EXTREMAL SOLUTIONS OF NEVANLINNA-PICK PROBLEMS AND CERTAIN CLASSES OF INNER FUNCTIONS.

NACHO MONREAL GALÁN AND ARTUR NICOLAU

ABSTRACT. Consider a scaled Nevanlinna-Pick interpolation problem and let Π be the Blaschke product whose zeros are the nodes of the problem. It is proved that if Π belongs to a certain class of inner functions, then the extremal solutions of the problem or most of them, are in the same class. Three different classical classes are considered: inner functions whose derivative is in a certain Hardy space, exponential Blaschke products and also the well known class of α-Blaschke products, for $0 < \alpha < 1$.

1. INTRODUCTION.

Let H^∞ be the space of bounded analytic functions in the open unit disc $\mathbb{D}$ of the complex plane, and let $\mathbb{B}$ be the set of functions $f \in H^\infty$ with $\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}\} \leq 1$. Given two sequences $\{z_n\}$ and $\{w_n\}$ in the unit disc $\mathbb{D}$, the Nevanlinna-Pick interpolation problem consists in the following:

$$\text{Find } f \in \mathbb{B} \text{ such that } f(z_n) = w_n \text{ for } n \in \mathbb{N}. \tag{1}$$

Nevanlinna [14] and Pick [17] independently considered this problem. It was proved that there exists a solution if and only if for any integer $N \geq 1$, the matrix

$$\left(\frac{1 - w_i \overline{w}_j}{1 - z_i \overline{z}_j} \right)_{i,j=1,\dots,N}$$

is positive semidefinite. When the problem (1) has more than one solution, Nevanlinna showed that all the solutions can be expressed in the following way:

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$$\{f \in \mathbb{B} \ : \ f \text{ solves } (1)\} = \left\{ \frac{P - Q\varphi}{R - S\varphi} : \varphi \in \mathbb{B} \right\}. \tag{2}$$

This parametrization arose from an iterative argument called Schur's algorithm, which is explained in detail for example in [11, p. 159]. The Nevanlinna coefficients P, Q, R and S are analytic functions in the Smirnov class, depend on the sequences $\{z_n\}$ and $\{w_n\}$ and are uniquely determined by these sequences if one normalizes them so that $S(0) = 0$ and $PS - QR = \Pi$. Here Π denotes the Blaschke product with zeros $\{z_n\}$, that is,

$$\Pi(z) = \prod_n \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z}_n z}.$$

Observe that the Blaschke condition $\sum(1 - |z_n|) < \infty$ holds because (1) has more than one solution. Among the solutions of (1), those of the form

$$I_\gamma(z) = \frac{P(z) - Q(z)e^{i\gamma}}{R(z) - S(z)e^{i\gamma}}, \text{ with } \gamma \in [0, 2\pi),$$

which are called extremal solutions, are of special interest. In [15] Nevanlinna proved that every extremal solution is an inner function. Recall that a function $I \in H^\infty$ is inner if the radial limits $I(e^{i\gamma}) = \lim_{r \to 1} I(re^{i\gamma})$ satisfy $|I(e^{i\gamma})| = 1$ for almost every $\gamma \in [0, 2\pi)$. This was refined by Stray in [19], who proved that I_γ is a Blaschke product for almost every (C) $\gamma \in [0, 2\pi)$, that is, for all $\gamma \in [0, 2\pi)$ except for a set of logarithmic capacity zero. This result is related to a classical theorem by Frotsman (see [11, p. 75]), which says that if I is an inner function, then for a.e.(C) $w \in \mathbb{D}$ the function $(I - w)/(1 - \overline{w}I)$ is a Blaschke product.

Consider a Nevanlinna-Pick interpolation problem (1) with more than one solution, and its corresponding Blaschke product Π. It is natural to expect that if a certain property is held by Π then it is also enjoyed by the corresponding extremal solutions of (1), or by most of them. This question will be considered for certain natural subclasses of inner functions that will be introduced below.

For $0 < \alpha < 1$, let $\mathcal{B}_\alpha$ denote the class of Blaschke products B for which

$$\sum_{z \,:\, B(z) = 0} (1 - |z|)^{1-\alpha} < \infty.$$

Carleson considered this class of Blaschke products in his doctoral thesis [3]. Among many other results he proved the following one (see [3, p. 28]).

Lemma A. *Let B be a Blaschke product. Fix $0 < \alpha < 1$. Then $B \in \mathcal{B}_\alpha$ if and only if*

$$\int_{\mathbb{D}} \frac{\log |B(z)|^{-1}}{(1-|z|)^{1+\alpha}} dA(z) < \infty.$$

Here $dA(z)$ denotes the area measure on $\mathbb{D}$. In the spirit of Frostman's result, Carleson proved also in [3, p. 56] that if $B \in \mathcal{B}_\alpha$, then its Frostman shifts $(B-w)/(1-\overline{w}B)$ are also in $\mathcal{B}_\alpha$ for a.e.(C) $w \in \mathbb{D}$.

Let $H^{1,\alpha}$ denote the class of analytic functions in the unit disc whose derivative is in the Hardy space H^α. Recall that an analytic function f is in H^α if

$$\|f\|_\alpha^\alpha = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^\alpha d\theta < +\infty.$$

When $\alpha \geq 1$, the only inner functions in $H^{1,\alpha}$ are the finite Blaschke products. When $\alpha < 1$, inner functions I in $H^{1,\alpha}$ have been extensively studied (see [2], [7], [8], [9], [10], [18] and [13]). In [18] (see also [2]), it was shown that $\mathcal{B}_\alpha \subset H^{1,\alpha}$ for $\alpha \in (1/2, 1)$. The converse is false, although, fixed $1/2 < \alpha < 1$, Ahern proved in [1] that an inner function I is in $H^{1,\alpha}$ if and only if there exists $w \in \mathbb{D}$ such that the Frostman shift $(I-w)/(1-\overline{w}I)$ is in $\mathcal{B}_\alpha$. There is another characterization of $H^{1,\alpha}$ for $\alpha \in (1/2, 1)$ in terms of Carleson contours given by Cohn in [6]. Carleson contours are described in the following result in [4] or [11, p.333]

Theorem B. *Let $f \in \mathbb{B}$. Given $0 < \varepsilon < 1$ there exists $0 < \delta(\varepsilon) < 1$ and a collection $\{\Gamma_j\}$ of closed curves in $\mathbb{D}$ such that*

- **(i)** $|f(z)| > \varepsilon$ *if* $z \in \mathbb{D} \setminus \bigcup_j \operatorname{Int} \Gamma_j$,
- **(ii)** $|f(z)| < \delta$ *if* $z \in \bigcup_j \operatorname{Int} \Gamma_j$,
- **(iii)** *Arc length on* $\bigcup_j \Gamma_j$ *is a Carleson measure.*

We will refer to $\Gamma = \bigcup_j \Gamma_j$ as ε-Carleson contour of the function f .

Let us consider another class of Blaschke products. A Blaschke product B is said to be exponential if there exists a constant $M = M(B) > 0$ such that for any $j = 1, 2, \ldots$, the annulus $\{z \in \mathbb{D} : 2^{-j-1} < 1 - |z| < 2^{-j}\}$ does not contain more than M zeros of B. Clearly, an exponential Blaschke product is in $\mathcal{B}_\alpha$ for any $0 < \alpha < 1$. Recall that an analytic function in the unit disc f is in the weak Hardy space H^1_w if its non-tangential maximal function is in $L^1_w(\partial\mathbb{D})$, that is, if there exists a constant $C = C(f) > 0$ such that for every $\lambda > 0$ one has

$$\left|\left\{e^{i\gamma} \in \partial\mathbb{D} \,:\, Mf(e^{i\gamma}) > \lambda)\right\}\right| \leq C/\lambda.$$

Here Mf denotes the non-tangential maximal function of f and $|E|$ denotes the length of the set $E \subset \partial\mathbb{D}$. In [5] the following characterization of exponential Blaschke products was given.

Lemma C. *An inner function I is an exponential Blaschke product if and only if I' is in the weak Hardy space H^1_w.*

A similar result holds in the context of weak Besov spaces. See [12].

Returning to the interpolation problem (1), one may ask what properties the extremal solutions will have when Π belongs to one of these classes of functions. The methods used here depend strongly on a property of a subclass of Nevanlinna-Pick problems, called scaled problems. A Nevanlinna-Pick problem with more than one solution is called scaled if there exists a solution f_0 of the problem with $\|f_0\|_\infty < 1$.

Theorem 1. *Let (1) be a scaled Nevanlinna-Pick problem and let Π be the Blaschke product with zeros $\{z_n\}$. Let I_γ, with $\gamma \in [0, 2\pi)$, be the extremal solutions of (1).*

- **(a)** *Fix $0 < \alpha < 1$. Assume that $\Pi \in H^{1,\alpha}$, then $I_\gamma \in H^{1,\alpha}$ for every $\gamma \in [0, 2\pi)$.*
- **(b)** *Assume that Π is an exponential Blaschke product, then I_γ is an exponential Blaschke product for every $\gamma \in [0, 2\pi)$.*
- **(c)** *Assume that*

$$\sum_n (1 - |z_n|)^{1-\alpha} |\log(1 - |z_n|)| < \infty, \tag{3}$$

then $I_\gamma \in \mathcal{B}_\alpha$ for a.e.(C) $\gamma \in [0, 2\pi)$.

The conclusion of part **(c)** may still hold under the weaker assumption $\Pi \in \mathcal{B}_\alpha$, but we do not know how to prove it. It is also worth mentioning that in contrast with the statements in **(a)** and **(b)**, in part **(c)** an exceptional set is needed. Actually, pick a Blaschke product $\Pi \in \mathcal{B}_\alpha$ with zeros $\{z_n\}$ such that for a certain $w \in \mathbb{D}$ one has $(\Pi + w)/(1 + \overline{w}\Pi) \notin \mathcal{B}_\alpha$. Consider the Nevanlinna-Pick problem (1) with $w_n = w$ for any n. For this specific problem one has that

$$\{f \in \mathbb{B} \;:\; f \text{ solves } (1)\} = \left\{\frac{w + \Pi\varphi}{1 + \overline{w}\Pi\varphi} : \varphi \in \mathbb{B}\right\}.$$

Hence not all the extremal solutions $I_\gamma = (w + \Pi e^{i\gamma})/(1 + \overline{w} e^{i\gamma}\Pi)$ can be in $\mathcal{B}_\alpha$ and an exceptional set is needed in the statement **(c)**. A direct consequence of the Theorem 1 is the following:

Corollary 2. *Let Π be a Blaschke product with zeros $\{z_n\}$ and let $\{w_n\}$ be a sequence of complex numbers such that*

$$M = \inf \{\|f\|_\infty \,:\, f \in H^\infty,\, f(z_n) = w_n,\, n = 1, 2, \dots\} < \infty.$$

Fix $\varepsilon > 0$ and $0 < \alpha < 1$.

(a) *If $\Pi \in H^{1,\alpha}$ then there exists a Blaschke product $I \in H^{1,\alpha}$ such that $(M + \varepsilon)I(z_n) = w_n$ for $n = 1, 2, \dots$*

(b) *If Π is an exponential Blaschke product then there exists an exponential Blaschke product I such that $(M + \varepsilon)I(z_n) = w_n$ for $n = 1, 2, \dots$*

(c) *If $\sum_n (1 - |z_n|)^{1-\alpha} |\log(1 - |z_n|)| < \infty$ then there exists $I \in \mathcal{B}_\alpha$ such that $(M + \varepsilon)I(z_n) = w_n$ for $n = 1, 2, \dots$*

Section 2 will be devoted to the proof of Theorem 1.

In the same spirit of Theorem 1, it will be proved in Section 3 that certain ratios of Nevanlinna's coefficients of a scaled problem belong to $H^{1,\alpha}$ (respectively, have derivatives in H^1_w) whenever $\Pi \in H^{1,\alpha}$ (respectively, $\Pi' \in H^1_w$). To this end, let $\mathcal{U}^{1,\alpha}$ (respectively, $\mathcal{U}^1_w$) denote the closed (in H^∞) linear hull of the inner functions in $H^{1,\alpha}$ (respectively, inner functions whose derivative is in H^1_w). The next theorem is analogous to the main result in [16].

Theorem 3. *Let (1) be a scaled Nevanlinna-Pick problem and consider Nevanlinna's parametrization (2). Let Π be the Blaschke product with zeros $\{z_n\}$.*

(a) *Fix $0 < \alpha < 1$. Assume that $\Pi \in H^{1,\alpha}$. Then Q/R is an inner function in $H^{1,\alpha}$ and the functions P/R, S/R and $1/R$ are in $\mathcal{U}^{1,\alpha}$.*

(b) *Assume that Π is an exponential Blaschke product. Then Q/R is an exponential Blaschke product and the functions P/R, S/R and $1/R$ are $\mathcal{U}^1_w$.*

From now on, the letter C will denote a universal constant, while $C(x)$ will denote a constant depending on the parameter x.

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2. Proof of Theorem 1.

Given two sequences $\{z_n\}$ and $\{w_n\}$ in the unit disc, consider the corresponding Nevanlinna-Pick problem (1) and suppose it has more than one solution. Then, if f denotes one of these solutions, according to Nevanlinna's parametrization, f may be expressed as

$$f(z) = \frac{P(z) - Q(z)\varphi(z)}{R(z) - S(z)\varphi(z)} \tag{4}$$

for a certain $\varphi \in \mathbb{B}$. Fix a point $z \in \mathbb{D}$ and consider the set $\Delta(z) = \{f(z) : f \text{ solves } (1)\}$. By Nevanlinna's parametrization one has that

$$\Delta(z) = \left\{ \frac{P(z) - Q(z)w}{R(z) - S(z)w},\ w \in \overline{\mathbb{D}} \right\}.$$

Then $\Delta(z)$ is a disc contained in $\mathbb{D}$, which is called *vertevorrat*, with center $c(z)$ and radius $\rho(z)$ given by

$$c(z) = \frac{P(z)\overline{R(z)} - Q(z)\overline{S(z)}}{|R(z)|^2 - |S(z)|^2}, \qquad \rho(z) = \frac{|\Pi(z)|}{|R(z)|^2 - |S(z)|^2} \tag{5}$$

Observe that $I_\gamma(z) \in \partial\Delta(z)$ for all $\gamma \in [0, 2\pi)$. Nevanlinna also proved that $|R(z)| > \max\{|Q(z)|, |P(z)|, |S(z)|, 1\}$ for any $z \in \mathbb{D}$ and that at almost every point of the unit circle one has $Q = -\Pi\overline{R}$ and $P = -\Pi\overline{S}$. In [20] Stray proved that if the problem is scaled the *vertevorrat*, and also Nevanlinna's coefficients, have certain nice properties.

Lemma D. *Assume that (1) is scaled. Then*

- **(i)** *There exists $\eta > 0$ such that $(1-\eta)|\Pi(z)| \le \rho(z) \le |\Pi(z)|$ for any $z \in \mathbb{D}$.*
- **(ii)** *The radius $\rho(z) \to 1$ as $|\Pi(z)| \to 1$.*
- **(iii)** *There exists $\epsilon > 0$ such that P, Q, R and S are in $H^{2+\epsilon}$.*

Let $f \in \mathbb{B}$ and $e^{i\theta} \in \partial\mathbb{D}$ such that $\lim_{r\to 1} f(re^{i\theta}) \in \partial\mathbb{D}$. If $f'(z)$ has limit as z approaches non-tangentially $e^{i\theta}$, the limit is called the angular derivative of f at $e^{i\theta}$ and it is denoted by $f'(e^{i\theta})$. It is a classical fact that if

$$\liminf_{z \to e^{i\theta}} \frac{1 - |f(z)|}{1 - |z|} < \infty\,,$$

then f has an angular derivative at $e^{i\theta}$ and

$$|f'(e^{i\theta})| = \lim_{r \to 1^-} \frac{1 - |f(re^{i\theta})|}{1 - r}$$

The following Lemma is a basic tool in the proof of Theorem 1.

Lemma 4. *Let B and I be two inner functions, and let Γ be an ε-Carleson contour of B. Suppose that $\inf_{z\in\mathbb{D}\setminus\operatorname{Int}\Gamma}|I(z)| \geq \eta > 0$. Then there exists a constant $C = C(\varepsilon, \eta) > 0$ such that for every $e^{i\theta} \in \partial\mathbb{D}$ for which $\lim_{r\to 1} B(re^{i\theta}) \in \partial\mathbb{D}$ and $B'(e^{i\theta})$ exists, one has that I has angular derivative at $e^{i\theta}$ and $|I'(e^{i\theta})| \leq C|B'(e^{i\theta})|$.*

Proof. Consider the domain $\Omega = \mathbb{D}\setminus\operatorname{Int}\Gamma$, where $\operatorname{Int}\Gamma = \cup\operatorname{Int}\Gamma_j$. The functions $\log|I|^{-1}$ and $\log|B|^{-1}$ are harmonic on Ω and they both vanish on $\partial\mathbb{D}$. By hypothesis there exists a constant $C = C(\varepsilon,\eta) > 0$ such that $\log|I|^{-1} \leq C\log|B|^{-1}$ on Γ. Applying the maximum principle to these functions in the domain Ω one gets

$$\log\frac{1}{|I(z)|} \leq C\log\frac{1}{|B(z)|}$$

when $z \in \Omega$. Then there exists a constant $C_1 > 0$ such that $1 - |I(re^{i\theta})| \leq C_1(1-|B(re^{i\theta})|)$ for all $re^{i\theta} \in \mathbb{D}\setminus\operatorname{Int}\Gamma$. Hence, dividing by $1-r$ on both sides and taking $r \to 1$ the Lemma is proved. □

Now we are ready to prove **(a)** and **(b)** in Theorem 1. Since the Nevanlinna-Pick problem (1) is scaled, one may apply Lemma D to see that there exists $\eta > 0$ such that $\rho(z) \geq 3/4$ when $|\Pi(z)| \geq 1-\eta$. For a fixed $\gamma \in [0, 2\pi)$, let $I_\gamma(z)$ be an extremal solution of (1), then $|I_\gamma(z)| \geq 1/4$ if $|\Pi(z)| \geq 1-\eta$. Considering a $(1-\eta)$-Carleson contour of Π one may apply Lemma 4 to the functions I_γ and Π to conclude that I_γ has angular derivative at almost every point $e^{i\theta} \in \partial\mathbb{D}$ and

$$|I'_\gamma(e^{i\theta})| \leq C|\Pi'(e^{i\theta})|. \tag{6}$$

Now one may prove the first two paragraphs of Theorem 1. In order to prove **(a)**, suppose that $\Pi \in H^{1,\alpha}$. Then clearly (6) implies that $I_\gamma \in H^{1,\alpha}$ for any $\gamma \in [0, 2\pi)$. To prove **(b)**, note that for every $\lambda > 0$ one has that

$$\{e^{i\theta} \in \partial\mathbb{D} : MI'_\gamma(e^{i\theta}) > \lambda\} \subseteq \{e^{i\theta} \in \partial\mathbb{D} : M\Pi'(e^{i\theta}) > \lambda/C\}. \tag{7}$$

Since Π is an exponential Blaschke product, one may apply Lemma C to deduce that $M\Pi' \in L^1_w(\partial\mathbb{D})$ and (7) and Lemma C again, to conclude that I_γ is an exponential Blaschke product for any $\gamma \in [0, 2\pi)$.

In order to prove **(c)** of Theorem 1 one needs an auxiliary result, which is analogous to the description of $\mathcal{B}_\alpha$ given by Carleson that was stated as Lemma A.

Lemma 5. *Let $\{z_n\}$ be a Blaschke sequence in $\mathbb{D}$ and let Π be the Blaschke product with zeros $\{z_n\}$. Fix $0 < \alpha < 1$. The following conditions are equivalent:*

(i) $\sum_n (1-|z_n|)^{1-\alpha} \log(1-|z_n|)^{-1} < \infty$.

(ii) $\int_{\mathbb{D}} \frac{\log|\Pi(z)|^{-1}}{(1-|z|)^{1+\alpha}} \log(1-|z|)^{-1} dA(z) < \infty$.

Proof. Let Π_N be the finite Blaschke product with zeros $\{z_n : n = 1, \dots, N\}$. Then by Green's formula

$$\int_{\mathbb{D}} (1-|z|^2)^{1-\alpha} \log(1-|z|^2) \triangle \log|\Pi_N(z)| dA(z) = \\ = \int_{\mathbb{D}} \triangle \left((1-|z|^2)^{1-\alpha} \log(1-|z|^2) \right) \log|\Pi_N(z)| dA(z).$$

Denoting by δ_{z_n} the Dirac measure at the point z_n, one may check that $\triangle \log|\Pi_N(z)| = C \sum_{n=1}^{N} \delta_{z_n}$, in the sense of distributions. Besides

$$\triangle \left((1-|z|^2)^{1-\alpha} \log(1-|z|^2) \right) = C(\alpha)(1-|z|^2)^{-1-\alpha} \log(1-|z|^2) + \\ + O\left((1-|z|^2)^{-1-\alpha} \right).$$

Hence,

$$C \sum_{n=1}^{N} (1-|z_n|^2)^{1-\alpha} \log(1-|z_n|^2) = \\ = C(\alpha) \int_{\mathbb{D}} \frac{\log|\Pi_N(z)|}{(1-|z|^2)^{1+\alpha}} \log(1-|z|^2) dA(z) + \\ + O(1) \int_{\mathbb{D}} \frac{\log|\Pi_N(z)|}{(1-|z|^2)^{1+\alpha}} dA(z).$$

The proof is finished by letting $N \to \infty$.

□

Return now to the scaled Nevalinna-Pick interpolation problem (1) in order to prove statement **(c)** of Theorem 1. Let I_γ be the corresponding extremal solutions of the problem. We follow an idea of Stray in [19]. Consider the set $E = \{e^{i\gamma} \in \partial\mathbb{D} : I_\gamma \notin \mathcal{B}_\alpha\}$. One wants to see that E has logarithmic capacity zero. So, given a positive measure μ supported on E with bounded logarithmic potential, that is,

$$K = \sup_{z \in \mathbb{C}} \int_E \log|z-w|^{-1} d\mu(w) < \infty,$$

one needs to show that $\mu \equiv 0$. To this end, Lemma A tells that it is enough to see that

$$\int_E \int_{\mathbb{D}} \frac{\log |I_\gamma(z)|^{-1}}{(1-|z|)^{1+\alpha}} dA(z) d\mu(\gamma) < \infty.$$

By Fubini this is equivalent to showing that

$$\int_{\mathbb{D}} \left(\int_E \log |I_\gamma(z)|^{-1} d\mu(\gamma) \right) \frac{dA(z)}{(1-|z|)^{1+\alpha}} < \infty. \tag{8}$$

As in paragraphs **(a)** and **(b)**, by (ii) of Lemma D one can pick $\eta > 0$ small enough such that the $(1-\eta)$-Carleson contour Γ of Π satisfies $\rho(z) > 3/4$ for any $z \in \mathbb{D} \setminus \operatorname{Int}\Gamma$. Hence for every $\gamma \in [0, 2\pi)$ one has that $|I_\gamma(z)| \geq 1/4$ for $z \in \mathbb{D} \setminus \operatorname{Int}\Gamma$. Split the integral in (8) over the disc into two parts, one over the set $\operatorname{Int}\Gamma$ and the other one over $\mathbb{D} \setminus \operatorname{Int}\Gamma$. Since for any $\gamma \in [0, 2\pi]$, the maximum principle gives that $\log |I_\gamma(z)|^{-1} \leq C(\eta) \log |\Pi(z)|^{-1}$ for $z \in \mathbb{D} \setminus \operatorname{Int}\Gamma$, one deduces

$$\int_{\mathbb{D}\setminus \operatorname{Int}\Gamma} \left(\int_E \log |I_\gamma(z)|^{-1} d\mu(\gamma) \right) \frac{dA(z)}{(1-|z|)^{1+\alpha}} \leq$$
$$\leq C(\eta)\mu(E) \int_{\mathbb{D}\setminus \operatorname{Int}\Gamma} \frac{\log |\Pi(z)|^{-1}}{(1-|z|)^{1+\alpha}} dA(z) < \infty$$

by Lemma A. Focused now on the other part of the integral, the fact that the logarithmic potential of μ is bounded yields

$$\int_E \log |I_\gamma(z)|^{-1} d\mu(\gamma) \leq \log \frac{1}{\max\{|(P/R)(z)|, |(Q/R)(z)|\}} + 2K + \log 2.$$

Since $S/R \in \mathbb{B}$ and $(Q/R)(S/R) - P/R = \Pi/R^2$ one has

$$\max\{|(P/R)(z)|, |(Q/R)(z)|\} \geq |\Pi(z)|/2|R(z)|^2$$

Hence

$$\int_{\operatorname{Int}\Gamma} \left(\int_E \log |I_\gamma(z)|^{-1} d\mu(\gamma) \right) \frac{dA(z)}{(1-|z|)^{1+\alpha}} \leq$$
$$\leq \int_{\operatorname{Int}\Gamma} \frac{\log |\Pi(z)|^{-1}}{(1-|z|)^{1+\alpha}} dA(z) + \int_{\operatorname{Int}\Gamma} \frac{\log |R(z)|^2}{(1-|z|)^{1+\alpha}} dA(z) +$$
$$+ \int_{\operatorname{Int}\Gamma} \frac{\log 4 + 2K}{(1-|z|)^{1+\alpha}} dA(z) = (A_1) + (A_2) + (A_3).$$

The first integral (A_1) is finite again by Lemma A. Lemma D implies that $R \in H^2$, and then $\log |R(z)| \leq C(1 + \log(1-|z|)^{-1})$. Hence

$$(A_2) + (A_3) \leq C \int_{\operatorname{Int}\Gamma} \frac{|\log(1-|z|)|}{(1-|z|)^{1+\alpha}} dA(z) + \int_{\operatorname{Int}\Gamma} \frac{C(K)}{(1-|z|)^{1+\alpha}} dA(z).$$

Since Γ is a $(1-\eta)$-Carleson contour of Π, by Lemma B there exists $0<\delta<1$ depending on η such that $|\Pi(z)| \leq \delta$ on $\operatorname{Int}\Gamma$. Then,

$$(A_2)+(A_3) \leq C(\delta)\int_{\operatorname{Int}\Gamma} \frac{\log|\Pi(z)|^{-1}}{(1-|z|)^{1+\alpha}}|\log(1-|z|)|dA(z),$$

which is finite according to Lemma 5. Then the integral (8) is finite, and this finishes the proof of Theorem 1. □

Remark 6. *In the last part of the proof it is shown actually that if the sequence* $\{z_n\}$ *satisfies condition (3) then*

$$\int_{\operatorname{Int}\Gamma} \frac{|\log(1-|z|)|}{(1-|z|)^{1+\alpha}}dA(z) < \infty,$$

where Γ *is a Carleson contour of* Π*. This is a direct consequence of Lemma 5, and it is an analogue to a condition introduced by Cohn in* [6] *for describing inner functions in* $H^{1,\alpha}$ *for* $\alpha \in (1/2,1)$*.*

Remark 7. *One may prove a more general result than* ***(c)*** *of Theorem 1. Let* $h:[0,1]\longrightarrow[0,1]$ *be a twice differentiable nonincreasing concave function such that* $h(1)=0$*. We say that a Blaschke product* B *is in the class* $\mathcal{B}_h$ *if*

$$\sum_{z:B(z)=0} h(|z|^2) < \infty$$

When $h(t)=(1-t)^{1-\alpha}$, $0<\alpha<1$ *the class* $\mathcal{B}_h$ *is* $\mathcal{B}_\alpha$*. Assume that* $\limsup_{r\to1}|h'(r)/h''(r)|<1$*. The proof of Lemma 5 shows that a Blaschke product* B *is in* $\mathcal{B}_h$ *if and only if*

$$\int_{\mathbb{D}} h''(|z|^2)\log|B(z)|^{-1}dA(z) < \infty.$$

Similarly, the zeros of a Blaschke product B *satisfy*

$$\sum_{z:B(z)=0} h(|z|^2)|\log(1-|z|^2)| < \infty$$

if and only if

$$\int_{\mathbb{D}} \log|B(z)|^{-1}h''(|z|^2)|\log(1-|z|^2)|dA(z) < \infty.$$

Now, following the proof of part ***(c)*** *of Theorem 1, one obtains the following result.*

Theorem 8. *Let $h : [0,1] \longrightarrow [0,1]$ be a twice differentiable nonincreasing concave function such that $h(1) = 0$. Assume that*

$$\limsup_{r \to 1} |h'(r)/h''(r)| < 1.$$

Let (1) be a scaled Nevanlinna-Pick problem and let Π be the Blaschke product with zeros $\{z_n\}$. Let I_γ, with $\gamma \in [0, 2\pi)$, be the extremal solutions of (1). Assume that

$$\sum_n h(|z_n|^2) |\log(1 - |z_n|^2)| < \infty \tag{9}$$

then $I_\gamma \in \mathcal{B}_h$ for a.e.(C) $\gamma \in [0, 2\pi)$.

3. Proof of Theorem 3.

Consider now H^∞ as a subalgebra of the space $L^\infty(\partial\mathbb{D})$. Let D_Π be the Douglas algebra generated by H^∞ and the restriction of $\overline{\Pi}$ to $\partial\mathbb{D}$. Let I be an inner function. The property of I being invertible in D_Π means that $|I(\xi_n)| \to 1$ whenever $\{\xi_n\}$ is a sequence satisfying $|\Pi(\xi_n)| \to 1$. Let CDA_Π be the subalgebra of H^∞ generated by all inner functions I invertible in D_Π. Consider again a scaled Nevanlinna-Pick problem (1) and let Π be the Blaschke product with zeros $\{z_n\}$. In [21] it was shown that all the extremal solutions of (1) are invertible in D_Π. Furthermore, recalling Nevalinna's parametrization in (2), in [16] it was proved that the functions P/R, Q/R, S/R and $1/R$ belong also to CDA_Π.

For proving paragraph **(a)** of Theorem 3, suppose that $\Pi \in H^{1,\alpha}$. It is well known that Q/R is an inner function. Observe

$$\frac{(Q/R)(z) - (P/R)(z) \cdot (S/R)(z)}{1 - |(S/R)(z)|^2} - (Q/R)(z) =$$

$$= (S/R)(z) \cdot \frac{(Q/R)(z) \cdot \overline{(S/R)(z)} - (P/R)(z)}{1 - |(S/R)(z)|^2}$$

Since the problem is scaled, Lemma D gives $c(z) \to 0$ and $\rho(z) \to 1$ when $|\Pi(z)| \to 1$. Hence if Γ is an $(1-\varepsilon)$-Carleson contour of Π for sufficiently small $0 < \varepsilon < 1$, then there exists an $0 < \eta < 1$ such that $|Q(z)/R(z)| > 1 - \eta$ when $z \in \mathbb{D} \setminus \operatorname{Int}\Gamma$. One may apply here Lemma 4 to conclude that $Q/R \in H^{1,\alpha}$.

In order to see that the function P/R is in $\mathcal{U}^{1,\alpha}$, fix $z \in \mathbb{D}$ and consider the following integral

$$\frac{1}{2\pi}\int_0^{2\pi} I_\gamma(z) d\gamma = \frac{1}{2\pi}\int_0^{2\pi} \frac{P(z) - Q(z)e^{i\gamma}}{R(z) - S(z)e^{i\gamma}} d\gamma,$$

where I_γ are the extremal solutions of the problem. Consider $I_w(z) = (P(z) - Q(z)w)/(R(z) - S(z)w)$. Since $|(S/R)(z)| < 1$, one has that $I_w(z)$ is analytic in $w \in \overline{\mathbb{D}}$. Then

$$(P/R)(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(P/R)(z) - (Q/R)(z)e^{i\gamma}}{1 - (S/R)(z)e^{i\gamma}} d\gamma.$$

Theorem 1 tells that there exists a constant $C > 0$ such that $\|I'_\gamma\|_\alpha \leq C\|\Pi'\|_\alpha$. In [16] it is shown that Riemann sums of the integral above converge uniformly. Hence $(P/R) \in \mathcal{U}^{1,\alpha}$.

Let $0 < \delta < 1$ be a constant to be fixed later. The function $\tilde{I}_w(z) = (\delta(S/R)(z) + (Q/R)(z)w)/(1 + \delta(P/R)(z)w)$ is analytic in $w \in \overline{\mathbb{D}}$. Consequently

$$\delta(S/R)(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\delta(S/R)(z) + (Q/R)(z)e^{i\gamma}}{1 + \delta(P/R)(z)e^{i\gamma}} d\gamma.$$

Observe that $\tilde{I}_w(z)$ is also analytic and uniformly bounded in $z \in \mathbb{D}$. In order to see that $\tilde{I}_w$ are inner functions in z for any $w \in \partial\mathbb{D}$, recall that $P\overline{R} - Q\overline{S} = 0$ a.e. on $\partial\mathbb{D}$. Then for $|w| = 1$ one has that

$$|\tilde{I}_w(e^{i\theta})| = \left| \frac{\delta(S/R)(e^{i\theta}) + (Q/R)(e^{i\theta})w}{1 + \delta\overline{(S/R)}(e^{i\theta})Q/R(e^{i\theta})w} \right| = 1 \text{ a.e. } \theta \in [0, 2\pi).$$

Take $\varepsilon > 0$ and let Γ be an ε-Carleson contour of (Q/R). Choosing $0 < \delta < \varepsilon$ one has $|\tilde{I}_w(z)| \geq (\varepsilon - \delta)/2$ when $z \in \mathbb{D} \setminus \operatorname{Int}\Gamma$. Then Lemma 4 yields that $\tilde{I}_w \in H^{1,\alpha}$ for all $w \in \partial\mathbb{D}$. As before, this shows that $(S/R) \in \mathcal{U}^{1,\alpha}$.

Consider now the function $(\delta(1/R)(z) + (Q/R)(z)w)/(1 - \delta(\Pi/R)(z)w)$, for a certain $0 < \delta < 1$ to be fixed, which is analytic in $w \in \overline{\mathbb{D}}$. A similar argument shows that for any $w \in \partial\mathbb{D}$, this function is inner in the variable z. This holds because $-(\Pi/R) = (Q/R)(1/\overline{R})$ a.e. on $\partial\mathbb{D}$. Since

$$\delta(1/R)(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\delta(1/R)(z) + (Q/R)(z)e^{i\gamma}}{1 - \delta(\Pi/R)(z)e^{i\gamma}} d\gamma,$$

one may conclude that $(1/R) \in \mathcal{U}^{1,\alpha}$.

The proof of paragraph **(b)** is an easy adaptation of the arguments above, so it will not be included here. □

References

[1] P. R. Ahern. The mean modulus and the derivative of an inner function. *Indiana Univ. Math. J.*, 28(2):311–347, 1979.

[2] P. R. Ahern and D. N. Clark. On inner functions with H^p-derivative. *Michigan Math. J.*, 21:115–127, 1974.

[3] L. Carleson. *On a class of meromorphic functions and its associated exceptional sets*. Uppsala, 1950.

[4] L. Carleson. Interpolations by bounded analytic functions and the corona problem. *Ann. of Math. (2)*, 76:547–559, 1962.

[5] J. A. Cima and A. Nicolau. Inner functions with derivatives in the weak hardy space. *Proc. Amer. Math. Soc.*

[6] W. S. Cohn. On the H^p classes of derivatives of functions orthogonal to invariant subspaces. *Michigan Math. J.*, 30(2):221–229, 1983.

[7] K. M. Dyakonov. Smooth functions in the range of a Hankel operator. *Indiana Univ. Math. J.*, 43(3):805–838, 1994.

[8] K. M. Dyakonov. Embedding theorems for star-invariant subspaces generated by smooth inner functions. *J. Funct. Anal.*, 157(2):588–598, 1998.

[9] K. M. Dyakonov. Self-improving behaviour of inner functions as multipliers. *J. Funct. Anal.*, 240(2):429–444, 2006.

[10] E. Fricain and J. Mashreghi. Integral means of the derivatives of Blaschke products. *Glasg. Math. J.*, 50(2):233–249, 2008.

[11] J. B. Garnett. *Bounded analytic functions*, volume 236 of *Graduate Texts in Mathematics*. Springer, New York, revised first edition, 2007.

[12] J. Gröhn and A. Nicolau. Inner functions in weak Besov spaces. *J. Funct. Anal.*, 266(6):3685–3700, 2014.

[13] J. Mashreghi. *Derivatives of inner functions*, volume 31 of *Fields Institute Monographs*. Springer, Fields Institute for Research in Mathematical Sciences, Toronto, ON, 2013.

[14] R. Nevanlinna. Über beschränkte Funktionen, die in gegebenen Punkten vorgeschrieben Werte annehmen. *Ann. Acad. Sci. Fenn. Ser. A*, 13(1), 1919.

[15] R. Nevanlinna. Über beschränkte analytische Funktionen. *Ann. Acad. Sci. Fenn.*, 32(7), 1929.

[16] A. Nicolau and A. Stray. Nevanlinna's coefficients and Douglas algebras. *Pacific J. Math.*, 172(2):541–552, 1996.

[17] G. Pick. Über die Beschränkungen analytischer Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden. *Math. Ann.*, 77(1):7–23, 1915.

[18] D. Protas. Blaschke products with derivative in H^p and B^p. *Michigan Math. J.*, 20:393–396, 1973.

[19] A. Stray. Minimal interpolation by Blaschke products. II. *Bull. London Math. Soc.*, 20(4):329–332, 1988.

[20] A. Stray. Interpolating sequences and the Nevanlinna Pick problem. *Publ. Mat.*, 35(2):507–516, 1991.

[21] V. Tolokonnikov. Extremal functions of the nevanlinna-pick problem and Douglas algebras. *Studia Math.*, 105(2):151–158, 1993.

Nacho Monreal Galán, Departament of Mathematics, University of Crete, Voutes Campus, 70013 Heraklion, Crete, Greece.
E-mail address: nacho.mgalan@gmail.com

Artur Nicolau, Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain.
E-mail address: artur@mat.uab.cat