \mathcal{I}_0\big(U \times (-\varepsilon, \varepsilon)\big).$$

Notice moreover that $T_1(s; \mu, \eta) \longrightarrow 0$, $T_3(s; \mu, \eta) \longrightarrow 0$ and $R_1(s; \mu, \eta) \longrightarrow s$ as $\eta \longrightarrow 0$. Therefore

(12) $$\lim_{\eta \to 0} \Delta_0^1(\mu, \eta) = \lim_{\eta \to 0} \Delta_1^1(\mu, \eta) = \lim_{\eta \to 0} \Delta_1^3(\mu, \eta) = 0 \text{ and } \lim_{\eta \to 0} \rho_1(\mu, \eta) = 1.$$

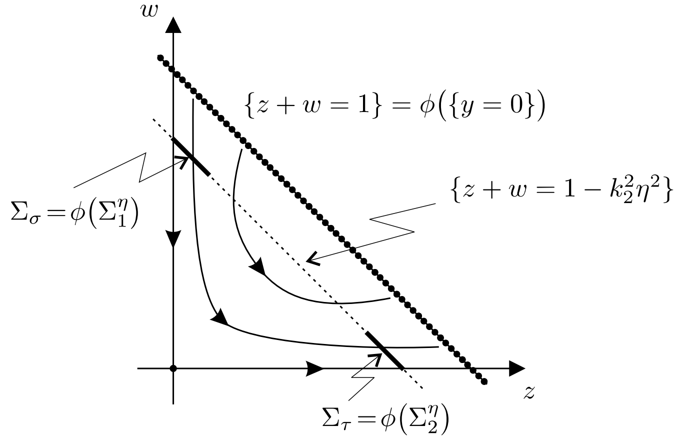

Figure 7: Passage through the saddle at infinity in (z, w)-coordinates.

The asymptotic developments of T_2 and R_2, which correspond to the passage through a saddle at infinity, are more delicate. To obtain them we shall apply Proposition 3.9, and to this end we must first perform a coordinate transformation that sends the separatrices of the saddle to straight lines. We thus consider the *singular* change of variables given by

$$(z,w)=\phi(x,y):=\left(\frac{2q_\mu(x)-y^2}{2a(p_2-x)^2}\,,\frac{p_2-p_1}{p_2-x}\right).$$

Setting $k_1:=p_2-p_1$ and $k_2:=1/\sqrt{2a}$ for the sake of shortness, some computations show that it brings (2) to the system given by the vector field

$$X_\mu=\frac{1}{w}\bigg(z\,P(z,w;\mu)\,\partial_z+w\,Q(z,w;\mu)\,\partial_w\bigg),$$

where

$$P(z,w;\mu)=\frac{2}{k_2}\sqrt{1-z-w}\,\Big(k_1(F-1)+(p_2-1)w\Big)$$

and

$$Q(z,w;\mu)=\frac{1}{k_2}\sqrt{1-z-w}\,\Big(-k_1+(p_2-1)w\Big).$$

One can verify (see Figure 7) that $\Sigma_\sigma:=\phi(\Sigma_1^\eta)$ and $\Sigma_\tau:=\phi(\Sigma_2^\eta)$ are in the straight line $z+w=1-k_2^2\eta^2$. We parameterize them with the transferred parameterizations of Σ_1^η and Σ_2^η. More concretely, we consider

$$\sigma(s;\mu,\eta):=\phi\big(\alpha_\eta^1(s)\big)=\left(\frac{s(1-k_2^2\eta^2)^2}{k_1+s(1-k_2^2\eta^2)}\,,\frac{k_1(1-k_2^2\eta^2)}{k_1+s(1-k_2^2\eta^2)}\right)$$

and

$$\tau(s;\mu,\eta):=\phi\big(\alpha_\eta^2(s)\big)=\left(\frac{sp_1+1}{sp_2+1}-k_2^2\eta^2\,,\frac{sk_1}{sp_2+1}\right).$$

Note therefore that $T_2(s;\mu,\eta)$ is precisely the time function between Σ_σ and Σ_τ, and that, on the other hand, the vector field X_μ is meromorphic on the region under consideration. (We point out that Proposition 3.9 can not be applied to compute $T(s;\mu)$ directly because $\phi(\Sigma_1)$ and $\phi(\Sigma_2)$ are in the straight line $z+w=1$ and the vector field X_μ is not meromorphic there.) Moreover, since $H_\mu\big(\phi^{-1}(z,w)\big)$ is a Darboux first integral of X_μ, from Remark 3.8 it follows that $\{X_\mu,\ \mu\in U\}$ is a family of vector fields verifying FLP. Consequently we can apply Proposition 3.9 to compute the asymptotic development of T_2 at $s=0$. As a matter of fact, to be precise, since the transverse sections depend also on η, we shall apply it to the family $\{X_{(\mu,\eta)}\,,\ (\mu,\eta)\in U\times(0,\varepsilon)\}$. Thus, following the notation of Proposition 3.9, since

$$\lambda(\mu) := -\frac{Q(0,0;\mu)}{P(0,0;\mu)} = \frac{1}{2(F-1)}$$

does not depend on η, it turns out that $W_i = U_i\times(0,\varepsilon)$ where

$$U_1=\{\mu\in U: F<3/2\}\,,\ U_2=\{\mu\in U: F>3/2\}\ \text{and}\ U_3=\{\mu\in U: 5/4<F<2\}.$$

In addition we can assert that

$$T_2(s;\mu,\eta)=\Delta_0^2(\mu,\eta)+\Delta_1^2(\mu,\eta)s+sf_2^1(s;\mu,\eta)\ \text{ if } \mu\in U_1, \tag{13}$$

$$T_2(s;\mu,\eta)=\Delta_0^2(\mu,\eta)+\Delta_2^2(\mu,\eta)s^\lambda+s^\lambda f_2^2(s;\mu,\eta)\ \text{ if } \mu\in U_2, \tag{14}$$

$$T_2(s;\mu,\eta)=\Delta_0^2(\mu,\eta)+\Delta_3^2(\mu,\eta)s\omega(s;\lambda)+\Delta_4^2(\mu,\eta)s+sf_2^3(s;\mu,\eta)\ \text{ if } \mu\in U_3, \tag{15}$$

where $f_2^i\in\mathcal{I}\big(U_i\times(0,\varepsilon)\big)$. Some computations show that

$$\Delta_0^2(\mu,\eta)=\int_{\sigma_2(0)}^0\frac{dw}{Q(0,w)}=\int_{1-k_2^2\eta^2}^0\frac{k_2}{(p_2-1)w-k_1}\frac{dw}{\sqrt{1-w}}\,. \tag{16}$$

Let us compute next the coefficient $\Delta_1^2(\mu,\eta)$. From Proposition 3.9 we know that it is given by

$$\Delta_1^2(\mu,\eta)=-\frac{\sigma_2'(0)}{Q\big(0,\sigma_2(0)\big)}+\sigma_1'(0)\sigma_2(0)^{1/\lambda}\int_0^{\sigma_2(0)}\frac{Q_z(0,w)L(w)}{Q(0,w)^2}\frac{dw}{w^{1/\lambda}}.$$

One can verify that

$$L(w)=\left(\frac{(1-p_2)w+k_1}{1-p_1-k_2^2\eta^2(1-p_2)}\right)^{2F}\ \text{ and }\ \frac{Q_z(0,w)}{Q(0,w)^2}=\frac{k_2}{2(1-w)^{3/2}\big((1-p_2)w+k_1\big)}\,.$$

Consequently, using also that

$$\frac{\sigma_2'(0)}{Q\big(0,\sigma_2(0)\big)}=\frac{(1-k_2^2\eta^2)^2}{k_1\eta\big(1-p_1-k_2^2\eta^2(1-p_2)\big)}\ \text{ and }\ \sigma_1'(0)\sigma_2(0)^{1/\lambda}=\frac{(1-k_2^2\eta^2)^{2F}}{k_1}\,,$$

it turns out that

$$\begin{aligned}\Delta_1^2(\mu,\eta)=&\frac{-(1-k_2^2\eta^2)^2}{k_1\eta\big(1-p_1-k_2^2\eta^2(1-p_2)\big)}\\&+\frac{k_2(1-k_2^2\eta^2)^{2F}}{2k_1\big(1-p_1-k_2^2\eta^2(1-p_2)\big)^{2F}}\int_0^{1-k_2^2\eta^2}\frac{G(w)}{(1-w)^{3/2}}\,dw,\end{aligned} \tag{17}$$

where $G(w):=w^{-1/\lambda}\big((1-p_2)w+k_1\big)^{2F-1}$. Let us turn now to the computation of $\Delta_2^2(\mu,\eta)$, which is given by

$$\Delta_2^2(\mu,\eta)=\sigma_1'(0)^\lambda\sigma_2(0)L(0)^\lambda\left\{\frac{\tau_1(0)^{-\lambda}}{Q(0,0)}+\int_0^{\tau_1(0)}\left(\frac{M(z)}{P(z,0)}-\frac{M(0)}{P(0,0)}\right)\frac{dz}{z^{\lambda+1}}\right\}.$$

In this case, since one can show that $M(z) \equiv 1$,

$$\sigma_1'(0)^\lambda \sigma_2(0) L(0)^\lambda = \frac{1}{k_1^\lambda} \left(\frac{k_1(1-k_2^2\eta^2)}{1-p_1-k_2^2\eta^2(1-p_2)} \right)^{2\lambda F} \quad \text{and} \quad \frac{\tau_1(0)^{-\lambda}}{Q(0,0)} = \frac{-k_2}{k_1(1-k_2^2\eta^2)^\lambda},$$

we conclude that

$$\Delta_2^2(\mu,\eta) = \frac{k_2 k_1^\lambda (1-k_2^2\eta^2)^{2\lambda F}}{\big(1-p_1-k_2^2\eta^2(1-p_2)\big)^{2\lambda F}} \left\{ \frac{-1}{(1-k_2^2\eta^2)^\lambda} + \lambda \int_0^{1-k_2^2\eta^2} \left(\frac{1}{\sqrt{1-z}} - 1 \right) \frac{dz}{z^{\lambda+1}} \right\}. \tag{18}$$

Concerning the coefficient $\Delta_3^2(\mu,\eta)$, we know that if $\lambda(\mu_0) = 1$ then

$$\Delta_3^2(\mu_0,\eta) = -\sigma_1'(0)\sigma_2(0)L(0)\frac{Q_z(0,0)}{P(0,0)^2}.$$

In our situation, using that $\lambda(\mu_0) = 1$ corresponds to $F = 3/2$, some computations show that

$$\Delta_3^2(\mu_0,\eta) = \frac{-k_1 k_2 (1-k_2^2\eta^2)^3}{2\big(1-p_1-k_2^2\eta^2(1-p_2)\big)^3}. \tag{19}$$

In order to study $R_2(s;\mu,\eta)$ we will also use (z,w)-coordinates. Notice (see Figure 7) that, taking the transferred parametrizations, it is precisely the Dulac map between $\phi(\Sigma_1)$ and $\phi(\Sigma_2^\eta)$. We point out that $\phi(\Sigma_1)$ is in the straight line $z+w=1$. However, in this case, this is not a problem for our purpose. Indeed, in order to study the Poincaré mapping we can apply Proposition 3.9 with the polynomial vector field

$$\widetilde{X}_\mu := \frac{w}{\sqrt{1-z-w}} X_\mu,$$

which provides the same foliation as X_μ and it is obviously analytic. So we can assert that

$$R_2(s;\mu,\eta) = \rho_2(\mu,\eta)s^\lambda + s^\lambda g_2(s;\mu,\eta), \tag{20}$$

where $g_2 \in \mathcal{I}\big(U \times (-\varepsilon,\varepsilon)\big)$ and $\rho_2(\mu,\eta)$ is an analytic function on $U \times (-\varepsilon,\varepsilon)$. For values of μ such that $\lambda(\mu) \approx 1$ we need more information about the remainder term in R_2. In fact, if $\mu \in U_3$ then

$$s^\lambda g_2(s;\mu,\eta) = s\widetilde{g}_2(s;\mu,\eta) \;\text{ with } \widetilde{g}_2 \in \mathcal{I}\big(U_3 \times (-\varepsilon,\varepsilon)\big). \tag{21}$$

This fact does not follow from Proposition 3.9 but it is easy to show and so, for the sake of brevity, we do not prove it here. As we shall see later on, we do not need the concrete expression of $\rho_2(\mu,\eta)$. We shall only use that it is convergent as $\eta \longrightarrow 0$ and this follows from its analyticity at $\eta = 0$.

We can now study the composition $T_3\big(R_2(s;\mu,\eta);\mu,\eta\big)$. Thus, on account of (10) and (20), by applying Lemma 3.10 we can assert that

$$T_3\big(R_2(s)\big) = \Delta_1^3\,\rho_2\,s^\lambda + s^\lambda h_1 \;\text{ with } h_1 \in \mathcal{I}\big(U \times (0,\varepsilon)\big). \tag{22}$$

In case that $\mu \in U_3$ we must be sharper. Since $f_3 \in \mathcal{I}_0\big(U \times (-\varepsilon,\varepsilon)\big)$, we have that $f_3 = s\widehat{f}_3$ where $\widehat{f}_3$ is an analytic function on $s = 0$. Hence, from (10) and (21), it follows that

$$\begin{aligned} T_3\big(R_2(s)\big) &= \Delta_1^3(\rho_2 s^\lambda + s\widetilde{g}_2) + (\rho_2 s^\lambda + s\widetilde{g}_2) f_3\big(\rho_2 s^\lambda + s\widetilde{g}_2\big) \\ &= \Delta_1^3(\rho_2 s^\lambda + s\widetilde{g}_2) + s(\rho_2 s^{\lambda-1/2} + s^{1/2}\widetilde{g}_2)^2 \widehat{f}_3\big(\rho_2 s^\lambda + s\widetilde{g}_2\big). \end{aligned}$$

Since $\lambda(\mu) > 1/2$ for $\mu \in U_3$, note that $\rho_2\, s^{\lambda-1/2}$ belongs to $\mathcal{I}\big(U_3 \times (0,\varepsilon)\big)$. Consequently the expression above shows that $T_3\big(R_2(s)\big) = \Delta_1^3\,\rho_2 s^\lambda + s\widetilde{h}_1$ with $\widetilde{h}_1 \in \mathcal{I}\big(U_3 \times (0,\varepsilon)\big)$. Hence, using that $s^\lambda = (\lambda-1)s\omega(s;\lambda) + s$, we obtain

$$T_3\big(R_2(s)\big) = \Delta_1^3\,\rho_2\,(\lambda-1)s\omega(s;\lambda) + \Delta_1^3\,\rho_2 s + s\widetilde{h}_1 \;\text{ with } \widetilde{h}_1 \in \mathcal{I}\big(U_3 \times (0,\varepsilon)\big). \tag{23}$$

We have now all the necessary ingredients to study $T(s;\mu)$. Let us consider first the case $\mu\in U_1$. In this case, from (11) and (13), by applying Lemma 3.10 we obtain

$$T_2\big(R_1(s)\big)=\Delta_0^2+\Delta_1^2\rho_1 s+sh_2\ \text{ with } h_2\in\mathcal{I}\big(U_1\times(0,\varepsilon)\big).$$

Therefore, taking (9) and (22) also into account, we get

$$T(s;\mu)=\Delta_0^1+\Delta_0^2+\big(\Delta_1^1+\Delta_1^2\rho_1\big)s+s\big(h_2+f_1+s^{\lambda-1}(\Delta_1^3\rho_2+h_1)\big).$$

Then, using that $\lambda(\mu)>1$ for $\mu\in U_1$, we conclude that

$$T(s;\mu)=\Delta_0^1(\mu,\eta)+\Delta_0^2(\mu,\eta)+\big(\Delta_1^1(\mu,\eta)+\Delta_1^2(\mu,\eta)\rho_1(\mu,\eta)\big)s+sh_3(s;\mu,\eta)$$

with $h_3\in\mathcal{I}\big(U_1\times(0,\varepsilon)\big)$. At this point we stress that the coefficients

$$\Delta_0(\mu):=\Delta_0^1(\mu,\eta)+\Delta_0^2(\mu,\eta)\ \text{ and }\ \Delta_1(\mu):=\Delta_1^1(\mu,\eta)+\Delta_1^2(\mu,\eta)\rho_1(\mu,\eta)$$

depend only on μ because $T(s;\mu)$ does not depend on η. This proves in particular that $h_3\in\mathcal{I}(U_1)$. In order to compute explicitly these coefficients we take advantage of (12). For the first one we get

$$\Delta_0(\mu)=\lim_{\eta\longrightarrow 0}\big(\Delta_0^1(\mu,\eta)+\Delta_0^2(\mu,\eta)\big)=\lim_{\eta\longrightarrow 0}\Delta_0^2(\mu,\eta).$$

Thus, by applying the Dominate Convergence Theorem to the expression of $\Delta_0^2(\mu,\eta)$ given in (16) we obtain

$$\Delta_0(\mu)=\int_0^1\frac{k_2}{(1-p_2)w+k_1}\frac{dw}{\sqrt{1-w}}=\frac{\sqrt{2}}{\sqrt{a+b+c}}\operatorname{arctanh}\left(\frac{2\,a+b-\sqrt{b^2-4\,ac}}{2\sqrt{a(a+b+c)}}\right).$$

The last equality above follows from direct integration and using the relation of p_2, k_1 and k_2 with the coefficients of $q_\mu(x)=ax^2+bx+c$. On the other hand, taking (12) into account again,

$$\Delta_1(\mu)=\lim_{\eta\longrightarrow 0}\big(\Delta_1^1(\mu,\eta)+\Delta_1^2(\mu,\eta)\rho_1(\mu,\eta)\big)=\lim_{\eta\longrightarrow 0}\Delta_1^2(\mu,\eta).$$

The computation of this limit is more delicate because $\Delta_1^2(\mu,\eta)$, which is given in (17), contains two terms that considered separately diverge as $\eta\longrightarrow 0$. To show that these divergences cancellate each other we proceed as follows.

$$\begin{aligned}\Delta_1^2(\mu,\eta)=&\frac{-(1-k_2^2\eta^2)^2}{k_1\eta\big(1-p_1-k_2^2\eta^2(1-p_2)\big)}\\&+\frac{k_2(1-k_2^2\eta^2)^{2F}}{2k_1\big(1-p_1-k_2^2\eta^2(1-p_2)\big)^{2F}}\left\{2G(1)\frac{1-k_2\eta}{k_2\eta}+\int_0^{1-k_2^2\eta^2}\frac{G(w)-G(1)}{(1-w)^{3/2}}\,dw\right\}\\=&\frac{(1-k_2^2\eta^2)^2}{k_1\eta\big(1-p_1-k_2^2\eta^2(1-p_2)\big)}\left\{-1+\frac{(1-k_2^2\eta^2)^{1/\lambda}(1-k_2\eta)(1-p_1)^{2F-1}}{\big(1-p_1-k_2^2\eta^2(1-p_2)\big)^{2F-1}}\right\}\\&+\frac{k_2(1-k_2^2\eta^2)^{2F}}{2k_1\big(1-p_1-k_2^2\eta^2(1-p_2)\big)^{2F}}\int_0^{1-k_2^2\eta^2}\frac{G(w)-G(1)}{(1-w)^{3/2}}\,dw.\end{aligned}$$

Now, to compute the limit we apply L'Hôpital's rule to the first term and the Dominate Convergence Theorem to the second one. It can be shown in this way that

$$\begin{aligned}\Delta_1(\mu)=\lim_{\eta\longrightarrow 0}\Delta_1^2(\mu,\eta)&=\frac{k_2}{k_1(p_1-1)}+\frac{k_2}{2k_1(1-p_1)^{2F}}\int_0^1\frac{G(w)-G(1)}{(1-w)^{3/2}}\,dw\\&=\frac{k_2}{2k_1(p_1-1)}\left\{2-\int_0^1\left(w^{-\frac{1}{\lambda}}\left(\frac{1-p_2}{1-p_1}(w-1)+1\right)^{2F-1}-1\right)\frac{dw}{(1-w)^{3/2}}\right\},\end{aligned}$$

and this concludes the proof of the assertion in (a).

Let us turn now to the case $\mu \in U_2$. In this case, from (11) and (14), by applying Lemma 3.10 we obtain that
$$T_2\big(R_1(s)\big) = \Delta_0^2 + \Delta_2^2\,\rho_1^{\lambda}\,s^{\lambda} + s^{\lambda}h_4 \ \text{ with } h_4 \in \mathcal{I}\big(U_2 \times (0,\varepsilon)\big).$$
The combination of (9) and (22) with the above expression shows that
$$T(s;\mu) = \Delta_0^1 + \Delta_0^2 + \big(\Delta_1^3\,\rho_2 + \Delta_2^2\,\rho_1^{\lambda}\big)s^{\lambda} + s^{\lambda}h_5,$$
where $h_5 := h_1 + h_4 + s^{1-\lambda}(\Delta_1^1 + f_1)$ is a function that belongs to $\mathcal{I}\big(U_2 \times (0,\varepsilon)\big)$. This assertion follows from using that $\lambda(\mu) < 1$ for $\mu \in U_2$. Note in addition that, since $T(s;\mu)$ and $s^{\lambda(\mu)}$ do not depend on η, the coefficients
$$\Delta_0(\mu) := \Delta_0^1(\mu,\eta) + \Delta_0^2(\mu,\eta) \ \text{ and } \ \Delta_2(\mu) := \Delta_1^3(\mu,\eta)\,\rho_2(\mu,\eta) + \Delta_2^2(\mu,\eta)\,\rho_1(\mu,\eta)^{\lambda(\mu)}$$
depend only on μ. On the other hand, since $\rho_2(\mu,\eta)$ is analytic at $\eta = 0$, from (12) it turns out that
$$\Delta_2(\mu) = \lim_{\eta \longrightarrow 0}\Big(\Delta_1^3(\mu,\eta)\,\rho_2(\mu,\eta) + \Delta_2^2(\mu,\eta)\,\rho_1(\mu,\eta)^{\lambda(\mu)}\Big) = \lim_{\eta \longrightarrow 0}\Delta_2^2(\mu,\eta).$$
Thus, by applying the Dominate Convergence Theorem to the expression of $\Delta_2^2(\mu,\eta)$ given in (18), one can easily show that
$$\Delta_2(\mu) = \frac{k_2 k_1^{\lambda}}{(1-p_1)^{2\lambda F}}\left\{\lambda\int_0^1\left(\frac{1}{\sqrt{1-z}} - 1\right)\frac{dz}{z^{\lambda+1}} - 1\right\} = \frac{k_2 k_1^{\lambda}}{\big(1-p_1\big)^{2\lambda F}}\,\frac{\lambda\sqrt{\pi}\,\Gamma(-\lambda)}{\Gamma\left(\frac{1}{2}-\lambda\right)}\,.$$
The last equality above follows from direct integration. This proves (b).

Let us study finally the case $\mu \in U_3$. In this case, from (11) and (15), by applying Lemma 3.10 we obtain
$$T_2\big(R_1(s)\big) = \Delta_0^2 + \big(\Delta_3^2\,\rho_1^{\lambda}\big)\,s\omega(s;\lambda) + \big(\Delta_3^2\,\rho_1\omega(\rho_1;\lambda) + \Delta_4^2\,\rho_1\big)\,s + sh_6$$
with $h_6 \in \mathcal{I}\big(U_3 \times (0,\varepsilon)\big)$. Therefore, taking (9) and (23) also into account, we get
$$\begin{aligned} T(s;\mu) = \Delta_0^1 + \Delta_0^2 &+ \big(\Delta_3^2\,\rho_1^{\lambda} + \Delta_1^3\,\rho_2(\lambda-1)\big)\,s\omega(s;\lambda) \\ &+ \big(\Delta_1^1 + \Delta_3^2\,\rho_1\omega(\rho_1;\lambda) + \Delta_4^2\,\rho_1 + \Delta_1^3\,\rho_2\big)\,s + sh_7, \end{aligned}$$
where $h_7 := f_1 + \widetilde{h}_1 + h_6$ is a function that belongs to $\mathcal{I}\big(U_3 \times (0,\varepsilon)\big)$. On the other hand, since $T(s;\mu)$ and $\omega\big(s;\lambda(\mu)\big)$ depend only on μ, the coefficients
$$\begin{aligned} \Delta_0(\mu) &:= \Delta_0^1(\mu,\eta) + \Delta_0^2(\mu,\eta), \\ \Delta_3(\mu) &:= \Delta_3^2(\mu,\eta)\,\rho_1(\mu,\eta)^{\lambda(\mu)} + \Delta_1^3(\mu,\eta)\,\rho_2(\mu,\eta)\big(\lambda(\mu) - 1\big) \end{aligned}$$
and
$$\begin{aligned} \Delta_4(\mu) := \Delta_1^1(\mu,\eta) &+ \Delta_3^2(\mu,\eta)\,\rho_1(\mu,\eta)\omega\big(\rho_1(\mu,\eta);\lambda(\mu)\big) \\ &+ \Delta_4^2(\mu,\eta)\,\rho_1(\mu,\eta) + \Delta_1^3(\mu,\eta)\,\rho_2(\mu,\eta) \end{aligned}$$
do not depend on η. This implies in particular that $h_7(s;\mu,\eta)$ does not depend on η, and so we can assert that $h_7 \in \mathcal{I}(U_3)$. Finally, if we consider some $\mu_0 \in U_3$ such that $\lambda(\mu_0) = 1$, then $\Delta_3(\mu_0) = \Delta_3^2(\mu_0,\eta)\,\rho_1(\mu_0,\eta)$ and consequently, from (12),
$$\Delta_3(\mu_0) = \lim_{\eta \longrightarrow 0}\Delta_3^2(\mu_0,\eta)\,\rho_1(\mu_0,\eta) = \lim_{\eta \longrightarrow 0}\Delta_3^2(\mu_0,\eta).$$

Thus, on account of (19), it follows that

$$\Delta_3(\mu_0) = \frac{k_1 k_2}{2(p_1-1)^3}\,.$$

This proves (c) and concludes the proof of the result. ■

It is clear that the sign of $P_s(s;\mu)$ for small positive s determines the monotonicity of the period function near the outer boundary of the period annulus. So we need to study the coefficients of the second monomial in the asymptotic development given in Theorem 3.6. To this end we introduce the sets

$$\begin{aligned}\Gamma_1 &:= \{\mu \in U_1 : \Delta_1(\mu) = 0\},\\ \Gamma_2 &:= \{\mu \in U_2 : \Delta_2(\mu) = 0\},\\ \Gamma_3 &:= \{\mu \in U_3 : \Delta_3(\mu) = 0 \text{ with } \lambda(\mu) = 1\}.\end{aligned} \tag{24}$$

One can easily verify that $\Gamma_2 = \{\mu \in U_2 : F = 2\}$ and that Γ_3 is empty. Figure 4 shows the set Γ_1 computed numerically. We are now in position to prove the main result of this subsection:

Proof of Theorem 3.3 Fix some $\mu^\star \in U\setminus\{\Gamma_1\cup\Gamma_2\}$ and note that, taking (6) and (24) into account, there are three different situations to consider:

(a) $\mu^\star \in U_1 \setminus \Gamma_1$,

(b) $\mu^\star \in U_2 \setminus \Gamma_2$,

(c) $\mu^\star \in U_3$ such that $\lambda(\mu^\star) = 1$.

The fact that $\mu^\star$ is a local regular value in the cases (a) and (b) follows exactly the same way as in the proof of Theorem 5.1 in [11]. So let us consider only the case (c), which corresponds to the values of $\mu \in U$ such that $F = 3/2$. Note first of all that, from (c) in Theorem 3.6, we can assert that if $\mu \in U_3$ then

$$P_s(s;\mu) = \Delta_3(\mu)\big(\lambda\omega(s;\lambda)+1\big) + \Delta_4(\mu) + s f_3'(s;\mu) + f_3(s;\mu),$$

where $f_3 \in \mathcal{I}(U_3)$. Here we used that, on account of Definition 3.5, $s\omega_s = (\lambda-1)\omega + 1$. On the other hand, since $\lambda(\mu) \longrightarrow 1$ as $\mu \longrightarrow \mu^\star$, it is clear that $\omega(s;\lambda) \longrightarrow -\infty$ as $(s,\mu) \longrightarrow (0,\mu^\star)$. Consequently, using also that $f_3 \in \mathcal{I}(U_3)$, from the above equality we obtain that

$$\frac{P_s(s;\mu)}{\lambda\omega(s;\lambda)+1} \longrightarrow \Delta_3(\mu^\star) \text{ as } (s,\mu) \longrightarrow (0,\mu^\star).$$

Therefore, since one can easily verify that $\Delta_3(\mu^\star) < 0$, we can assert that there exists a neighbourhood $U^\star$ of $\mu^\star$ and $\varepsilon > 0$ such that $P_s(s;\mu) > 0$ for all $s \in (0,\varepsilon)$ and $\mu \in U^\star$. According to (b) in Remark 2.6, this proves that $\mu^\star$ is a local regular value. It also shows that the period function is monotonous decreasing on the outer boundary of $\mathcal{P}_\mu$. Indeed, $P(s;\mu)$ is by definition the period of the periodic orbit of X_μ passing through the point $(p_1 - s, 0)$, which approaches to the outer boundary as s *decreases.*

The assertions concerning the monotonicity in the cases (a) and (b) follow exactly the same way taking into account the sign of $\Delta_1(\mu^\star)$ and $\Delta_2(\mu^\star)$ respectively. ■

The rest of the subsection is devoted to show some properties of Γ_1. We prove the following:

Proposition 3.11. *The set Γ_1 is the graphic of an analytic function $D = \mathcal{G}(F)$ defined for $F \in (1, 3/2)$ that has the following properties:*

(*a*) $-F < \mathcal{G}(F) < -1/2$ *for all* $F \in (1, 3/2)$,

(*b*) $\mathcal{G}(F) \longrightarrow -3/2$ *as* $F \nearrow 3/2$,

(*c*) $\mathcal{G}(F) \longrightarrow -1/2$ *as* $F \searrow 1$,

(*d*) $\mathcal{G}(5/4) = -1$.

This result follows almost directly from Lemma 3.13 bellow. However, in order to prove Lemma 3.13 we shall need a previous result concerning a general property of the coefficients in Proposition 3.9.

Lemma 3.12. *Under the hypothesis of Proposition 3.9, let* $\{\mu_k\}$ *be a sequence of parameters in* W_1 (*respectively* W_2) *such that* $\mu_k \longrightarrow \widehat{\mu}$ *with* $\lambda(\widehat{\mu}) = 1/n$ *and* $\Delta_3(\widehat{\mu}) \neq 0$.

(*a*) *If* $\Delta_3(\widehat{\mu}) > 0$ *then* $\Delta_1(\mu_k)$ (*respectively* $\Delta_2(\mu_k)$) *tends to* $-\infty$ *as* $\mu_k \longrightarrow \widehat{\mu}$.

(*b*) *If* $\Delta_3(\widehat{\mu}) < 0$ *then* $\Delta_1(\mu_k)$ (*respectively* $\Delta_2(\mu_k)$) *tends to* $+\infty$ *as* $\mu_k \longrightarrow \widehat{\mu}$.

Proof. We shall prove (*a*) and (*b*) for a sequence $\{\mu_k\}$ in W_1 (the other case follows exactly the same way). Notice first that, on account of $\mu_k \in W_1$, we have $\lambda(\mu_k) > 1/n$ and

$$T(s;\mu_k) = \Delta_0(\mu_k) + \Delta_1(\mu_k)s + sf_1(s;\mu_k) \ \text{ with } f_1 \in \mathcal{I}(W_1). \tag{25}$$

On the other hand, note that $\mu_k \in W_3$ for k large enough because $\mu_k \longrightarrow \widehat{\mu} \in W_3$. Therefore

$$T(s;\mu_k) = \Delta_0(\mu_k) + \Delta_3(\mu_k)s\omega\big(s;\lambda(\mu_k)n\big) + \Delta_4(\mu_k)s + sf_3(s;\mu_k) \ \text{ with } f_3 \in \mathcal{I}(W_3). \tag{26}$$

By definition

$$s\omega\big(s;\lambda(\mu_k)n\big) = \frac{s^{\lambda(\mu_k)\,n} - s}{\lambda(\mu_k)n - 1}$$

and consequently, from (26),

$$T(s;\mu_k) = \Delta_0(\mu_k) + \left(\Delta_4(\mu_k) - \frac{\Delta_3(\mu_k)}{\lambda(\mu_k)n-1}\right)s + sg(s;\mu_k),$$

where

$$g(s;\mu) := f_3(s;\mu) + \frac{\Delta_3(\mu)}{\lambda(\mu)n-1}\, s^{\lambda(\mu)\,n-1}$$

is a function that belongs to $\mathcal{I}(W_1)$. Consequently the combination of this expression for $T(s;\mu_k)$ and the one in (25) shows that

$$\Delta_1(\mu_k) = \Delta_4(\mu_k) - \frac{\Delta_3(\mu_k)}{\lambda(\mu_k)n-1}. \tag{27}$$

Note also that, since Δ_3 and Δ_4 are analytic on W_3, $\Delta_3(\mu_k) \longrightarrow \Delta_3(\widehat{\mu})$ and $\Delta_4(\mu_k) \longrightarrow \Delta_4(\widehat{\mu})$ as $\mu_k \longrightarrow \widehat{\mu}$. In addition, due to $\mu_k \in W_1$, it turns out that $\lambda(\mu_k)n - 1 \searrow 0$ as $\mu_k \longrightarrow \widehat{\mu}$. Hence, from (27), we conclude that

$$\lim_{\mu_k \to \widehat{\mu}} \Delta_1(\mu_k) = -\infty \ \text{ if } \Delta_3(\widehat{\mu}) > 0 \text{ and } \lim_{\mu_k \to \widehat{\mu}} \Delta_1(\mu_k) = +\infty \ \text{ if } \Delta_3(\widehat{\mu}) < 0$$

as claimed. ■

In what follows we shall use the notation $k_1 = p_2 - p_1$ and $k_2 = 1/\sqrt{2a}$ introduced in the proof of Theorem 3.6. Let us also define $\Psi(\mu)$ by means of the relation $\Delta_1(\mu) = \frac{k_2}{2k_1(p_1-1)}\,\Psi(\mu)$, that is,

$$\Psi(\mu) := 2 - \int_0^1 \left(u^{2(1-F)}\big((u-1)\kappa+1\big)^{2F-1} - 1\right)\frac{du}{(1-u)^{3/2}} \ \text{ where } \kappa(\mu) := \frac{1-p_2}{1-p_1}. \tag{28}$$

Concerning this function we prove the following:

Lemma 3.13. *If* $(D,F) \in U_1$ *then the following holds:*

(a) $\Psi_D(D,F) < 0$,

(b) $\Psi(D,F) < 0$ *for* $D \geq -1/2$,

(c) $\Psi(-1,5/4) = 0$,

(d) $\Psi(D,F) \longrightarrow 4$ *as* $(D,F) \longrightarrow (-q,q)$ *with* $1 < q < 3/2$,

(e) $\Psi(D,F) \longrightarrow -\infty$ *as* $(D,F) \longrightarrow (q,3/2)$ *with* $-3/2 < q < 0$,

(f) $\dfrac{\Psi\big(D,1+(D+1/2)^2\big)}{(D+1/2)^2} \longrightarrow 4(4-\pi)$ *as* $D \nearrow -1/2$.

Proof. Some computations show that

$$\kappa = \frac{(2D+1)\sqrt{F(F-1)} + \sqrt{(F+D)(F-D-1)}}{(2D+1)\sqrt{F(F-1)} - \sqrt{(F+D)(F-D-1)}} \tag{29}$$

and

$$\frac{d\kappa}{dD} = \frac{-(2F-1)^2}{\big((2D+1)\sqrt{F(F-1)} - \sqrt{(F+D)(F-D-1)}\,\big)^2}\sqrt{\frac{F(F-1)}{(F+D)(F-D-1)}}\,.$$

Thus, from the last expression above it follows that

$$\Psi_{\mathrm{D}}(\mu) = (2F-1)\,\frac{d\kappa}{dD}\int_0^1 \left(\frac{(u-1)\kappa+1}{u}\right)^{2(F-1)} \frac{du}{(1-u)^{1/2}}$$

is negative for $\mu \in U_1$. This proves (a). Let us turn next to the assertion in (b). Notice first that

$$\Psi(\mu) = \int_0^1 \frac{2-u-u^{2(1-F)}\big((u-1)\kappa+1\big)^{2F-1}}{(1-u)^{3/2}}\,du = \int_0^1 \frac{2-u\big(h(u;\mu)+1\big)}{(1-u)^{3/2}}\,du, \tag{30}$$

where

$$h(u;\mu) := \left(\frac{(u-1)\kappa+1}{u}\right)^{2F-1}.$$

On the other hand one can verify that $\kappa \leq -1$ for $D \geq -1/2$. Taking this into account it is easy to show that

$$\frac{(u-1)\kappa+1}{u} > \frac{2-u}{u} > 1 \text{ for } u \in (0,1).$$

Therefore, since $F > 1$, we have that

$$h(u;\mu) > \left(\frac{2-u}{u}\right)^{2F-1} > \frac{2-u}{u}$$

and this, on account of (30), proves (b). The assertion in (c) is straightforward because $\kappa(-1,F) = 0$ and direct integration yields

$$\Psi(-1,5/4) = 2 - \int_0^1 \frac{u^{-1/2}-1}{(1-u)^{3/2}} = 0.$$

To show (d) notice first that $\kappa(-q,q)=1$. Thus, if $(D,F)\longrightarrow(-q,q)$ with $1<q<3/2$, then

$$\Psi(D,F)\longrightarrow 2+\int_0^1\frac{du}{\sqrt{1-u}}=4.$$

Here we apply the Dominate Convergence Theorem and to do so it is necessary that $1<q<3/2$. In order to prove (e) we shall apply Lemma 3.12 to the results obtained in Theorem 3.6. To this end notice first that the parameter $\widehat{\mu}:=(q,3/2)$ satisfies $\lambda(\widehat{\mu})=1$. Hence (c) in Theorem 3.6 shows that

$$\Delta_3(\widehat{\mu})=\frac{k_1k_2}{2(p_1-1)^3},$$

which is negative because $0<p_1<1$ and $k_i>0$. Thus, if we consider any sequence $\{\mu_k\}$ in U_1 with $\mu_k\longrightarrow\widehat{\mu}$ then, by applying Lemma 3.12, it follows that

$$\Delta_1(\mu_k)\longrightarrow+\infty\ \text{ as }\mu_k\longrightarrow\widehat{\mu}. \tag{31}$$

Recall at this point that, by definition,

$$\Delta_1(\mu)=\frac{k_2}{2k_1(p_1-1)}\,\Psi(\mu). \tag{32}$$

One can verify moreover that if $(D,F)\longrightarrow(q,3/2)$ with $-3/2<q<0$, then

$$\frac{k_2}{2k_1(p_1-1)}\longrightarrow\frac{18}{6q+3-\sqrt{9-12q-12q^2}}\left(\frac{2q^3}{12q^2+12q-9}\right)^{1/2}<0.$$

Hence, on account of (31) and (32), we conclude that $\Psi(D,F)\longrightarrow-\infty$ as $(D,F)\longrightarrow(q,3/2)$. This proves (e). Finally, to show (f) we shall take advantage of the expression of $\Psi(\mu)$ given in (30). Let us define

$$g(u;D):=h\Big(u;\big(D,1+(D+1/2)^2\big)\Big).$$

The idea will be to use the Taylor's development of $g(u;D)$ at $D=-1/2$. We point out however that the term $\sqrt{F-1}$ in κ, see (29), makes that $\kappa\big(D,1+(D+1/2)^2\big)$ is not smooth enough at $D=-1/2$. Nevertheless, this will not be a problem for our purpose because we only need to study its behaviour as $D\nearrow-1/2$, and it is clear that this function coincides on $(-1/2-\varepsilon,-1/2]$ with an analytic function at $D=-1/2$. In the sequel, when we study $g(u;D)$, we will use this analytic function instead of the original $\kappa\big(D,1+(D+1/2)^2\big)$. Keeping this in mind, to avoid cumbersome notation we shall maintain the name of the functions. Now, since one can check that

$$g(u;D)=\frac{2-u}{u}+\frac{1}{2}\frac{d^2}{dD^2}\,g\big(u;\xi_{\mathrm{D}}\big)(D+1/2)^2\ \text{ with }\xi_{\mathrm{D}}\in(D,-1/2),$$

from (30) we obtain that

$$\frac{\Psi\big(D,1+(D+1/2)^2\big)}{(D+1/2)^2}=-\frac{1}{2}\int_0^1 u\,\frac{d^2}{dD^2}\,g\big(u;\xi_{\mathrm{D}}\big)\,\frac{du}{(1-u)^{3/2}}. \tag{33}$$

Lengthy computations, which are not included here for the sake of brevity, allow to verify that, for all $D\in(-1/2-\varepsilon,-1/2]$,

$$\left|\frac{u}{(1-u)^{3/2}}\frac{d^2}{dD^2}\,g(u;D)\right|<f(u)\ \text{ with }f\in L^1\big((0,1)\big).$$

Therefore, by applying the Dominate Convergence Theorem, from (33) it turns out that

$$\frac{\Psi\big(D,1+(D+1/2)^2\big)}{(D+1/2)^2}\longrightarrow-\frac{1}{2}\int_0^1 u\,\frac{d^2}{dD^2}\,g\big(u;-1/2\big)\,\frac{du}{(1-u)^{3/2}}\ \text{ as }D\nearrow-1/2.$$

Finally, since one can check that

$$\frac{d^2}{dD^2}\, g\big(u;-1/2\big) = \frac{4}{u}\left((2-u)\ln\Big(\frac{2-u}{u}\Big) + 4(u-1)\right)$$

and

$$-2\int_0^1 \left((2-u)\ln\Big(\frac{2-u}{u}\Big) + 4(u-1)\right)\frac{du}{(1-u)^{3/2}} = 4(4-\pi),$$

the result follows. ■

Proof of Proposition 3.11 Recall that, by definition,

$$\Delta_1(\mu) = \frac{k_2}{2k_1(p_1-1)}\,\Psi(\mu).$$

Then, since one can verify that $\frac{k_2}{2k_1(p_1-1)}$ does not vanish on U_1, it suffices to study $\{\mu \in U_1 : \Psi(\mu) = 0\}$. Notice first that, by applying the Implicit Function Theorem, (a) and (c) in Lemma 3.13 show that this set is the graphic of an analytic function $D = \mathcal{G}(F)$ with $\mathcal{G}(5/4) = -1$. The fact that $\mathcal{G}$ is defined for all $F \in (1, 3/2)$ and that $-F < \mathcal{G}(F) < -1/2$ follow from (b) and (d) in Lemma 3.13. On the other hand, by applying (e) in Lemma 3.13 we can assert that $\mathcal{G}(F) \longrightarrow -3/2$ as $F \nearrow 3/2$. So it only remains to prove (c). To this end note that (f) in Lemma 3.13 implies that

$$\Psi\big(D, 1 + (D+1/2)^2\big) > 0 \text{ for all } D \in (-1/2-\varepsilon, -1/2).$$

In addition, from (d) in Lemma 3.13, we have that $\Psi(-1/2, F) < 0$ for all $F \in (1, 1+\varepsilon)$. Consequently, by Bolzano's Theorem, in any neighbourhood V of $(D,F) = (-1/2, 1)$ there exists some $\mu \in U_1 \cap V$ such that $\Psi(\mu) = 0$. This shows (c) and concludes the proof of the result. ■

3.2.2 The case $0 < F < 1$ and $-1 < D < 0$.

The aim of this subsection is only to recall the results that we obtain in [11] concerning the period function of the center at the origin of X_μ in case (see Figure 8) that μ belongs to

$$W := \big\{(D,F) \in \mathbb{R}^2 :\ -1 < D < 0 \text{ and } 0 < F < 1\big\}.$$

Setting

$$\Gamma_3 := \big\{\mu \in W : D = -\tfrac{1}{2}, F \in \big(\tfrac{1}{2}, 1\big)\big\} \text{ and } \Gamma_4 := \big\{\mu \in W :\ F = \tfrac{1}{2}\big\},$$

from Theorem 5.1 and Proposition 5.2 in [11] it follows the next result:

Theorem 3.14. *Let $\{X_\mu,\, \mu \in \mathbb{R}^2\}$ be the family of vector fields in (2) and consider the period function of the center at the origin. Then the open set $W \setminus \{\Gamma_3 \cup \Gamma_4\}$ corresponds to local regular values of the period function at the outer boundary of the period annulus. Moreover, for these parameters, the period function is monotonous near the outer boundary and the corresponding character is shown in Figure 8.*

3.3 Bounded period annulus

In this section we study the period function of the center at the origin for the parameter values μ such that $\mathcal{P}_\mu$ is bounded. Notice that among these parameters there are two main situations to consider (see Figure 3). The first one are those parameters such that the outer boundary of P_μ is a saddle loop, which corresponds to

$$M := \big\{\mu \in \mathbb{R}^2 : D < -1 \text{ and } F + D < 0\big\} \cup \big\{\mu \in \mathbb{R}^2 : D > 0 \text{ and } F + D > 0\big\}.$$

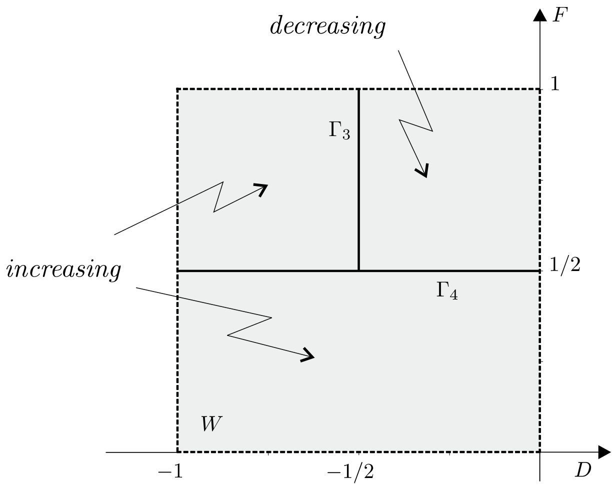

Figure 8: Monotonicity of the period function at the outer boundary of $\mathcal{P}_\mu$.

The second one are those parameters such that it is a bicycle, which corresponds to

$$N := \{\mu \in \mathbb{R}^2 : -1 < D < 0 \text{ and } F < 0\} \cup \{\mu \in \mathbb{R}^2 : D > 0 \text{ and } F + D < 0\}.$$

Now, with this definitions (see Figure 9), our goal is to prove the following result:

Theorem 3.15. *Let $\{X_\mu,\, \mu \in \mathbb{R}^2\}$ be the family of vector fields in (2) and consider the period function of the center at the origin. Then the open set $M \cup N$ corresponds to local regular values of the period function at the outer boundary of the period annulus. Moreover, for these parameters, the period function is monotonous increasing near the outer boundary.*

Proof. Let us begin by recalling some facts about the normal form of a family of vector fields with a hyperbolic saddle. So let us consider a $\mathcal{C}^\infty$-family of vector fields $\{X_\mu : \mu \in W\}$ with a hyperbolic saddle p_μ. Let the eigenvalues of X_μ at p_μ be $\lambda_1(\mu)$ and $\lambda_2(\mu)$, with $\lambda_2 < 0 < \lambda_1$, and let $r(\mu) := -\frac{\lambda_2(\mu)}{\lambda_1(\mu)}$ be its ratio of hyperbolicity. Fix some μ_0 and assume first that $r(\mu_0)$ is rational, i.e., $r(\mu_0) = \frac{p}{q}$ with $(p, q) = 1$. Then, for each $k \in \mathbb{N}$, there exists a $\mathcal{C}^k$-diffeomorphism Φ such that, in some neighbourhood of p_μ and for $\mu \approx \mu_0$,

$$X_\mu = \Phi_* \left(\frac{1}{f(u;\mu)} \left(x\partial_x - y g(u;\mu)\partial_y \right) \right), \tag{34}$$

where $u = x^p y^q$ and

$$f(u;\mu) = \tfrac{1}{\lambda_1(\mu)} + \beta_1(\mu)u + \ldots + \beta_{n_k} u^{n_k},$$
$$g(u;\mu) = r(\mu) + \alpha_1(\mu)u + \ldots + \alpha_{n_k} u^{n_k}.$$

If $r(\mu_0) \notin \mathbb{Q}$ then the above results holds with $\alpha_i(\mu) \equiv 0$ and $\beta_i(\mu) \equiv 0$ for all i.

Let us first take a parameter $\mu^\star$ such that the period annulus is bounded by a saddle loop (i.e., $\mu \in M$). Notice that, according to Remark 3.1, there is a neigbourhood $U^\star$ of $\mu^\star$ such that the saddle loop persists and it is the outer boundary of $\mathcal{P}_\mu$ for all $\mu \in U^\star$. Moreover in section 3.1 we showed that the saddle is located at $(-\frac{1}{D}, 0)$ and has eigenvalues

$$\lambda_1(\mu) = \sqrt{1 + \tfrac{1}{D}} \text{ and } \lambda_2(\mu) = -\sqrt{1 + \tfrac{1}{D}},$$

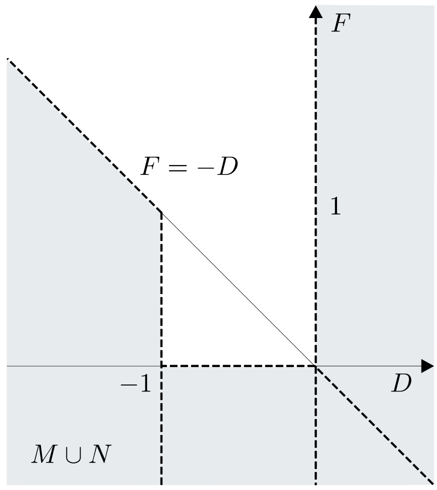

Figure 9: Parameters corresponding to bounded period annulus.

and so $r(\mu) \equiv 1$. Since in particular $r(\mu^\star) = 1$, there exists a local diffeomorphism Φ such that it holds (34) with $p = q = 1$. We introduce two transversal sections Σ_σ and Σ_τ given by $s \longmapsto \Phi(s,1)$ and $s \longmapsto \Phi(1,s)$ respectively. Let us also define $P(s;\mu)$ as the period of the periodic orbit of X_μ passing through $\Phi(s,1)$. We split it as

$$P(s;\mu) = T_1(s;\mu) + T_2(s;\mu),$$

where T_1 is the time function for $-X_\mu$ from Σ_σ to Σ_τ and T_2 is the time function for X_μ from Σ_σ to Σ_τ (i.e., the passage through the saddle). To be more precise, $T_1(s;\mu)$ is the minimum *positive* time necessary so that the solution of $-X_\mu$ passing through $\Phi(s,1) \in \Sigma_\sigma$ reaches Σ_τ. Note that $T_1(s;\mu)$ is a smooth function on $s = 0$. The time function associated to the passage through a saddle with $r(\mu^\star) = 1$ has already been studied. Indeed, Lemma 2 in [1] shows that

$$T_2(s;\mu) = -\frac{1}{\lambda_1(\mu)} \log s - \beta_1(\mu) s \omega\big(s; r(\mu)\big) + \psi(s;\mu), \tag{35}$$

where ψ is a $\mathcal{C}^1$ function at $(s,\mu) = (0,\mu^\star)$ and 1-flat at $s = 0$ for all μ. In our situation $\omega\big(s; r(\mu)\big) = \log s$ because in fact $r(\mu) \equiv 1$. From the above expression we obtain that

$$s\frac{d}{ds} T_2(s;\mu) \longrightarrow -1/\lambda_1(\mu^\star) \;\text{ as } (s,\mu) \longrightarrow (0,\mu^\star).$$

This shows, since T_1 is smooth at $s = 0$, that $sP_s(s;\mu) \longrightarrow -1/\lambda_1(\mu^\star)$ as $(s,\mu) \longrightarrow (0,\mu^\star)$. Therefore, due to $\lambda_1(\mu^\star) > 0$, we can assert that there exists $\varepsilon > 0$ such that $P_s(s;\mu) < 0$ for all $s \in (0,\varepsilon)$ and $\mu \approx \mu^\star$. Consequently, by (b) in Remark 2.6, $\mu^\star$ is a local regular value of the period function at the outer boundary. We can conclude in addition, noting that $\Phi(s,1)$ approaches to the saddle loop as s *decreases*, that the period function is increasing near the outer boundary.

Let us consider next a parameter $\mu^\star$ such that the outer boundary of $\mathcal{P}_{\mu^\star}$ is a bicycle (i.e., $\mu \in N$). As before, according to Remark 3.1, the bicycle persists and it is the outer boundary of $\mathcal{P}_\mu$ for $\mu \approx \mu^\star$. We shall take advantage of the symmetry of the Loud's systems with respect to $\{y = 0\}$ to study only the passage through one of the saddles. For instance, let us consider the saddle in $\{y > 0\}$. Recall (see section 3.1) that this saddle is located at

$$p_\mu := \left(1, \sqrt{-\tfrac{(D+1)}{F}}\right) \text{ with } \lambda_1(\mu) = \sqrt{-\tfrac{(D+1)}{F}} \text{ and } \lambda_2(\mu) = 2F\sqrt{-\tfrac{(D+1)}{F}}\,,$$

and so its ratio of hyperbolicity is $r(\mu) = -2F$. We take now a local diffeomorphism Φ that conjugates X_μ for $\mu \approx \mu^\star$ with its normal form, which depends on $r(\mu^\star) \in \mathbb{Q}$ or $r(\mu^\star) \notin \mathbb{Q}$. As before, we shall take two transversal sections Σ_σ and Σ_τ given by $s \longmapsto \Phi(s,1)$ and $s \longmapsto \Phi(1,s)$ respectively. Denote by $P(s;\mu)$ the period of the periodic orbit of X_μ passing through $\Phi(s,1)$. We decompose it as

$$P(s;\mu) = 2T_1(s;\mu) + 2T_2(s;\mu)\big) + 2T_3\big(R(s;\mu);\mu\big),$$

where T_1 is the time function for $-X_\mu$ from Σ_σ to $\{y=0\}$, R and T_2 are respectively the Dulac map and the time function for X_μ from Σ_σ to Σ_τ and, finally, T_3 is the time function for X_μ from Σ_τ to $\{y=0\}$. It is clear that T_1 and T_3 are smooth functions on $s=0$. On the other hand, it is well known (see [13, 15] for instance) that

$$R(s;\mu) = s^{r(\mu)}\big(\rho(\mu) + \psi_1(s;\mu)\big) \text{ where } \psi_1 \in \mathcal{I}\big(U^\star\big)$$

for some neighbourhood $U^\star$ of $\mu^\star$. Concerning T_2, if $r(\mu^\star) \notin \mathbb{Q}$ then the normal form (34) is linear and one can easily verify that $T_2(s;\mu) = -\frac{1}{\lambda_1(\mu)} \log s$. The expression is not so easy when $r(\mu^\star) = \frac{p}{q}$. The case $p = q = 1$ is treated in [1] and, as we already mentioned, one obtains the expression given in (35). In the general case, following the same approach it can be shown that

$$T_2(s;\mu) = -\tfrac{1}{\lambda_1(\mu)} \log s - \tfrac{1}{p}\,\beta_1(\mu)\, s^p \omega\big(s^p; \tfrac{q}{p}\, r(\mu)\big) + \psi_2(s;\mu),$$

where ψ_2 is $\mathcal{C}^1$ at $(s,\mu) = (0,\mu^\star)$ and 1-flat at $s=0$ for all μ. Some computations show that

$$\tfrac{d}{ds}\, T_2(s;\mu) = -\tfrac{1}{\lambda_1(\mu)}\,\tfrac{1}{s} - \beta_1(\mu)\, s^{p-1}\left(\tfrac{q}{p}\, r(\mu)\,\omega\big(s^p; \tfrac{q}{p}\, r(\mu)\big) + 1\right) + \psi_2'(s;\mu).$$

Therefore we can assert that in both cases, $r(\mu^\star)$ rational or irrational, it holds

$$s\frac{d}{ds}\, T_2(s;\mu) \longrightarrow -1/\lambda_1(\mu^\star) \text{ as } (s,\mu) \longrightarrow (0,\mu^\star).$$

Finally this implies that $sP_s(s;\mu) \longrightarrow -2/\lambda_1(\mu^\star)$ as $(s,\mu) \longrightarrow (0,\mu^\star)$ because T_1 and T_3 are smooth at $s=0$ and $\psi_1 \in \mathcal{I}\big(U^\star\big)$. Exactly as in the saddle loop case, this proves that $\mu^\star$ is a local regular value at the outer boundary and that the period function is monotonous increasing there. ■

Remark 3.16 From the proof of Theorem 3.15 it follows that if $\mu \in M \cup N$ then the period function of X_μ tends to $+\infty$ as we approach to the outer boundary of $\mathcal{P}_\mu$. □

3.4 Proof of the main result

Proof of Theorem A The fact that the parameters in $\mathbb{R}^2 \setminus \{\Gamma_B \cup \Gamma_U\}$ are local regular values follows from the application of Theorems 3.3, 3.14 and 3.15 in the corresponding regions that cover. These theorems also show the assertions in (*a*) and (*b*) concerning the monotonicity near the outer boundary. Consider now a parameter $\mu_0 \in \Gamma_B$ and note that any neighbourhood of μ_0 intersects $\mathcal{D}_B \setminus \Gamma_U$ and $\mathcal{I}_B \setminus \Gamma_U$. Consequently, any neighbourhood of μ_0 contains two parameters μ^+ and μ^- such that the respective period functions have different monotonicity in the outer boundary. (Here we use, recall Remark 2.5, that the character increasing or decreasing does not depend on the particular parametrization of the period function used.) This clearly implies that μ_0 is a local bifurcation value at the outer boundary and so the result is proved. ■

4 Bifurcation in the interior

In this section we determine some local bifurcation values of the period function in the interior of the period annulus of the dehomogenized Loud's systems (2). To be more precise, we prove that there are three parameter values, namely

$$
(36)\qquad L_1=\left(-\frac{3}{2},\frac{5}{2}\right),\quad L_2=\left(\frac{-11+\sqrt{105}}{20},\frac{15-\sqrt{105}}{20}\right)\ \text{ and }\ L_3=\left(\frac{-11-\sqrt{105}}{20},\frac{15+\sqrt{105}}{20}\right),
$$

such that at each L_i there exists a germ of analytic curve corresponding to this type of bifurcation. We describe moreover the relative position of this curve with respect to other bifurcation curves. The result is based on the work of Chicone and Jacobs in [4].

Setting $\mu=(D,F)$ as usual, let us denote by $P(s;\mu)$ the period of the periodic orbit of system (2) passing through the point $(s,0)$. Note that $P(s;\mu)$ is a well defined analytic function for $(s,\mu)\in(0,\varepsilon)\times\mathbb{R}^2$ because the center is nondegenerate. Moreover, since the eigenvalues of the linear part of X_μ at the origin are $\pm i$, it can be extended analytically to $s=0$ by setting $P(0;\mu):=2\pi$. We can thus consider the Taylor expansion of $P(s;\mu)$ at $s=0$,

$$
(37)\qquad P(s;\mu)=2\pi+P_2(\mu)s^2+P_3(\mu)s^3+P_4(\mu)s^4+P_5(\mu)s^5+P_6(\mu)s^6+\dots.
$$

The coefficients $P_k(\mu)$, which are real polynomials in the parameters of the system, are called the period constants of the center. For instance (see [4]),

$$
\begin{aligned}
P_2(D,F)=&\frac{\pi}{12}(10D^2+10DF-D+4F^2-5F+1),\\
P_4(D,F)=&\frac{\pi}{1152}\big(1540D^4+4040D^3F+1180D^3+4692D^2F^2+1992D^2F+453D^2\\
&+2768DF^3+228DF^2+318DF-2D+784F^4-616F^3-63F^2-154F+49\big).
\end{aligned}
$$

Chicone and Jacobs prove in [4] that the ideals generated by the period constants verify that

$$
(38)\qquad (P_2)=(P_2,P_3)\subsetneq(P_2,P_4)=(P_2,P_4,P_5)\subsetneq(P_2,P_4,P_6)=(P_i,\ i\in\mathbb{N}).
$$

They also show that the ideal (P_2,P_4,P_6) determines the points

$$
S_1=(-1/2,1/2),\qquad S_2=(0,1),\qquad S_3=(0,1/4),\qquad S_4=(-1/2,2),
$$

which correspond to the four nonlinear quadratic isochronous centers. The ideal (P_2,P_4) determines, apart from the four isochronous centers, the three weak centers L_i given in (36). Note in particular that these seven parameter values are over the conic $\Gamma_C:=\{\mu\in\mathbb{R}^2:\ P_2(\mu)=0\}$ (see Figure 10).

In [4] the authors use a notion of bifurcation which differs from the one introduced in section 2. Indeed, they say that k critical periods bifurcate from the center corresponding to the parameter μ_0 if for every $\varepsilon>0$ and every neighbourhood U of μ_0 there is a point $\mu_1\in U$ such that the equation $P'(s;\mu_1)=0$ has k solutions in the interval $(0,\varepsilon)$. With this definition and the notation introduced above we can now summarize their result concerning the dehomogenized Loud's systems:

Theorem 4.1 (Chicone-Jacobs). *The maximal number of critical periods bifurcating from the center at the origin of the dehomogenized Loud's family is two. In addition,*

(a) *If $\mu\notin\Gamma_C$ then no critical period bifurcates from the center.*

(b) *If $\mu\in\Gamma_C\setminus\{L_1,L_2,L_3\}$ then at most one critical period bifurcates from the center and there are perturbations with exactly one critical period.*

(c) *If $\mu\in\{L_1,L_2,L_3\}$ then at most two critical periods bifurcate from the center and there are perturbations with exactly one and exactly two critical periods.*

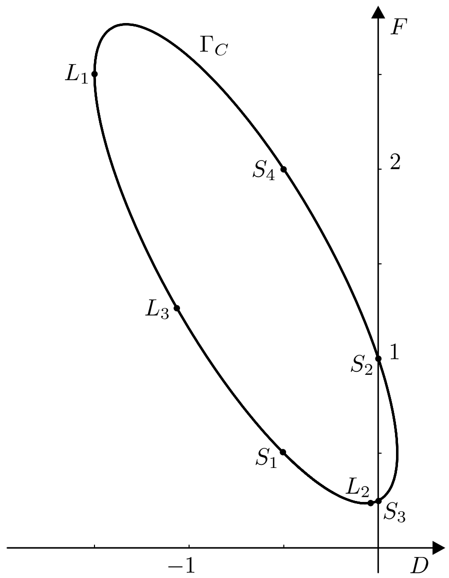

Figure 10: The ellipse $\Gamma_C = \{\mu \in \mathbb{R}^2 : \, P_2(\mu) = 0\}$.

Remark 4.2 It is clear, on account of (37), that if $\mu \notin \Gamma_C$ then the monotonicity of $P(s;\mu)$ for $s > 0$ small enough is given by the sign of $P_2(\mu)$. Note that Γ_C is a Jordan curve. We denote the bounded and unbounded component of $\mathbb{R}^2 \setminus \Gamma_C$ by $\mathcal{D}_C$ and $\mathcal{I}_C$ respectively. One can check then that $P_2(\mu)$ is positive for $\mu \in \mathcal{I}_C$ and negative for $\mu \in \mathcal{D}_C$. Thus, if μ belongs to $\mathcal{I}_C$ (respectively, $\mathcal{D}_C$) then the period function of X_μ is monotonous increasing (respectively, decreasing) at the inner boundary (i.e., the center). Therefore, according to (a) in Remark 2.6, the parameters in Γ_C are local bifurcation values of the period function at the inner boundary. On the other hand, since the expansion in (37) is uniform with respect to μ, from (b) in Remark 2.6 we can assert that any parameter $\mu \notin \Gamma_C$ is a local regular value of the period function at the inner boundary. In short, the regular and bifurcation values at the center are the same with the definition in [4] and Definition 2.4. □

One can easily verify that $\Gamma_C = \{\mu \in \mathbb{R}^2 : P_2(\mu) = 0\}$ and $\{\mu \in \mathbb{R}^2 : P_4(\mu) = 0\}$ are analytic curves that intersect transversally at each L_i. We shall prove next the following result.

Theorem 4.3. *For each $i = 1, 2, 3$ there exist a neighbourhood U_i of L_i and an analytic curve δ_i which is tangent to Γ_C at L_i such that the arc $\delta_i \cap \{\mu \in U_i : P_4(\mu) < 0\}$ corresponds to local bifurcation values of the period function in the interior. Moreover this arc is inside $\{\mu \in \mathbb{R}^2 : P_2(\mu) > 0\}$.*

Proof. To study the period function near the center it is more convenient to parametrize the periodic orbits by means of the first integral $H_\mu(x,y) - H_\mu(0,0)$. (Recall that H_μ is given in section 3.1.) This will eliminate the rather artificial property that only the even coefficients in (37) are significant, which is due to the fact that each periodic orbit intersects twice the x-axis. So for each $h > 0$ denote by $\widehat{P}(h;\mu)$ the period of the periodic orbit of X_μ inside the energy level $\{H_\mu(x,y) - H_\mu(0,0) = h\}$. Thus, since one can verify that

$$h = H_\mu(s,0) - H(0,0) = \frac{s^2}{2} + (1 + D + 2F)\frac{s^3}{3} + (1 + D + F)(1 + 2F)\frac{s^4}{4} + \ldots,$$

from (37) it follows that $\widehat{P}(h;\mu) = 2\pi + Q_0(\mu)h + Q_1(\mu)\frac{h^2}{2} + Q_2(\mu)\frac{h^3}{3} + \dots$ with

$$
\begin{aligned}
Q_0(\mu) &= 2P_2(\mu),\\
Q_1(\mu) &= 8P_4(\mu) - 4(1+D+F)(1+2F)P_2(\mu),\\
Q_2(\mu) &= 24P_6(\mu) \mod\big(P_2(\mu), P_4(\mu)\big).
\end{aligned}
\tag{39}
$$

It is clear moreover that the critical periods coincide with the positive zeros of the function

$$
Z(h;\mu) := \widehat{P}_h(h;\mu) = Q_0(\mu) + Q_1(\mu)h + Q_2(\mu)h^2 + \dots.
\tag{40}
$$

Since $P_2(L_i) = P_4(L_i) = 0$, from (39) and using also the expression for P_6 given in [4], it follows that $Q_2(L_i) = 24P_6(L_i) > 0$ for each $i = 1,2,3$. One can also verify, taking (39) into account again, that the gradients $\nabla Q_0(L_i)$ and $\nabla Q_1(L_i)$ are linearly independent for each $i = 1,2,3$. From now on let us only consider L_1 for the sake of simplicity in the exposition.

By applying the Weierstrass Preparation Theorem, there exist a neighbourhood U of L_1, two analytic functions a_0 and a_1 with $a_i\colon U \longrightarrow \mathbb{R}$ and a positive analytic function $K\colon(-\varepsilon,\varepsilon)\times U \longrightarrow \mathbb{R}$ such that

$$
Z(h;\mu) = K(h;\mu)\big(h^2 + a_1(\mu)h + a_0(\mu)\big).
\tag{41}
$$

Accordingly, if $K(h;\mu) = k_0(\mu) + k_1(\mu)h + \mathrm{o}(h)$ then $k_0(L_1) > 0$ and, from (40),

$$
a_0(\mu)k_0(\mu) = Q_0(\mu) \text{ and } a_0(\mu)k_1(\mu) + a_1(\mu)k_0(\mu) = Q_1(\mu).
\tag{42}
$$

So it turns out that $a_0(L_1) = a_1(L_1) = 0$. Therefore

$$
\nabla Q_0(L_1) = k_0(L_1)\nabla a_0(L_1) \text{ and } \nabla Q_1(L_1) = k_1(L_1)\nabla a_0(L_1) + k_0(L_1)\nabla a_1(L_1),
$$

and we can thus assert that the gradients $\nabla a_0(L_1)$ and $\nabla a_1(L_1)$ are linearly independent. Consequently $\psi(\mu) := \big(a_0(\mu), a_1(\mu)\big)$ is a local diffeomorphism between U and some neighbourhood V of $(0,0)$. We define δ_1 as the preimage by ψ of the analytic curve $\{(a_0,a_1)\in V\colon a_1^2 - 4a_0 = 0\}$. Note in particular that $L_1 = \psi^{-1}(0,0)$ belongs to δ_1. On the other hand, since $\Gamma_C \cap U = \{\mu \in U\colon Q_0(\mu) = 0\}$ is the preimage by ψ of $\{(a_0,a_1)\in V\colon a_0 = 0\}$, we conclude that δ_1 is tangent to Γ_C at L_1.

Now we shall prove, recall Definition 2.4, that any parameter in $\delta_1 \cap \{\mu \in U : P_4(\mu) < 0\}$ is a local bifurcation value of the period function in the interior. To this end fix some $\mu^\star \in \delta_1 \cap U$ with $P_4(\mu^\star) < 0$ and define $h^\star := -\frac{a_1(\mu^\star)}{2}$. Observe that if $\mu \in \delta_1$ then

$$
\begin{aligned}
8P_4(\mu) &= Q_1(\mu) + 2(1+D+F)(1+2F)Q_0(\mu)\\
&= a_1(\mu)k_0(\mu) + a_0(\mu)\big(k_1(\mu) + 2(1+D+F)(1+2F)k_0(\mu)\big)\\
&= a_1(\mu)\Big\{k_0(\mu) + \tfrac{1}{4}a_1(\mu)\big(k_1(\mu) + 2(1+D+F)(1+2F)k_0(\mu)\big)\Big\}.
\end{aligned}
$$

Here we use (39) in the first equality, (42) in the second one and that $\mu \in \delta_1$ in the third one. Therefore, since $k_0(L_1) > 0$ and $a_1(L_1) = 0$, the above equality shows that $P_4(\mu)$ and $a_1(\mu)$ have the same sign for $\mu \approx L_1$. Thus, shrinking the neighbourhood U of L_1 if necessary, we have that $h^\star > 0$. Consider now any neighbourhood $U^\star$ of $\mu^\star$. Due to $\mu^\star \in \delta_1$, there exist $\bar\mu \in U^\star$ such that $a_1^2(\bar\mu) - 4a_0(\bar\mu) > 0$. Hence (41) implies that $Z\big(-a_1(\bar\mu)/2;\bar\mu\big) < 0$. Note also that $Z(h;\mu^\star) > 0$ for $h \neq h^\star$. Therefore, since $-\frac{a_1(\bar\mu)}{2} \longrightarrow h^\star$ as $\bar\mu \longrightarrow \mu^\star$, it turns out that the relation (3) can not be verified in any neighbourhood of $\mu^\star$. So $\mu^\star$ is a local bifurcation value in the interior because Definition 2.4 is not fulfilled for $c = h^\star$. Finally, from (39) and (42) and taking $\mu^\star \in \delta_1$ into account,

$$
2P_2(\mu^\star) = a_0(\mu^\star)k_0(\mu^\star) = a_1(\mu^\star)^2 k_0(\mu^\star)/4
$$

and this implies that $P_2(\mu^\star) > 0$. The proof is completed. ■

Remark 4.4 In fact the preceding proof provides a stronger result than Theorem 4.3. Namely, that for each $i = 1,2,3$ there exists a neighbourhood U_i of L_i and a local analytic equivalence $\psi_i : U_i \longrightarrow V_i$ between the local bifurcation diagrams of the families $\widehat{P}_h(h;\mu)$ with $\mu \in U_i$ and $\mathcal{C}(h;a) := h^2 + a_1 h + a_0$ with $a \in V_i$ for $h \in (0,\varepsilon)$. $\square$

On the other hand it is also possible to obtain the asymptotic expansion of the curves δ_i. To do so it is enough to find the expansion of the functions $a_0(\mu)$ and $a_1(\mu)$ in terms of the coefficients $Q_k(\mu)$ and substitute them into the equation $a_1^2 - 4a_0 = 0$. In this way one can obtain for instance the second order expansion of the curve δ_1 at the point $L_1 = (-3/2, 5/2)$,

$$\tfrac{189}{10}\left(D+\tfrac{3}{2}\right) - \tfrac{63}{2}\left(D+\tfrac{3}{2}\right)^2 - \tfrac{63}{2}\left(F-\tfrac{5}{2}\right)\left(D+\tfrac{3}{2}\right) + \tfrac{17}{5}\left(F-\tfrac{5}{2}\right)^2 + \mathrm{O}\left(\left\|\left(D+\tfrac{3}{2}, F-\tfrac{5}{2}\right)\right\|^3\right) = 0.$$

5 Existence of critical periods

In this section we determine two subsets of the parameter space such that the corresponding period function has at least one critical period and at least two critical periods respectively. We shall use the notation $\mathcal{D}_B$ and $\mathcal{I}_B$, introduced in section 1, for the bounded and unbounded components of $\mathbb{R}^2 \setminus \Gamma_B$, and the notation $\mathcal{D}_C$ and $\mathcal{I}_C$, introduced in Remark 4.2, for the bounded and unbounded components of $\mathbb{R}^2 \setminus \Gamma_C$. Let us note that $\mathcal{D}$ and $\mathcal{I}$ stand for decreasing and increasing respectively.

Theorem 5.1. *Consider a parameter μ_0 inside $\mathcal{I}_C \cap \mathcal{D}_B$ or $\mathcal{D}_C \cap \mathcal{I}_B$. If $\mu_0 \notin \Gamma_U$ then the period function of X_{μ_0} has at least one critical period.*

Proof. Let us prove for instance the assertion concerning $\mathcal{I}_C \cap \mathcal{D}_B$ (the other one follows exactly the same way). So consider some $\mu_0 \in \mathcal{I}_C \cap \mathcal{D}_B \setminus \Gamma_U$ and let $P_{\mu_0} : (0,1) \longrightarrow \mathbb{R}$ be a parametrization of the period function of X_{μ_0}. Then, on account of Remark 4.2, we have that P'_{μ_0} is positive near $s = 0$ because $\mu_0 \in \mathcal{I}_C$. On the other hand, using that $\mu_0 \in \mathcal{D}_B$, by applying Theorem A it follows that P'_{μ_0} is negative near $s = 1$. Therefore, by Bolzano's Theorem, we can assert that there exists $s_0 \in (0,1)$ such that $P'_{\mu_0}(s_0) = 0$. ■

Consider now the subsets U, W, M and N introduced in section 3 and let $\{P_\mu : (0,1) \longrightarrow \mathbb{R},\ \mu \in \mathbb{R}^2\}$ be any parametrization of the period function. It follows then that $L(\mu) := \lim_{s \longrightarrow 1} P_\mu(s)$ is a well defined function on $U \cup W \cup M \cup N \setminus \Gamma_4$, where Γ_4 is the segment represented in Figure 8. Indeed, Theorem 3.6 shows that

$$L(\mu) = \frac{2\sqrt{2}}{\sqrt{a+b+c}} \operatorname{arctanh}\left(\frac{2\,a + b - \sqrt{b^2 - 4\,ac}}{2\sqrt{a(a+b+c)}}\right) \quad \text{if } \mu \in U,$$

Proposition 5.2 in [11] that $L(\mu) = \frac{\pi}{\sqrt{F(D+1)}}$ if $\mu \in W \setminus \Gamma_4$ and Remark 3.16 that $L(\mu) = +\infty$ if $\mu \in M \cup N$. Some easy computations, that are not included here for the sake of brevity, show that the set

$$\{\mu \in U \cup W \setminus \Gamma_4 : L(\mu) - 2\pi = 0\}$$

together with the points $(-3/4, 1)$ and $(-1/2, 1/2)$ and the segment $\{0\} \times [1/4, 1]$ form a Jordan curve, say Γ_0 (see Figure 11). We can thus consider the bounded and unbounded components of $\mathbb{R}^2 \setminus \Gamma_0$, which we denote by $\mathcal{J}_-$ and $\mathcal{J}_+$ respectively. The subscripts are chosen in this way because one can verify that $L(\mu) - 2\pi$ is negative for $\mu \in \mathcal{J}_-$ and positive for $\mu \in \mathcal{J}_+$. With this notation we obtain the following result:

Theorem 5.2. *Consider a parameter μ_0 inside $\mathcal{I}_C \cap \mathcal{I}_B \cap \mathcal{J}_-$ or $\mathcal{D}_C \cap \mathcal{D}_B \cap \mathcal{J}_+$. If $\mu_0 \notin \Gamma_U$ then the period function of X_{μ_0} has at least two critical periods.*

Proof. Let us prove for instance the assertion concerning $\mathcal{I}_C \cap \mathcal{I}_B \cap \mathcal{J}_-$ (the other one follows exactly the same way). Fix $\mu_0 \notin \Gamma_U$ and consider a parametrization $P_{\mu_0} : (0,1) \longrightarrow \mathbb{R}$ of the period function of X_{μ_0}.

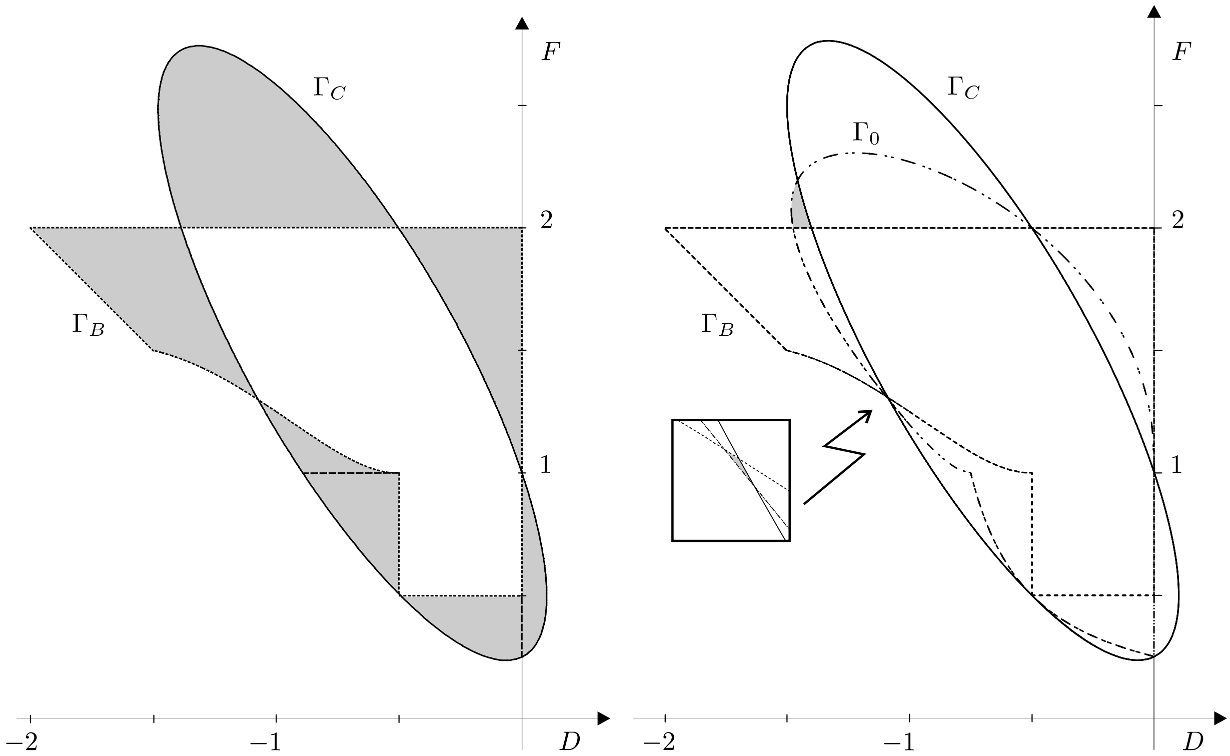

Figure 11: Numerical drawing of the regions in Theorem 5.1 (left) and Theorem 5.2 (right).

Then, using that the eigenvalues of the linear part of X_{μ_0} are $\pm i$, it follows that $\lim_{s\longrightarrow 0} P_{\mu_0}(s) = 2\pi$. We have in addition, recall Remark 4.2, that P_{μ_0} is increasing near $s = 0$ because $\mu_0 \in \mathcal{I}_C$. Then, since $\lim_{s\longrightarrow 1} P_{\mu_0}(s) < 2\pi$ due to $\mu_0 \in \mathcal{J}_-$, there exists $\bar{s} \in (0, 1)$ such that $P_{\mu_0}(\bar{s}) = 2\pi$ and $P'_{\mu_0}(\bar{s}) \leq 0$. Thus, since P'_{μ_0} is positive near $s = 0$ and, on account of $\mu_0 \in \mathcal{I}_B$, also near $s = 1$ by Theorem A, we conclude that there exist $s_1 \in (0, \bar{s})$ and $s_2 \in [\bar{s}, 1)$ such that $P'_{\mu_0}(s_i) = 0$. ■

Unfortunately it is very difficult to provide an explicit and simple analytical description of the regions in Theorems 5.1 and 5.2. We prefer instead to make a numerical drawing of them using the exact analytic expressions that define the curves Γ_B, Γ_C, and Γ_0. The picture on the left in Figure 11 shows several regions, which can be clearly observed, with at least one critical period. The picture on the right shows two regions, one of them very tiny, with at least two critical periods. Both regions corresponds to the set $\mathcal{I}_C \cap \mathcal{I}_B \cap \mathcal{J}_-$, the other one seems to be empty. It is to be referred here the result of Chicone and Dumortier in [3]. They proved that there exists some $D^\star \approx -1.47$ such if $\mu \in (D^\star, -1.4) \times \{2\}$ then the period function of X_μ has at least one critical point. Observe in Figure 11 that this is the horizontal segment in the boundary of the biggest component of $\mathcal{I}_C \cap \mathcal{I}_B \cap \mathcal{J}_-$. Their results follows from the fact that this segment is inside $\mathcal{I}_C \cap \mathcal{J}_-$.

6 Conjectures and open problems

In this section we give the complete conjectural diagram of the period function of the dehomogenized Loud's family. We explain how do we come to this conjecture and comment on various steps that should be done in order to prove it. Some of them seem feasible, while others seem out of reach for the moment.

6.1 Geometrical picture

Consider the parametrization of the set of periodic orbits in the period annulus that provides the first integral H_μ given in (4). Let us assume that we normalize it in order that $h = 0$ corresponds to the center (i.e., the inner boundary) and $h = 1$ to the polycycle (i.e., the outer boundary). Then the parametrization of the period function $P(h;\mu)$ that we obtain is defined on $(0,1)$ for all $\mu \in \mathbb{R}^2$. Recall (see section 4) that it can be extended analytically to $h = 0$ by setting $P(0;\mu) := 2\pi$ and that

$$P'(h;\mu) = Q_0(\mu) + Q_1(\mu)h + Q_2(\mu)h^2 + \dots \quad \text{for } h \approx 0.$$

Let $M \subset [0,1] \times \mathbb{R}^2$ be the set of points (h,μ) verifying $P'(h;\mu) = 0$ for $h \in [0,1)$ and extended to $h = 1$ by continuity using an asymptotic development at the polycycle. We conjecture that M is a smooth surface for $h \neq 1$ and that it is fibred by simple closed curves M_h given by $\{\mu \in \mathbb{R}^2 : P'(h;\mu) = 0\}$ for each fixed $h \in [0,1)$. This last condition is equivalent to require that $\frac{\partial P'(h;\mu)}{\partial D}$ and $\frac{\partial P'(h;\mu)}{\partial F}$ do not vanish simultaneously. Thus M_0 is the ellipse Γ_C in Figure 10 and M_1 should correspond to the local bifurcation values at the outer boundary, that we conjecture to be the curve Γ_B in Theorem A together with the segment $\{0\} \times [0,1/2]$. In fact the strange ellipse-like figure that appears in the numerical bifurcation diagram of Chicone and Jacobs (see Figure 1) would correspond approximately to some curve M_h with $h \approx 1$. The curves that correspond to local bifurcation values of the period function in the interior are obtained by the projection of M on the μ-plane. More precisely, these curves would be given as envelopes of the family $\{M_h,\, h \in [0,1]\}$. We obtained our conjectural bifurcation diagram by trying to interpolate a continuous family of curves M_h starting at Γ_C for $h = 0$ and ending for $h = 1$ at our conjectural bifurcation diagram at the polycycle. Figure 12 shows two intermediate curves M_h and $M_{h'}$ with $0 < h < h' < 1$. Note that every curve M_h must pass through S_1, S_2, S_3 and S_4, the parameters corresponding to the four isochronous centers of the family. On the other hand, according to [4], $Q_0(\mu)$, $Q_1(\mu)$ and $Q_2(\mu)$ generate the ideal of the coefficients of $P'(h;\mu)$ at $h = 0$. This gives that for $\mu \approx S_i$ and $h \approx 0$ the family of curves M_h is approximately of the form of the pencil $Q_0(\mu) + Q_1(\mu)h = 0$. Since the curves $Q_0(\mu) = 0$ and $Q_1(\mu) = 0$ are transverse at S_2, S_3 and S_4, it follows that, in a neighbourhood of these three parameters and at least for h small, the curves M_h look like a pencil of straight lines passing through S_i. At the other isochronous center S_1, since $Q_0(\mu) = 0$ and $Q_1(\mu) = 0$ have quadratic contact, the curves M_h look like a pencil of parabolas tangent to Γ_C. Consider finally the curves δ_i passing through the three weak centers L_i that we obtain in Theorem 4.3. From Remark 4.4 it follows that, in a neighbourhood of each L_i, the curves M_h are tangent to δ_i for $h \approx 0$ and that these curves correspond to parameters in which two critical periods collapse disappearing in the interior. In other words, near each L_i and for $h \approx 0$ the curves δ_i are double bifurcation curves in the interior. We conjecture that this behaviour holds for the entire curve δ_i and that they separate the regions with two critical periods from the region in which the period function is globally monotonous increasing. Since these bifurcation curves δ_i begin at the weak centers $L_i \in \Gamma_C$, which correspond to double bifurcations at the inner boundary, we presume that they end at three special points in the curve $\Gamma_B \cup \big(\{0\} \times [0,1/2]\big)$, which would play the role of double bifurcation parameters at the outer boundary.

Let us precise all this in the following conjecture about the complete bifurcation diagram (see Figure 13) of the period function of the dehomogenized Loud's systems:

Conjecture. *The bifurcation diagram of the period function of the dehomogenized Loud's family consists in the union of the following curves:*

(*a*) *The ellipse* $\Gamma_C = \{\mu \in \mathbb{R}^2 : P_2(\mu) = 0\}$, *which corresponds to the local bifurcation values at the inner boundary.*

(*b*) *The Jordan curve* Γ_B *given in Theorem A together with the segment* $\{0\} \times [0,1/2]$, *which corresponds to the local bifurcation values at the outer boundary.*

(*c*) *Three simple curves* δ_1, δ_2 *and* δ_3 *that connect* L_1 *with* $(-2,2)$, L_2 *with* $(0,0)$ *and* L_3 *with* $(-3/2,3/2)$ *respectively, which correspond to the local bifurcation values in the interior.*

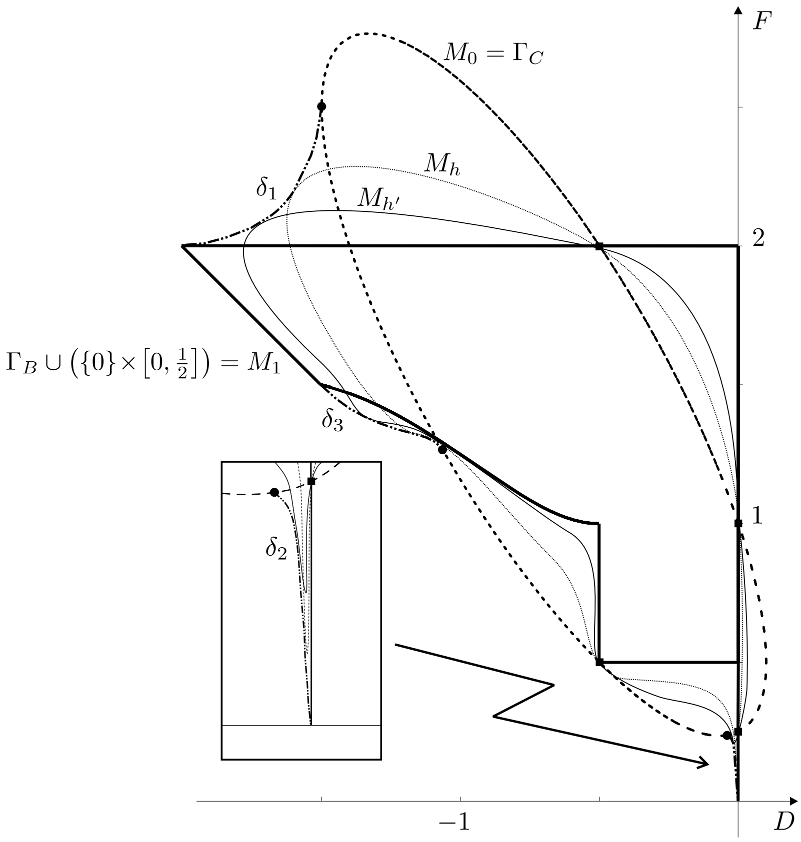

Figure 12: The intermediate curves M_h.

We think of course that the period function of the dehomogenized Loud's systems has at most two critical periods. Figure 13 shows the regions where we conjecture that there are 0, 1 and 2 critical periods. We also sketch there the changes in the monotonicity of the period function. Let us note finally that in the segment $\{0\} \times [0, 1/2]$ occur two different types of bifurcation at the outer boundary. Indeed, crossing from left to right the segment $\{0\}\times [0, 1/4]$ corresponds to the "disappearance" of two critical periods, while crossing $\{0\} \times [1/4, 1/2]$ corresponds to a "rebound" of a critical period.

6.2 Bifurcation at the outer boundary

Our main result determines two sets Γ_B and Γ_U such that the parameters in Γ_B are local bifurcation values of the period function at the outer boundary and the ones in $\mathbb{R}^2 \setminus (\Gamma_B \cup \Gamma_U)$ are local regular values. The character of the parameters in Γ_U remains unspecified in our work. The first natural problem that raises is to determine the character of the parameters in Γ_U. As we already mention, we conjecture that they are all regular values except for the segment $\{0\} \times [0, 1/2]$, whose conjectural bifurcation is described below.

The set Γ_U is stratified as a union of open segments and a few points at the intersection of them. Probably one has to treat first the open segments and next the points (higher codimension strata). Our main tool, Proposition 3.9, does not apply along some curves of Γ_U because the singular points at infinity of the polycycle are saddle-nodes or resonant non-linearizable saddles. It seems reasonable to think that an analogue of Proposition 3.9 can be developed in all these cases and that one could determine the behaviour

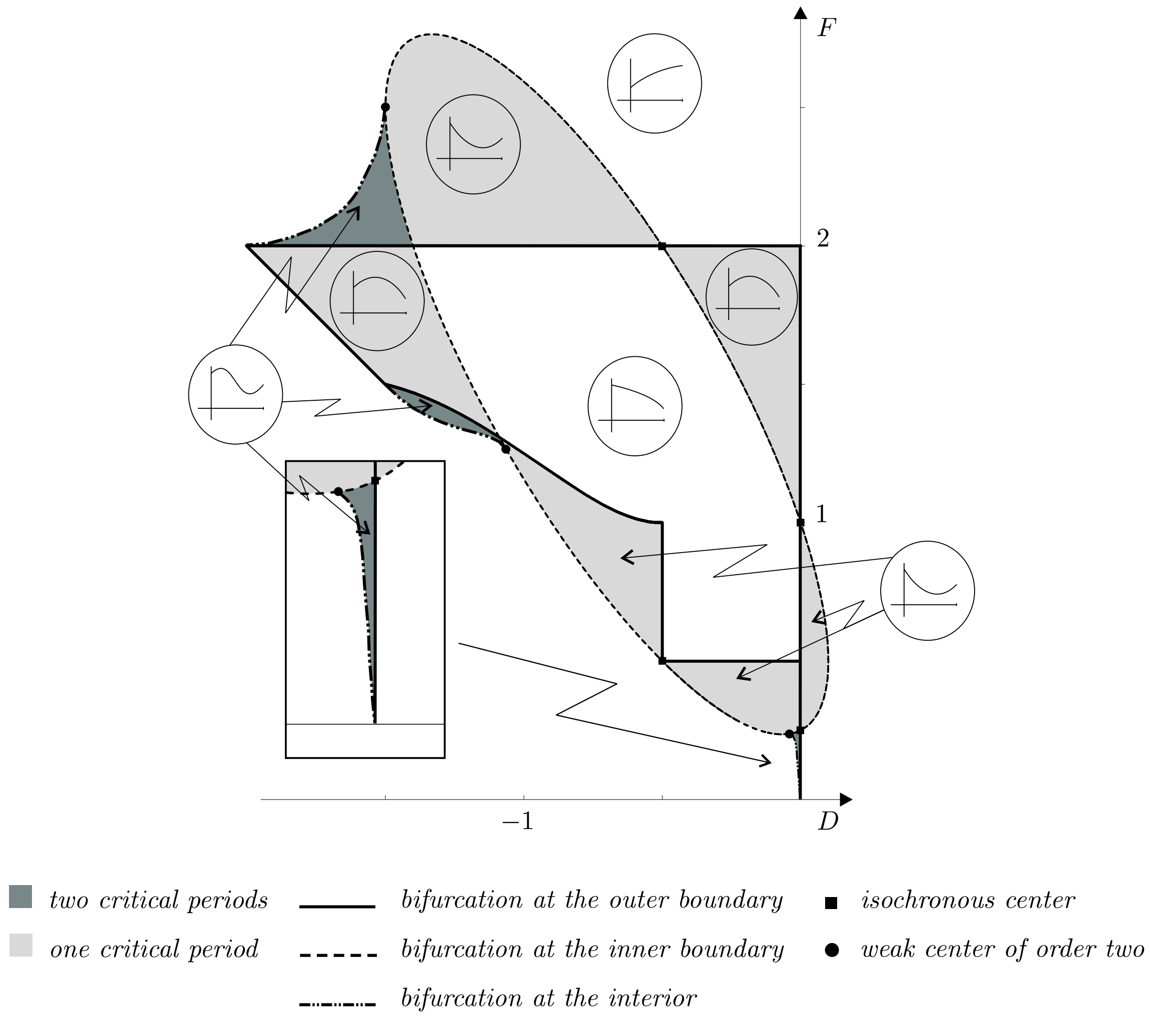

Figure 13: Conjectural bifurcation diagram of the period function.

of the period function at the polycycle in a neighbourhood of these curves. Next, a specific study should be done for each of the codimension two points.

Our study of the bifurcation diagram of the period function does not deal with higher order bifurcation parameters. The so called weak centers of order one and two that appear in the study of the period function near the center (see [4]) have a counterpart near the polycycle. The determination of these parameters and their study requires the knowledge of at least one more coefficient in the asymptotic expansion of the period function near the polycycle. There is no theoretical obstacle in doing so, but the technicalities seem prohibitive. Once these parameters are determined, using the derivation-division process as in the study of Chebyshev systems (see [10] for instance), one should be able to prove the equivalence of the local bifurcation at the outer boundary with some polynomial model.

Let us say a few words about the study at $D = 0$. In this case the polycycle in the boundary of the period annulus has a degenerate singularity at infinity. Blowing-up this singularity and applying a sill unpublished generalization of Proposition 3.9, we hope to obtain the beginning of the asymptotic expansion of the period function $P(h; \mu)$ at the outer boundary. It seems feasible to prove in this way that the segment $\{0\} \times (0, 1/2)$

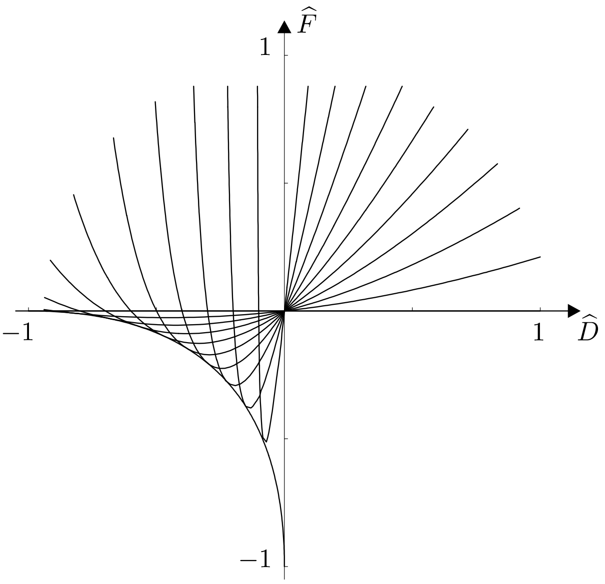

Figure 14: Bifurcation diagram of the conjectured polynomial model near $D = 0$.

consists of local bifurcation values at the outer boundary. On the other hand, we have a polynomial family of functions that we think is a good model for the family $P'(h; D, F)$ near $S_3 = (0, 1/4)$. To be more precise, we conjecture that the families

$$Z(h; D, F) = P'(h; D, F), \qquad h \in [0, 1),\ (D, F) \in \mathbb{R}^2,$$

$$\widehat{Z}(h; \widehat{D}, \widehat{F}) = (\widehat{F} + \widehat{D} - \widehat{F}\widehat{D})\, h^2 + (\widehat{D}\widehat{F} - 2\widehat{F} - \widehat{D} - \widehat{D}^2)\, h + \widehat{F}, \qquad h \in [0, 1],\ (\widehat{D}, \widehat{F}) \in \mathbb{R}^2,$$

have locally equivalent bifurcation diagrams near the points $(0; 0, 1/4)$ and $(0; 0, 0)$ respectively. By this equivalence the curve Γ_C would correspond to $\{\widehat{F} = 0\}$, $\{D = 0\}$ to $\{\widehat{D} = 0\}$ and the curve δ_2 to the curve defined by $\widehat{F}^2 + \widehat{D}^2 - 2\widehat{F}\widehat{D} + 2\widehat{D} + 2\widehat{F} + 1 = 0$. (Note that $\widehat{D}^2\big(\widehat{F}^2 + \widehat{D}^2 - 2\widehat{F}\widehat{D} + 2\widehat{D} + 2\widehat{F} + 1\big)$ is the discriminant of $\widehat{Z}(h; \widehat{D}, \widehat{F})$ with respect to h.) Furthermore the intermediate curves $M_h = \{Z(h; D, F) = 0\}$ would correspond to the hyperbolic branches $\widehat{M_h} = \{\widehat{Z}(h; \widehat{D}, \widehat{F}) = 0\}$. In Figure 14 we show the bifurcation diagram of this polynomial model.

6.3 Bifurcation in the interior

The study of the local bifurcation values of the period function in the interior is equivalent to the study of the behaviour of the zeros of $P'(h; \mu)$ for $h \in (0, 1)$. Here we assume again that h is, up to a normalization, the energy of the first integral H_μ given in (4). In order to study these zeros one hopes to apply methods which proved successful for the abelian integrals. However, in this case the situation is more complicated for two reasons. The first one is that the first integral is not rational but only of Darboux-type. The methods used for abelian integrals have not yet been successfully adapted to this situation despite several efforts [8, 20]. The second reason is that in the usual setting of the abelian integrals, the parameters enter linearly in the study as linear coefficients of the form that one integrates. In our situation the dependence on the parameters is highly nonlinear.

Due to the form of the first integral it can be verified that the complex fibers $\{H_\mu(x, y) = h\}$ given by a fixed (h, μ) are generically of infinite genus. This suggests complicated study unless, for some reason (the symmetry of the system for instance), one can project to some smaller space. The study of asymptotic cycles probably plays some role too.

Let us finally refer a method developed in [6]. The authors obtain a general formula for the derivative of the period function that can be applied to determine the critical periods that persists after the perturbation of an isochronous center. This formula can be viewed as an analogous of the first Melnikov function used to study limit cycles. As an example of application they study the isochronous center $S_2 = (0, 1)$. Proposition 5 in [6] shows that for each closed interval I inside $(0, 1)$ there exists a neighbourhood U of S_2 such that if $\mu_0 \in U$ then $P(h; \mu_0)$ has at most one critical period in I and that this critical period exists only in the case $\frac{1-F_0}{D_0} > 3$. (Observe in Figure 11 that this region is precisely the "linear approximation" at S_2 of the region in Theorem 5.1.) This implies, since one can also verify that the critical period is simple, that in a punctured neighbourhood of S_2 there are no local bifurcation values of the period function in the interior.

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