SMALLNESS SETS FOR BOUNDED HOLOMORPHIC FUNCTIONS

By

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0 Notation and definitions

Let $\{a_k\}$ be a discrete sequence of points in the unit disk $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$. The following notion was introduced in [Th], motivated by problems of sampling in the usual Hardy spaces H^p, $0 < p < \infty$, and in the space H^∞ of bounded analytic functions in $\mathbb{D}$.

Definition. $\{a_k\}$ is (H^∞)-*thin* if and only if there exists a nonidentically zero function $f \in H^\infty$ such that

$$\sum_k (1 - |a_k|)|f(a_k)| < \infty.$$

A nonthin sequence is said to be *thick*.

A thin sequence is thus one over which the values of a nonzero bounded analytic function may decrease fast enough. This is a weaker analogue of the Blaschke property $\sum(1 - |a_k|) < \infty$, in the sense that any sequence on which some function in H^∞ vanishes is obviously thin. However, the class of thin sequences is much larger. An analogous problem, involving more general function spaces, has been studied by Eiderman [Ei]. On the other hand, Hayman [Ha] has characterized another type of decrease of bounded analytic functions.

Theorem (*Hayman*). *Given a discrete sequence* $\{a_k\}$ *of points in the unit disk, there exists a nonidentically zero function* $f \in H^\infty$ *such that* $\lim_{k\to\infty} f(a_k) = 0$ *if and only if* $|NT(a)| = 0$.

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In the above statement, for any sequence $a = \{a_k\} \subset \mathbb{D}$, $NT(a)$ denotes the ***nontangential accumulation set***, that is to say, the set of points $\zeta \in \partial\mathbb{D}$ which are limit points of the intersection of $\{a_k\}$ with some Stolz angle having vertex at ζ, and $|\cdot|$ is Lebesgue measure on the circle $\partial\mathbb{D}$.

We restrict our attention to sequences $\{a_k\}$ which are separated in the Gleason distance

$$d(z,w) = \left|\frac{z-w}{1-\bar{w}z}\right|, \quad z,w \in \mathbb{D},$$

that is, those for which $\inf_{k\neq j} d(a_k, a_j) > 0$.

Given a point $a \in \mathbb{D}\setminus\{0\}$ and $\gamma > 0$, we denote by $I_\gamma(a)$ the arc of the unit circle centered at $a/|a|$ of length $\gamma(1-|a|)$. The reader may at first disregard the indices γ, since the result, as it turns out, does not depend on them. Given a sequence $\{a_k\}$ of points in the unit disk, we consider the function $\Gamma_\gamma = \Gamma_\gamma(\{a_k\})$ given by

$$\Gamma_\gamma(\xi) = \#\{k\colon \xi \in I_\gamma(a_k)\} = \sum \chi_k(\xi), \quad \xi \in \partial\mathbb{D}, \tag{0.1}$$

where χ_k stands for characteristic function of the arc $I_\gamma(a_k)$. Also, $\Gamma_\gamma(\xi)$ can be viewed as the number of points of the sequence $\{a_k\}$ in the Stolz angle with vertex at $\xi \in \partial\mathbb{D}$ of aperture depending on γ. Observe that the Blaschke condition can be rephrased in terms of Γ_γ because

$$\gamma\sum_k(1-|a_k|) = \int_{\partial\mathbb{D}} \Gamma_\gamma .$$

Also, the nontangential accumulation set $NT(\{a_k\})$ of the sequence $\{a_k\}$ and the set $\{\xi \in \partial\mathbb{D}\colon \Gamma_\gamma(\xi) = \infty\}$ differ only in a set of measure zero. So Hayman's theorem can also be rephrased in terms of Γ_γ. Our first result is also stated in these terms.

Theorem 1. *Let $\{a_k\}$ be a separated sequence of points in the unit disk. Let $\Gamma_\gamma = \Gamma_\gamma(\{a_k\})$ be the function given by* (0.1).

(a) *If there exists $\gamma > 0$ such that $\log_+ \Gamma_\gamma \in L^1(\partial\mathbb{D})$, then $\{a_k\}$ is thin.*

(b) *If $\{a_k\}$ is thin, then $\log_+ \Gamma_\gamma$ is weak L^1, that is, there exists a constant $C = C(\gamma) > 0$ such that*

$$\left|\{\theta \in [0,2\pi) : \log_+ \Gamma_\gamma(e^{i\theta}) \geq \lambda\}\right| \leq C/\lambda, \quad \textit{for all } \lambda > 0,$$

for any $\gamma > 0$.

So, we do not get a complete characterization of the thin separated sequences. However, neither the sufficient condition (a) nor the necessary condition (b) can be improved if we reason only in terms of the size of the function $\Gamma = \Gamma_\gamma$. Furthermore, it turns out that condition (a) is not necessary, nor is (b) sufficient. A precise statement is given in Section 2. The conditions in Theorem 1 are, roughly speaking, a "logarithm away" from the Blaschke condition $\Gamma \in L^1(\partial\mathbb{D})$.

Pushing the parallel between thin sequences and Blaschke sequences a bit further, we recall that, by a celebrated result of L. Carleson [Ca], a sequence $\{a_k\}$ is separated and "invariantly Blaschke", that is to say,

$$\sup \sum_k \left(1 - |\phi(a_k)|^2\right) < \infty,$$

where the supremum is taken over all automorphisms ϕ from the unit disk onto itself, if and only if it is an interpolating sequence, that is, if for any bounded sequence $\{w_k\}$ of complex numbers, there exists $f \in H^\infty(\mathbb{D})$ such that $f(a_k) = w_k$, $k = 1, 2, \ldots$.

Using the identity

$$1 - |\phi_\alpha(a_k)|^2 = \frac{(1 - |a_k|^2)(1 - |\alpha|^2)}{|1 - \bar{a}_k\alpha|^2},$$

where ϕ_α is the automorphism of the unit disk interchanging 0 and $\alpha \in \mathbb{D}$, the geometric fact that $\{a_k\}$ is separated and invariantly Blaschke can be rephrased by saying that

$$\inf_k \prod_{j:j\neq k} |\phi_{a_k}(a_j)| > 0;$$

this is actually the way Carleson's theorem is often stated (see [Ga, pp. 284–287] for an overview of equivalent conditions).

The condition that $\{a_k\}$ be invariantly Blaschke can also be expressed as requiring that the measure

$$\mu = \sum (1 - |a_k|)\delta_k$$

(where δ_k denotes the Dirac mass at the point a_k) be a Carleson measure, that is,

$$\mu(Q) < C\, l(Q)$$

for any Q of the form

$$Q = \{re^{i\theta} : 0 < 1 - r < l(Q), \quad |\theta - \theta_0| < l(Q)\}.$$

An equivalent characterization is that for any $p \in (0, \infty)$,

$$\sum (1 - |a_k|)|f(a_k)|^p = \int_{\mathbb{D}} |f|^p d\mu \leq C(\mu)\|f\|_{H^p}^p, \quad \text{for all } f \in H^p(\mathbb{D});$$

this condition is to be compared with the fact that when the sequence $\{a_k\}$ is thin, *there exists* a function $f \in H^p(\mathbb{D})$ such that $\sum(1 - |a_k|)|f(a_k)|^p < \infty$ (see [Ga, p. 33]).

It is then natural to wonder what happens if we require a sequence of points to be invariantly thin. The expression for $1 - |\phi_\alpha(a_k)|^2$ given above implies that the image under any automorphism of a thin sequence will again be thin. We define "invariantly thin" by the stronger property that there exists a nontrivial bounded analytic function $f \in H^\infty(\mathbb{D})$ such that

$$\sup \sum (1 - |\phi(a_k)|)|f(\phi(a_k))| < \infty,$$

where the supremum is taken over all automorphisms ϕ from the unit disk onto itself.

The somewhat surprising fact is that, whereas invariantly Blaschke sequences have the same quantitative behaviour as Blaschke sequences and are merely more uniformly distributed, and although thin sequences are typically much "bigger" than Blaschke sequences, our strengthening of "thin" to "invariantly thin", in the case of separated sequences, reduces us to the same class as invariantly Blaschke sequences.

Theorem 2. *An invariantly thin separated sequence is an interpolating sequence.*

Lyubarskii and Seip [Lu-Se] say that a nonincreasing function g from $[0, 1)$ to $(0, \infty)$, tending to 0 as x tends to 1, is an *essential minorant* for H^∞ if and only if, given any non-Blaschke separated sequence $\{a_k\} \subset \mathbb{D}$, any $f \in H^\infty$ verifying $|f(a_k)| \leq g(|a_k|)$ must vanish identically. Such g are characterized by the condition

$$\int_0^1 \frac{dr}{(1 - r) \log \frac{1}{g(r)}} < \infty.$$

Similarly, a nonincreasing function g from $[0, 1)$ to $(0, \infty)$ is an *essential minorant on thick sets* for H^∞ if and only if, given any thick separated sequence $\{a_k\} \subset \mathbb{D}$, any $f \in H^\infty$ verifying $|f(a_k)| \leq g(|a_k|)$ must vanish identically. Notice that such a function $g(r)$ must tend to 0 as r tends to 1.

The answer to the question of determining essential minorants on thick sets was given to us by our colleague Alexander Borichev after reading an earlier version of our paper.

Theorem 3 (***Borichev***). *A nonincreasing function* g *from* $[0,1)$ *to* $(0,\infty)$ *is an essential minorant on thick sets if and only if*

$$\liminf_{r\to 1} \frac{\log \frac{1}{g(r)}}{\log|\log \frac{1}{1-r}|} > 0.$$

The paper is organized as follows. In Section 1, some preliminary results are presented. Section 2 contains the notion of generations which arise naturally in our situation and a version of Theorem 1 (and its sharpness) in terms of generations, which is proved in Section 3 and 4. Section 5 is devoted to the proof of Theorem 2. Section 6 concerns essential minorants on thick sets. Section 7 contains some remarks and questions.

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1 Preliminary results

The following lemma shows that the study of thin separated sequences can be reduced to the case of zero-free bounded holomorphic functions, and thus is really a problem about positive harmonic functions in the disk. We denote by P_z the Poisson kernel associated to the point $z \in \mathbb{D}$,

$$P_z(\theta) = \frac{1-|z|^2}{|e^{i\theta}-z|^2}, \quad 0 \le \theta < 2\pi.$$

Lemma 4. *If* $\{a_k\}$ *is thin and separated, then there exists a holomorphic function* g *in* $\mathbb{D}$ *taking values in the right half plane such that*

$$\sum_k (1-|a_k|)|e^{-g(a_k)}| < \infty.$$

Proof. Let $f \in H^\infty(\mathbb{D})$ be such that $\|f\|_\infty \le 1$ and $\sum_k (1-|a_k|)|f(a_k)| < \infty$. Then $f = Bh$, where h is zero-free and B is a Blaschke product,

$$B(z) = z^m \prod_k \frac{|b_k|}{b_k}\frac{b_k - z}{1-\bar{b}_k z},$$

$b_k \in \mathbb{D}$, $\sum_k (1-|b_k|) < \infty$. Let $\delta = \inf\{d(a_k,a_j)\colon k \ne j\} > 0$. First notice that, for each k, there is at most one point a_j such that $d(b_k,a_j) < \delta/2$. Thus for any $f_1 \in H^\infty$,

$$\sum (1-|a_j|)|f_1(a_j)| \le C\|f_1\|_\infty \sum_k (1-|b_k|),$$

where the sum in the left hand side is taken over the indices j satisfying $d(a_j, B^{-1}(0)) < \delta/2$ and C is a constant depending on δ.

Now we define a holomorphic function h_1 in the unit disk such that $|e^{h_1}(z)| \leq |B(z)|$ for any z such that $d(z, B^{-1}(0)) \geq \delta/2$. Let h_1 be the unique holomorphic function such that Im $h_1(0) = 0$ and

$$\text{Re } h_1(z) = -c_0 \int_0^{2\pi} P_z(\theta) \sum_k \chi_{J_k}(\theta) \frac{d\theta}{2\pi},$$

where $J_k := (\arg b_k - (1 - |b_k|), \arg b_k + (1 - |b_k|))$, and c_0 is a constant to be determined. Then, using the estimate $\log x^{-2} \leq C_\delta(1 - x^2)$ if $1 \geq x \geq \delta$, one has

$$\begin{aligned} -\log|B(z)|^2 &= -\sum_k \log\left|\frac{b_k - z}{1 - \bar{b}_k z}\right|^2 \\ &\leq C_\delta \sum_k \frac{(1 - |b_k|^2)(1 - |z|^2)}{|1 - \bar{b}_k z|^2}. \end{aligned}$$

Observe that there exists an absolute constant $c_1 > 0$ such that for any $\theta \in J_k$,

$$|1 - \bar{b}_k z| \geq c_1 |e^{i\theta} - z|.$$

Thus the last sum can be estimated by

$$c_2 C_\delta \sum_k \int_{J_k} \frac{(1 - |z|^2)}{|e^{i\theta} - z|^2} \frac{d\theta}{2\pi} = -\frac{C_\delta c_2}{c_0} \text{Re } h_1(z),$$

where $c_2 := 2\pi/c_1^2$. Therefore, we can pick c_0 sufficiently large to obtain the desired inequality. If we split the sum $\sum(1 - |a_j|)|f_1(a_j)|$ into the cases $d(a_j, B^{-1}(0)) < \delta/2$ and $d(a_j, B^{-1}(0)) \geq \delta/2$, it is straightforward to see that $f_1 := he^{h_1} = e^{-g}$ satisfies our requirements. □

Corollary 5. *Let $\{z_n\}$ be a thin separated sequence. Let $0 < m < 1$ and $\{w_n\} \subset \mathbb{D}$ satisfy $d(z_n, w_n) \leq m$ for all n. Then $\{w_n\}$ is thin.*

Proof. Let g be the function given by Lemma 4. Harnack's inequality implies that

$$\text{Re } g(w_n) \geq c(m) \text{Re } g(z_n), \qquad n = 1, 2, \dots .$$

Since $d(z_n, w_n) \leq m$, we deduce that $(1 - |w_n|) \leq C(m)(1 - |z_n|)$ for all n. Thus

$$\sum_n (1 - |w_n|)|e^{-g(w_n)/c(m)}| \leq C(m) \sum_n (1 - |z_n|)|e^{-g(z_n)}|.$$

□

2 Generations

It will be expedient to carry the problem of describing the thin sequences over to the upper half plane, using the Cayley map

$$\Psi(z) := i\frac{1-z}{1+z}.$$

The Gleason distance in the upper half plane is then given by

$$d(z,w) := \left|\frac{z-w}{z-\bar{w}}\right|, \qquad \text{Im } z > 0, \quad \text{Im } w > 0.$$

Let $\{a_k\}$ be a sequence of points in the unit disk. Since a finite union of thin sequences is thin, it is no loss of generality to assume that $|\arg a_k| \leq \pi/2$. We denote $\Psi(a_k)$ by a'_k for brevity, and notice that the sequence $\{a'_k\}$ is contained in some bounded neighbourhood of the origin in the upper half plane. Furthermore, $\{a_k\}$ is thin if and only if $\{a'_k\}$ is thin on the upper half plane, that is, there exists a bounded holomorphic function f on the upper half plane such that $\sum_k (\text{Im } a'_k)|f(a'_k)| < \infty$. So it is no loss of generality to consider bounded sequences in the upper half plane. When no confusion is possible, we shall write $\{a_k\}$ for $\{a'_k\}$ in order to simplify notation.

We now consider the following standard dyadic partition of the half plane: for $n \geq 0, j \in \mathbb{Z}$,

$$Q_{n,j} := \{z \in \mathbb{C}\colon 2^{-n-1} < \text{Im } z \leq 2^{-n}, \quad j2^{-n} \leq \text{Re } z < (j+1)2^{-n}\}.$$

Notice that $Q_{n,j}$ is the top half of the "Carleson square" with base the projection of $Q_{n,j}$ to the real line, i.e.,

$$I_{n,j} := [j2^{-n}, (j+1)2^{-n}).$$

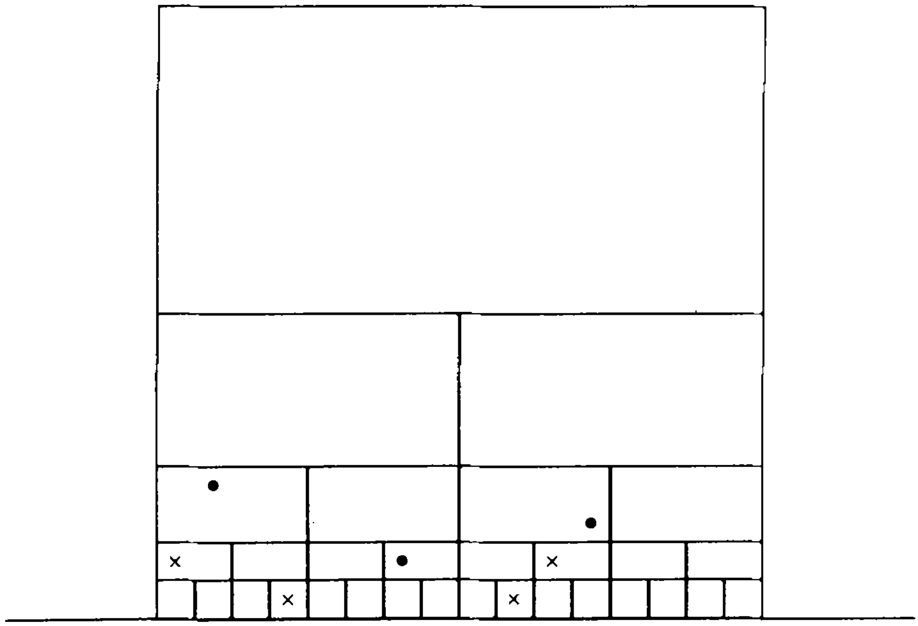
For any given n, the intervals $\{I_{n,j} \colon j \in \mathbb{Z}\}$ form a partition of $\mathbb{R}$.

Since the Gleason distance between the points in $Q_{n,j}$ is uniformly bounded away from 1, a separated sequence admits only a uniformly bounded number of points in each $Q_{n,j}$. Since a finite union of thin sequences is thin, it will be enough to consider sequences in the upper half plane admitting at most one point in each $Q_{n,j}$.

Given such a sequence $\{a_k\}$ in the upper half plane, following [Ga, §7.3, p. 299], we define the *generations* in the following way. Denote by Q_k the unique dyadic box $Q_{n,j}$ such that $a_k \in Q_{n,j}$, and by I_k the corresponding dyadic interval $I_{n,j} \subset \mathbb{R}$. The first generation $\mathcal{G}_1$ is made up of the indices so that the corresponding points of the sequence have no other points of the sequence above them, that is, an index

k is in the first generation if there exists no I_j, $j \neq k$, such that $I_k \subset I_j$. Then we define the second generation $\mathcal{G}_2$ as the first generation of the remainder sequence $\{a_j : j \notin \mathcal{G}_1\}$. The later generations $\mathcal{G}_3, \mathcal{G}_4, \ldots$ are defined recursively by

$$\mathcal{G}_{k+1} = \mathcal{G}_1(\{a_j : j \notin \bigcup_{l=0}^{k} \mathcal{G}_l\}).$$



The points marked with dots correspond to indices in the first generation $\mathcal{G}_1$, while the crosses are indices in the second generation $\mathcal{G}_2$.

Equivalently, $k \in \mathcal{G}_n$ if and only if $n = \#\{k' : I_k \subset I_{k'}\}$. Each given generation thus defines a disjoint family of dyadic intervals on the line, and we write

$$G_n := \bigcup_{k \in \mathcal{G}_n} I_k.$$

Thus

$$|G_n| = \sum_{k \in \mathcal{G}_n} |I_k|.$$

Note that one can also define analogous generations from a partition of the disk, which are not exactly the pull-back under Ψ of the above. Notation turns out to be simpler in the half plane case.

Let $\{a_k\}$ be a bounded and separated sequence in the upper half plane. Then $\{a_k\}$ is a Blaschke sequence, that is,

$$\sum_k \text{Im } a_k < \infty,$$

if and only if $\Gamma_\gamma(\{a_k\}) \in L^1(\mathbb{R})$, where

$$\Gamma_\gamma(\{a_k\})(x) := \#\{a_k : x \in I_\gamma(a_k)\} \quad \text{and} \quad I_\gamma(z) := (x - \gamma y, x + \gamma y).$$

In terms of generations, $\{a_k\}$ is Blaschke if and only if $\sum |G_n| < \infty$.

Also, $|NT\{a_k\}| = 0$ or equivalently $\Gamma_\gamma(\{a_k\})$ is finite almost everywhere, if and only if $|G_n| \to 0$, as $n \to \infty$. Here $NT(\{a_k\}) \subset \mathbb{R}$ is the nontangential accumulation set of the sequence $\{a_k\}$.

Consider

$$\widetilde{\Gamma} = \sum \chi_{G_n}.$$

We have not proved that $\log_+ \Gamma(\{a_k\}) \in L^1(\mathbb{R})$ if and only if $\log_+ \widetilde{\Gamma} \in L^1(\mathbb{R})$. However, the function $\widetilde{\Gamma}$ is a precise enough tool to prove Theorem 1. A summation by parts shows that $\log_+ \widetilde{\Gamma} \in L^1(\mathbb{R})$ if and only if

$$\sum_{n=1}^{\infty} \frac{|G_n|}{n} < \infty.$$

Also, $\log_+ \widetilde{\Gamma}$ is weak L^1 if and only if $|G_n| = |\{\log_+ \widetilde{\Gamma} \geq \log n\}| \leq C/\log n$, for any $n \geq 2$.

Theorem 1 (and its sharpness) will follow from

Theorem 6. (a) *Let $\{a_k\}$ be a separated sequence of points in the upper half plane and let $\{G_n\}$ be the corresponding generations. If $\sum_{n\geq 1}(1/n)|G_n| < \infty$, then $\{a_k\}$ is thin.*

(b) *Given any nonincreasing sequence $\gamma_n > 0$ such that $\sum_{n\geq 1}(\gamma_n/n) = \infty$, there exists a thick separated sequence $\{a_k\}$ in the upper half plane such that $\gamma_n \leq |G_n| \leq \gamma_n + 2^{-n}$, for n sufficiently large. Here $\{G_n\}$ are the generations corresponding to $\{a_k\}$.*

(c) *If $\{a_k\}$ is a bounded sequence which is thin and separated and $\{G_n\}$ its corresponding generations, there exists a constant $C > 0$ such that*

$$|G_n| \leq C/\log n.$$

(d) *There exists a thin separated sequence $\{a_k\}$ such that*

$$C^{-1}/\log n \leq |G_n| \leq C/\log n,$$

where $\{G_n\}$ are the generations corresponding to $\{a_k\}$ and $C > 1$ is a numerical constant.

Thus we have a sufficient condition (a) and a necessary condition (c). Part (b) tells us that the sufficient condition cannot be improved if we reason only in terms of the quantities $|G_n|$. Similarly, part (d) tells us that the necessary condition in (c) cannot be improved using the quantities $|G_n|$. Furthermore, (d) shows that the condition in (a) is not necessary; and (b) applied with $\gamma_n = (\log n)^{-1}$ shows that the one in (c) is not sufficient.

Accepting Theorem 6 momentarily, we can prove the main result.

Proof of Theorem 1. We recall the notation $a'_k = \Psi(a_k)$, where Ψ is the Cayley map from the unit disk to the upper half plane. Also, we recall that we may assume that the sequence $\{a'_k\}$ is bounded, and $\{a_k\}$ is thin if and only if $\{a'_k\}$ is.

For a point $z = x + iy$ in the upper half plane, recall the notation

$$I_\gamma(z) := (x - \gamma y, x + \gamma y).$$

Given any $\gamma > 0$, there exist $0 < \gamma' < \gamma''$ such that, for all points a_k as above, one has

$$I_{\gamma'}(a'_k) \subset \Psi(I_\gamma(a_k)) \subset I_{\gamma''}(a'_k).$$

So, instead of the original function $\Gamma_\gamma(\{a_k\})$, we may consider its analogue defined from the intervals $I_{\gamma'}(a'_k)$, which we denote again by $\Gamma_{\gamma'}(\{a'_k\})$. Since $\{a'_k\}$ is bounded, the function $\Gamma_{\gamma'}(\{a'_k\})$ is supported on a bounded interval of the real line.

(a) For any a_k, by choosing the smallest n such that $2^{-n} < \gamma' \operatorname{Im} a'_k$, we can make sure that there exists a $j = j(k)$ such that $\operatorname{Re} a'_k \in I_{n,j} \subset I_{\gamma'}(a'_k) \subset \Psi(I_\gamma(a_k))$. Set

$$b'_k := (j + \tfrac{1}{2})2^{-n} + 2^{-n}i.$$

It is easy to show $d(b'_k, a'_k) \le m < 1$, where m depends only on γ. Denote by I_k the arc $I_{n,j}$ determined above. Now clearly,

$$\widetilde{\Gamma}(\{b'_k\}) = \sum_k \chi_{I_k} \le \Gamma_{\gamma'}(\{a'_k\}) \le \Gamma_\gamma(\{a_k\}) \circ \Psi^{-1}.$$

Therefore, the hypothesis implies that $\log_+ \widetilde{\Gamma}(\{b'_k\}) \in L^1(\mathbb{R})$; and part (a) of Theorem 6 implies that $\{b'_k\}$ is thin. Since $d(b'_k, a'_k) \le m$, Corollary 5 implies that $\{a_k\}$ is thin.

(b) We now construct a new thin sequence $\{b'_k\}$ such that the corresponding $\widetilde{\Gamma}(\{b'_k\})$ dominates $\Gamma_\gamma(\{a_k\}) \circ \Psi^{-1}$.

First we must reduce ourselves to the case where for any $j \in \mathbb{Z}$, $\{a'_k\}$ has at most one point within the set $\widetilde{Q}_{n,j} := Q_{n,j-1} \cup Q_{n,j} \cup Q_{n,j+1}$. Since there exists $m \in (0,1)$ such that for any $a \in Q_{n,j}$, $\widetilde{Q}_{n,j} \subset \{z \in \mathbb{C} : \text{Im } z > 0, d(z,a) \leq m\}$, this can be achieved by splitting the sequence into a finite union to increase the separation constant. Note that the result we are proving is stable under finite unions, since $\Gamma(\{a_k\} \cup \{b_k\}) = \Gamma(\{a_k\}) + \Gamma(\{b_k\})$.

Now for any $a'_k \in Q_{n,j}$, define $a_k^{(1)}$ to be the "center" of $Q_{n,j-1}$, i.e.,

$$a_k^{(1)} := (j - \tfrac{1}{2})\, 2^{-n} + \tfrac{3}{4}\, 2^{-n} i,$$

and likewise $a_k^{(2)}$ to be the "center" of $Q_{n,j+1}$,

$$a_k^{(2)} := (j + \tfrac{3}{2})\, 2^{-n} + \tfrac{3}{4}\, 2^{-n} i.$$

Denote $\{b'_k\}_{k\geq 1} = \{a'_k\}_{k\geq 1} \cup \{a_k^{(1)}\}_{k\geq 1} \cup \{a_k^{(2)}\}_{k\geq 1}$. The sequence $\{b'_k\}$ is separated and has at most one point in each $Q_{n,j}$. Both sequences $\{a_k^{(1)}\}$ and $\{a_k^{(2)}\}$ are thin by Corollary 5 (since $d(a'_k, a_k^{(i)}) \leq m$, $i = 1, 2$, by the remark above). Therefore, $\{b'_k\}$ is thin and, by part (c) of Theorem 6, $\log_+ \widetilde{\Gamma}(\{b'_k\})$ is in weak L^1.

For any γ sufficiently small, one can choose $\gamma'' < 1$ and thus

$$\Psi(I_\gamma(a_k)) \subset I_{\gamma''}(a_k) = [\text{Re } a'_k - \gamma'' \text{Im } a'_k, \text{Re } a'_k + \gamma'' \text{Im } a'_k)$$
$$\subset I_{n,j-1} \cup I_{n,j} \cup I_{n,j+1}.$$

Therefore,

$$\Gamma_\gamma(\{a_k\}) \circ \Psi^{-1} \leq \Gamma_{\gamma''}(\{b'_k\}) \leq \widetilde{\Gamma}(\{a'_k\}) + \widetilde{\Gamma}(\{a_k^{(1)}\}) + \widetilde{\Gamma}(\{a_k^{(2)}\}) = \widetilde{\Gamma}(\{b'_k\}).$$

To deal with larger values of γ, one would have to add more companion sequences on each side of $\{a_k\}$. Details are left to the reader. □

Finally, to see that Theorem 1 is sharp, observe that the sequences in examples (b) and (d) are chosen so that $\Gamma_{1/2} = \widetilde{\Gamma}$ (and easy modifications of those examples would deal with different apertures).

3 Proof of the Sufficient Condition in Theorem 6

Proof of (a). Consider $\widetilde{\Gamma} = \sum_n \chi_{G_n}$ and the harmonic extension to the half plane of $c_0 \log_+ \widetilde{\Gamma}$,

$$H(z) = c_0 \int_{\mathbb{R}} P_z(t) \log_+ \widetilde{\Gamma}(t)\, dt,$$

where c_0 is a constant to be chosen, and $P_z(t)$ denotes the Poisson kernel for the upper half plane,

$$P_z(t) = \frac{1}{\pi}\frac{y}{(x-t)^2+y^2}, \qquad \text{for } z = x+iy.$$

The hypothesis means that $\log_+ \widetilde{\Gamma} \in L^1(\mathbb{R})$ and ensures that H is well-defined. Let h be the unique holomorphic function in the upper half plane such that Im $h(i) = 0$ and Re $h = -H$, and let $f = e^h$. To estimate $|f(a_k)|$, notice that if $k \in \mathcal{G}_n$, then $\widetilde{\Gamma}(t) \geq n$ for all $t \in I_k := I_{n,j}(a_k)$; for such t, one has $P_{a_k}(t) \geq c/|I_k|$, where c is a numerical constant. So

$$H(a_k) \geq c_0 \int_{I_k} P_{a_k}(t) \log_+ \widetilde{\Gamma}(t)\, dt \geq \log n,$$

if we choose c_0 sufficiently large. Therefore,

$$\sum_k (\text{Im } a_k)|f(a_k)| \leq \sum_n \frac{1}{n} \sum_{k\in\mathcal{G}_n} \text{Im } a_k \leq \sum_n \frac{1}{n}|G_n| < \infty. \qquad \square$$

Proof of (b). Define a sequence in the upper half plane by

$$a_{n,j} := 2^{-n}(j-1/2) + 2^{-n}i, \quad 1 \leq j \leq J_n := [2^n\gamma_n], \quad n = 1, 2, \ldots$$

where, in this proof only, $[x]$ denotes the smallest integer greater than or equal to the real number x. Note that it is no loss of generality to assume that $\gamma_n \leq 1$ for all n.

We have $a_{n,j} \in Q_{n,j}$ and

$$\sum_{j=1}^{j=J_n} |I_{n,j}| = 2^{-n}J_n \leq 2^{-(n-1)}J_{(n-1)},$$

which ensures that the generation $\mathcal{G}_n$ is made up exactly of the indices (n, j), $1 \leq j \leq J_n$, and $|G_n| = 2^{-n}J_n$. Thus the construction implies that $\gamma_n \leq |G_n| \leq \gamma_n + 2^{-n}$.

We shall proceed by contradiction. By Lemma 4, we know that if $\{a_k\}$ were thin, there would exist a positive harmonic function H in the upper half-plane such that

$$\sum_n \sum_{j=1}^{J_n} 2^{-n} e^{-H(a_{n,j})} < \infty.$$

Now H can be written as

$$H(z) = cy + \int_{-\infty}^{\infty} \frac{1}{\pi}\frac{y}{(x-t)^2+y^2}\, d\mu(t), \quad z = x+iy,$$

where c is a positive constant and μ is some positive Borel measure on the real line such that $\int_{\mathbb{R}}(1+t^2)^{-1}d\mu(t) < \infty$ [Ga, p. 18]. Set $d\mu_1(t) = \chi_{[-2,+2]}(t)d\mu(t)$ and

$$H_1(z) := \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{y}{(x-t)^2+y^2}\, d\mu_1(t) .$$

Since $|\operatorname{Re} a_{n,j}| \leq 1$ and $\operatorname{Im} a_{n,j} \leq 1$, we have, for any $t \geq 2$,

$$P_{a_{n,j}}(t) \leq \frac{9}{\pi}\frac{1}{1+t^2};$$

therefore, for all points $a_{n,j}$ in the sequence,

$$|H_1(a_{n,j}) - H(a_{n,j})| \leq |c| + \frac{9}{\pi}\int \frac{1}{1+t^2} d\mu(t) \leq C.$$

So we may replace H by H_1, which will have the same properties with respect to our sequence, that is,

$$\sum_{n\geq 0} g_n < \infty, \quad \text{where } g_n = \sum_{j=1}^{J_n} 2^{-n} e^{-H_1(a_{n,j})}.$$

We may henceforth assume that H is given by the Poisson integral of a measure μ with finite mass supported on a compact interval of the real line.

Claim. There exists a constant $C > 0$ such that, for all integers $n \geq 0$ such that $|G_n| \geq 8 \cdot 2^{-n/2}$ and $g_n \leq (1/n)|G_n|$, one has

$$\int_{|G_n|/2}^{|G_n|} d\mu(t) \geq C|G_n| \log n.$$

Accepting this Claim, to be proved below, we are going to bound $\sum |G_n|/n$, which will finish the proof by contradiction. First we want to see that the indices n that do not satisfy the hypotheses in the claim do not contribute much to the sum. Define the following sets of indices:

$$E = \{n \geq 0 : |G_n| \geq n g_n\}, \quad F = \{n \geq 0 : |G_n| \geq 8 \cdot 2^{-n/2}\}.$$

Then

$$\sum_{n \notin F} |G_n|/n \leq 8 \sum_{n\geq 1} 2^{-n/2}/n \quad \text{and} \quad \sum_{n\notin E} |G_n|/n \leq \sum_{n} g_n ,$$

which converges by assumption. So we only need to bound the term

$$\sum_{n \in E\cap F} |G_n|/n.$$

To do so, we regroup the terms in such a way that the size of generations drops by a factor of at least a half from one group of indices to the next. For a fixed $N > 0$, define by downward induction the indices

$$n_1 := \max E \cap F \cap \{1, \dots, N\},$$
$$n_{k+1} := \max\{n \in E \cap F \cap \{1, \dots, n_k - 1\} : |G_n| > 2|G_{n_k}|\},$$

where the induction stops and we set $n_{k+1} = 0$ when the set over which the "max" is taken becomes empty. Then, the Claim gives

$$\begin{aligned}\sum_{n \in E \cap F} |G_n|/n &= \sum_k \sum_{n_{k+1} < n \le n_k} |G_n|/n \le 2 \sum_k |G_{n_k}| \log n_k \\ &\le C^{-1} \sum_k \int_{|G_{n_k}|/2}^{|G_{n_k}|} d\mu(t) \le C^{-1} \mu(\mathbb{R}) < \infty,\end{aligned}$$

since our definition of the indices n_k's implies that $\frac{1}{2}|G_{n_{k+1}}| > |G_{n_k}|$, so the domains of integration given above are disjoint. □

Proof of the Claim. Roughly speaking, the idea of the proof is that whenever the value of H is large at a given point, there must be enough mass coming from the measure μ "below" the point. Since the total mass of μ is finite, this will put a cap on the number of separated points we can put into a given generation.

We now make this precise. In order to avoid "boundary effects" in the convolution with the Poisson kernel, we want to consider points $a_{n,j}$ the real part of which stays well inside $(\frac{1}{2}|G_n|, |G_n|)$. Let

$$R_n := \{j \ge 0 : 2^{n-1}|G_n| + 2^{n/2} < j < 2^n|G_n| - 2^{n/2}\}.$$

Recall that $J_n = 2^n|G_n|$. The fact that $|G_n| \ge 8 \cdot 2^{-n/2}$ ensures that $\#R_n \ge J_n/4$. For $j \in R_n$, one has

$$\begin{aligned}\int_{|G_n|/2}^{|G_n|} P_{2^{-n}(i+j)}(t)\, d\mu(t) &\ge \int_{j2^{-n}-2^{-n/2}}^{j2^{-n}+2^{-n/2}} P_{2^{-n}(i+j)}(t)\, d\mu(t) \\ &\ge \int_{-\infty}^{\infty} P_{2^{-n}(i+j)}(t)\, d\mu(t) - \frac{1}{\pi}\mu(\mathbb{R}) \qquad (3.1) \\ &= H(2^{-n}(i+j)) - \frac{1}{\pi}\mu(\mathbb{R}).\end{aligned}$$

Here, in the second inequality, we have used the estimate

$$P_{2^{-n}(i+j)}(t) \le \frac{1}{\pi} \qquad \text{for } t \notin [j2^{-n} - 2^{-n/2}, j2^{-n} + 2^{-n/2}].$$

We want to restrict attention to indices corresponding to points where the values of H are large enough, to derive the result from (3.1) by summing over j (i.e., over the points in a single generation). The hypothesis that $|G_n| \geq ng_n$ means that

$$\sum_{j=1}^{J_n} \exp(-H(2^{-n}(i+j))) \leq J_n/n. \tag{3.2}$$

Let

$$S_n := \{j \in R_n : e^{-H(2^{-n}(i+j))} \leq 8/n\}.$$

Chebyshev's inequality applied to (3.2) yields

$$\#(R_n \setminus S_n) \leq \frac{J_n/n}{8/n} = J_n/8,$$

whence $\#S_n \geq J_n/8$. Thus by (3.2), one has

$$\begin{aligned}\left(\log\frac{n}{8} - \frac{1}{\pi}\mu(\mathbb{R})\right)\frac{J_n}{8} &\leq \sum_{j\in S_n}\int_{|G_n|/2}^{|G_n|} P_{2^{-n}(i+j)}(t)\,d\mu(t) \\ &\leq \int_{|G_n|/2}^{|G_n|}\sum_{j\in\mathbf{Z}} P_{2^{-n}(i+j)}(t)\,d\mu(t) \leq C2^n\int_{|G_n|/2}^{|G_n|} d\mu(t),\end{aligned}$$

by an explicit estimation of the last series. Recalling that $|G_n| = 2^{-n}J_n$, we get the desired result. □

4 Proof of the necessary condition in Theorem 6

We actually prove part (c) under the following equivalent, but slightly more cumbersome form.

Claim. If $\{a_k\}$ is a bounded sequence which is thin and separated, $\{G_n\}$ its corresponding generations, and ϕ is any nonincreasing function from $[2,\infty)$ to $(0,\infty)$ such that

$$\int_2^\infty \frac{\phi(x)}{\log x}\,dx < \infty,$$

then $\sum_{n\geq 1}\phi(n)|G_n| < \infty$.

Proof of (c) assuming the Claim. We proceed by contradiction and assume that the sequence of positive numbers $|G_n| =: \gamma_n$ does not satisfy the desired conclusion. It will be enough to show that given a nonincreasing sequence of positive numbers $\{\gamma_n\}$ such that

$$\limsup_{n\to\infty} \gamma_n \log n = \infty,$$

there exists a nonincreasing function ϕ from $[2,\infty)$ to $(0,\infty)$ such that

$$\sum_{n\geq 2}\frac{\phi(n)}{\log n}<\infty \quad\text{and}\quad \sum_{n\geq 2}\phi(n)\gamma_n=\infty .$$

We write $\phi(n)=\sum_{k\geq n}\varepsilon_k$, where $\varepsilon_k\geq 0$ are the terms of a convergent series to be determined. Set

$$L(n):=\sum_{k=2}^{n}\frac{1}{\log k},\quad \Gamma(n):=\sum_{k=2}^{n}\gamma_k .$$

Then summation by parts shows that

$$\sum_{k=2}^{n}\frac{\phi(k)}{\log k}=\sum_{k=2}^{n}\varepsilon_k L(k)+\phi(n+1)L(n),$$

and

$$\sum_{k=2}^{n}\phi(k)\gamma_k=\sum_{k=2}^{n}\varepsilon_k\Gamma(k)+\phi(n+1)\Gamma(n).$$

Since $\{\gamma_n\}$ is nonincreasing, $\Gamma(n)\geq n\gamma_n$; and an elementary argument shows that $L(n)\leq Cn/\log n$. Therefore,

$$\limsup_{n\to\infty}\frac{\Gamma(n)}{L(n)}=\infty .$$

Pick an increasing sequence of integers $\{k_j, j\geq 1\}$ such that $\Gamma(k_j)\geq jL(k_j)$, and let

$$\varepsilon_{k_j}:=\frac{1}{j^2L(k_j)},\quad \varepsilon_k=0 \quad \text{for } k\notin\{k_j, j\geq 1\}.$$

Then the series $\sum\varepsilon_k$ converges,

$$\sum_{2}^{\infty}\phi(k)\gamma_k\geq\sum_{2}^{\infty}\varepsilon_k\Gamma(k)=\infty,$$

and

$$\sum_{k=2}^{n}\frac{\phi(k)}{\log k}\leq\sum_{j:k_j\leq n}\frac{1}{j^2}+L(n)\sum_{j:k_j>n}\frac{1}{j^2L(k_j)}\leq\sum_{1}^{\infty}\frac{1}{j^2}<\infty . \qquad \square$$

Proof of the Claim. Again, we want to regroup generations so that the typical decrease from a $|G_{n_k}|$ to the next is halving. Define by induction the indices

$$n_1:=2,$$
$$n_{k+1}:=\min\{n\colon 2|G_n|\leq|G_{n_k}|\}.$$

So $|G_n| \geq |G_{n_k}|/2$ for $n_k \leq n < n_{k+1}$ and $|G_{n_{k+1}}| \leq |G_{n_k}|/2$. Consider the following sets of indices:

$$E_1 = \{k\colon |G_{n_k}| \leq 1/\log n_{k+1}\},$$
$$E_2 = \{k \notin E_1 \colon n_{k+1} - n_k \leq \sqrt{n_k}\}.$$

They correspond, respectively, to indices where generations are too small to matter and to those where they decrease too fast. We can take care of the part of the sum corresponding to $k \in E_1$ right away:

$$\sum_{k\in E_1} \sum_{n_k}^{n_{k+1}-1} |G_n|\phi(n) \leq \sum_{n\geq 2} \sum_{n_k}^{n_{k+1}-1} \frac{\phi(n)}{\log n_{k+1}} \leq \sum_{n\geq 2} \frac{\phi(n)}{\log n} < +\infty.$$

It takes only a little bit more work to bound the sum for the indices $k \in E_2$, that is, such that $n_{k+1} - n_k \leq \sqrt{n_k}$ and $|G_{n_k}| \geq 1/(\log n_{k+1})$. From the fact that ϕ is nonincreasing and

$$\int_2^\infty \frac{\phi(x)}{\log x} dx < \infty,$$

we can deduce that $\phi(x) \leq C(\log x)/x$ for some constant C if x is sufficiently large. Then if $k \in E_2$, one has

$$\sum_{n_k}^{n_{k+1}-1} |G_n|\phi(n) \leq C|G_{n_k}|\frac{\log n_k}{\sqrt{n_k}}.$$

Now, since $(\log n_{k+1})^{-1} \leq |G_{n_k}| \leq 2^{-k}$ and the function $\log x/\sqrt{x}$ is decreasing for x large, we have

$$\sum_{k\in E_2} |G_{n_k}|\frac{\log n_k}{\sqrt{n_k}} \leq C\sum_{k\geq 1} \exp(-C_1 2^{k-1}) < \infty,$$

where $C_1 > 0$ is an absolute constant.

Now suppose $k \notin E_1 \cup E_2$. By Lemma 4, there exists a positive harmonic function H in the upper half plane such that $\sum_n (\text{Im } a_n)e^{-H(a_n)} < \infty$. Denote

$$g_m := \sum_{n\in \mathcal{G}_m} (\text{Im } a_n)e^{-H(a_n)}.$$

We can assume that

$$\sum_{n_k}^{n_{k+1}} g_m < 1$$

and then there exists $l = l(k) \in [n_k, n_{k+1})$ such that $g_l \le (n_{k+1} - n_k)^{-1}$. Let

$$\mathcal{F}_k := \left\{ n \in \mathcal{G}_l : H(a_n) \ge \log\left(\frac{|G_l|}{2}(n_{k+1} - n_k)\right)\right\}.$$

By Chebyshev's inequality,

$$\sum_{n \in \mathcal{G}_l, n \notin \mathcal{F}_k} \operatorname{Im} a_n \le \frac{(n_{k+1} - n_k)}{2} |G_l| g_l \le \frac{|G_l|}{2}.$$

Thus,

$$\sum_{n \in \mathcal{F}_k} \operatorname{Im} a_n \ge \tfrac{1}{2}|G_l| \ge \tfrac{1}{4}|G_{n_k}|.$$

We define a subsequence of the original sequence first by considering points only within the generations $G_{l(k)}$, then by restricting attention within those to the indices in the sets $\mathcal{F}_k$, and finally by further restricting attention to the sets

$$\mathcal{S}_k := \{n \in \mathcal{F}_k : \sum_{a_j \in Q(a_n)} \operatorname{Im} a_j \le N \operatorname{Im} a_n\},$$

where $N > 0$ is a constant to be chosen below. Here the sum is taken only over points a_j such that $j \in \mathcal{F}_m$, for some $m \ge k$; and

$$Q(a_n) := \{x + iy : 0 < y < \operatorname{Im} a_n, |x - \operatorname{Re} a_n| < \operatorname{Im} a_n\}.$$

This last choice forces the measure

$$\mu := \sum_k \sum_{n \in \mathcal{S}_k} (\operatorname{Im} a_n) \delta_{a_n}$$

to be a Carleson measure [Ga, p. 31].

We need to see that, although we have removed points from the sets $\mathcal{F}_k$ in order to ensure that we obtain a Carleson measure, the points we are left with are numerous enough to account for the whole sequence. The definition of $\mathcal{S}_k$ implies that

$$\sum_{n \in \mathcal{F}_k \setminus \mathcal{S}_k} \operatorname{Im} a_n \le \frac{1}{N} \sum_{n \in \mathcal{F}_k \setminus \mathcal{S}_k} \sum_{a_j \in Q(a_n)} \operatorname{Im} a_j,$$

itself trivially bounded by

$$\frac{1}{N} \sum_{r > k} \sum_{j \in \mathcal{F}_r} \operatorname{Im} a_j.$$

Since $|G_{l(k+2)}| \le |G_{l(k)}|/2$, and $\mathcal{F}_k \subset \mathcal{G}_{l(k)}$, the above sum converges and is bounded by $\frac{C}{N}|G_{l(k)}|$, where C is an absolute constant. Taking $N > 4C$, we have

$$\sum_{n\in\mathcal{S}_k} \text{Im } a_n \ge \sum_{n\in\mathcal{F}_k} \text{Im } a_n - \tfrac{1}{4}|G_{l(k)}| \ge \tfrac{1}{4}|G_{l(k)}|.$$

Thus the sequence $\{a_n : n \in \mathcal{S}_k\}$ satisfies

(i) $\sum_{n\in\mathcal{S}_k} \text{Im } a_n \ge \frac{1}{8}|G_{n_k}|$,

(ii) $H(a_n) \ge \log(|G_l|(n_{k+1} - n_k)/2)$ if $n \in \mathcal{S}_k$.

Since $k \notin E_1 \cup E_2$, then also $n_{k+1} - n_k \ge \sqrt{n_{k+1}}$. Thus if $n \in \mathcal{F}_k$, where k is sufficiently large, we also have

(ii)′ $H(a_n) \ge C' \log n_{k+1}$,

where C' is a numerical constant.

Denote by $\mathcal{M}f(t)$ the nontangential maximal function [Ga, p. 28], that is,

$$\mathcal{M}f(t) := \sup_{z\in\Gamma_\alpha(t)} |f(z)|,$$

where $\Gamma_\alpha(t)$ is the Stolz angle with vertex at t of aperture α. Now we need the following lemma.

Lemma 7. *Let h be a positive harmonic function on the upper half plane and let $\psi : [0, +\infty) \to (0, +\infty)$ be an increasing function such that*

$$\int_1^\infty \frac{\psi'(x)}{x} dx < \infty.$$

Then $\mathcal{M}(\psi \circ h)$ is locally integrable.

Before proving the lemma, we finish the proof of (c). Recall that if μ is a Carleson measure and $\mathcal{M}f \in L^1(\mathbb{R})$, then $f \in L^1(\mu)$ [Ga, p. 32]. Similarly, if μ is a Carleson measure, with compact support and $\mathcal{M}f$ is locally integrable, then $f \in L^1(\mu)$. Thus, by (i), (ii)′ and the fact that μ is a Carleson measure, we have

$$\sum_{k\notin E_1\cup E_2} |G_{n_k}|\psi(\log n_{k+1}) \le 8 \sum_{k\notin E_1\cup E_2} \sum_{n\in\mathcal{S}_k} (\text{Im } a_n)\psi(H(a_n)/C') < \infty$$

for any function ψ satisfying the conditions in Lemma 7. If we take

$$\psi(x) = \int_1^{e^x} \phi(t)dt,$$

for $x \geq 1$, say, and $\psi(x)$ is defined on $[0,1)$ in such a way that ψ is increasing, we deduce

$$\sum_{k \notin E_1 \cup E_2} \sum_{n_k}^{n_{k+1}-1} |G_n| \phi(n) < \infty ;$$

and the proof of (c) is complete. □

Proof of Lemma 7. Let $I \subset \mathbb{R}$ be an interval. We observe that

$$\{t \in I \colon \mathcal{M}(\psi \circ h)(t) > \lambda\} = \{t \in I \colon \mathcal{M}h(t) > \psi^{-1}(\lambda)\}.$$

Thus

$$\int_I \mathcal{M}(\psi \circ h)(t) dt \leq \int_{\psi(1)}^{\infty} |\{t : \mathcal{M}h(t) > \psi^{-1}(\lambda)\}| d\lambda + \psi(1)|I|.$$

Since $\mathcal{M}h$ is weak L^1 [Ga, p. 28], the last integral is bounded by a multiple of

$$\int_{\psi(1)}^{\infty} \frac{d\lambda}{\psi^{-1}(\lambda)} = \int_1^{\infty} \frac{\psi'(t)}{t} dt < \infty;$$

and the lemma is proved. □

Proof of (d). Let

$$I_{n,j} := [(j-1)2^{-n}, j2^{-n}), \quad S_0 := \{a_{n,j}\} := \{2^{-n}(j - \tfrac{1}{2} + i), 1 \leq j \leq 2^n, n \geq 0\},$$

so that Re $a_{n,j}$ is the center of $I_{n,j}$ and Im $a_{n,j} = |I_{n,j}| = 2^{-n}$. Observe that the intervals forming the n-th generation of S_0 are $\{I_{n,j}, 1 \leq j \leq 2^n\}$, which form a dyadic partition of $[0,1)$. Define $n_0 := 1$ and

$$n_k := 2^{2^k}, \quad k = 1, 2, \dots .$$

We define a subsequence S of S_0 together with its generations recursively as follows. The point $a_{0,1} \in S$. This defines G_1. For $n_k < n < n_{k+1}$, $a_{n,j} \in S$ if and only if $I_{n,j} \subset G_{n_k}$. So, for $n_k \leq n < n_{k+1}$, one has $|G_n| = |G_{n_k}|$. On the other hand, the generations drop their total length by a half at the indices n_k. Precisely, if $n = n_{k+1}$, then $a_{n,j} \in S$ if and only if $I_{n,j} \subset G_{n_k}$ and j is even. So $|G_{n_{k+1}}| = |G_{n_k}|/2$, and $G_{n_{k+1}}$ is uniformly distributed in each interval of the previous G_n, that is, for $n_k \leq n < n_{k+1}$,

$$|G_{n_{k+1}} \cap I_{n,j}| = \tfrac{1}{2}|I_{n,j}|.$$

Observe also that

$$|G_{n_k}| = 2^{-k} = \log 2 / \log n_k, \quad k = 1, 2, \dots,$$

and for $n_k \leq n < n_{k+1}$, we have

$$\log 2/\log n \leq |G_n| \leq |G_{n_k}| \leq 2\log 2/\log n\,.$$

Hence, the estimate in the statement is fulfilled.

The fact that the sequence S is thin will be obtained by the following construction.

Claim. There exists a probability measure μ on $[0,1)$ such that for any n and any interval $I_{n,j}$ of G_n one has

$$\mu(I_{n,j}) = \frac{|I_{n,j}|}{|G_n|}.$$

Proof of the Claim. The measure μ is constructed as a weak* limit of the measures μ_n defined below. This is analogous to the classical construction of a measure on the Cantor set: here we concentrate the mass uniformly on the intervals of each generation. This method is a simplified version of the one used to prove the necessity part in Theorem 2 of [Lu-Se]. Set

$$\mu_n := \frac{1}{|G_n|}\sum_{j:(n,j)\in\mathcal{G}_n} \chi_{I_{n,j}} m,$$

where m is the Lebesgue measure and χ_I the characteristic function of the interval I. It is clear that μ_n is a probability measure and $\mu_n(I_{n,j}) = |I_{n,j}|/|G_n|$ for any interval $I_{n,j}$ of G_n. The Claim will follow as soon as we show

$$\mu_n(I_{k,j}) = \frac{|I_{k,j}|}{|G_k|},$$

for any interval $I_{k,j}$ of G_k, $k \leq n$. The above construction implies that G_n is uniformly distributed on any interval of G_k, $k \leq n$; therefore,

$$|G_n \cap I_{k,j}| = \frac{|G_n|}{|G_k|}|I_{k,j}|.$$

So

$$\mu_n(I_{k,j}) = \frac{1}{|G_n|}|G_n \cap I_{k,j}| = \frac{|I_{k,j}|}{|G_k|},$$

and the claim is proved. □

To show that the sequence S is thin, consider a holomorphic function f on the upper half plane such that

$$\log|f(z)| = -c\int_{\mathbb{R}} P_z(t)d\mu(t),$$

where c is a constant to be chosen later.

For $t \in [x-y, x+y]$, we have

$$P_{x+iy}(t) \geq c_0/y\,,$$

where c_0 is an absolute constant. We choose $c = (2\log 2)/c_0$. Then, if $a_{n,j} \in G_n$, applying the above estimate to $x+iy = a_{n,j}$, we have

$$|f(a_{n,j})| \leq \exp\left(-c\frac{c_0}{|I_{n,j}|}\mu(I_{n,j})\right) = 2^{-2/|G_n|}\,.$$

Thus

$$\sum_{(n,j)} (\text{Im } a_{n,j})|f(a_{n,j})| \leq \sum_n |G_n| 2^{-2/|G_n|} = \sum_k |G_{n_k}|(n_{k+1}-n_k)2^{-2/|G_{n_k}|}\,.$$

Since $|G_{n_k}| = 2^{-k}$ and $n_k = 2^{2^k}$, this last sum converges. □

5 Invariantly thin sets

This section is devoted to the proof of Theorem 2. Here, we work in the unit disk, with the generations defined there.

Let $\{a_k\}$ be an invariantly thin separated sequence. Let $f \in H^\infty$, $||f||_\infty < 1$, $f(0) \neq 0$, satisfy

$$\sup \sum (1-|\phi(a_k)|)\,|f(\phi(a_k))| < 1\,,$$

where the supremum is taken over all automorphisms ϕ from the unit disk onto itself. Assume that $\{a_k\}$ is not an interpolating sequence, that is, the measure $\mu = \sum(1-|a_k|)\delta_k$ is not Carleson. Such sequences can also be described in terms of generations (see [Ga, p. 200]).

Given a point a_k of the sequence with $k \in \mathcal{G}_j$, we denote by $\mathcal{G}_1(a_k)$ the indices $l \in \mathcal{G}_{j+1}$ such that $I_l \subset I_k$. Given a point a_l with $l \in \mathcal{G}_1(a_k)$, we consider $\mathcal{G}_1(a_l)$ and define

$$\mathcal{G}_2(a_k) = \bigcup_{l \in \mathcal{G}_1(a_k)} \mathcal{G}_1(a_l)\,.$$

Subsequent generations are defined recursively:

$$\mathcal{G}_{n+1}(a_k) = \bigcup_{l \in \mathcal{G}_n(a_k)} \mathcal{G}_1(a_l)\,.$$

Then, if μ is not a Carleson measure, there exists a positive number $\eta > 0$, a subsequence $\{b_j\}$ of $\{a_k\}$ and a sequence $\{m(j)\}$ of positive numbers, $m(j) \to \infty$ as $j \to \infty$, such that

$$\sum_{a_k \in \mathcal{G}_n(b_j)} (1 - |a_k|) \geq \eta(1 - |b_j|), \quad n = 1, \dots, m(j) \tag{5.1}$$

for any j. In particular, we get

$$\frac{1}{1 - |b_j|} \sum_{a_k \in Q(b_j)} (1 - |a_k|) \to \infty \quad \text{as } j \to \infty .$$

Here $Q(b_j) = \{re^{i\theta} : 0 < 1 - r < 2(1 - |b_j|), |\theta - \arg b_j| < 2(1 - |b_j|)\}$.

Now, considering the automorphism ϕ sending b_j to the origin, we see that for a fixed $\varepsilon > 0$, one has, if j is large enough,

$$\sum_{a_k \in \mathcal{F}} (1 - |a_k|) \geq \frac{1}{2} \sum_{a_k \in Q(b_j)} (1 - |a_k|) .$$

Here $\mathcal{F} = \mathcal{F}(b_j, \varepsilon)$ is the family of points $a_k \in Q(b_j)$, $a_k \neq b_j$, such that $|f(\phi(a_k))| < \varepsilon$. We now consider the subfamily $\mathcal{G} = \mathcal{G}(b_j, \varepsilon)$ of $\mathcal{F}$ formed by those points $a_k \in \mathcal{F}$ for which

$$|f(\phi(z))| < \varepsilon ,$$

for any z in the disc $D_k = \{z \in \mathbb{D}\colon d(z, a_k) < 1/2\}$. We claim that for fixed $\varepsilon > 0$, we have

$$\frac{1}{1 - |b_j|} \sum_{\mathcal{G}} (1 - |a_k|) \underset{j \to \infty}{\longrightarrow} \infty. \tag{5.2}$$

This estimate follows from

$$\frac{1}{1 - |b_j|} \sum_{a_k \in \mathcal{F}} (1 - |a_k|) \underset{j \to \infty}{\longrightarrow} \infty ,$$

the fact that if $a_k \in \mathcal{F} \setminus \mathcal{G}$ and $g = f \circ \phi$, then there exists z_k, $d(z_k, a_k) \leq \frac{1}{2}$ such that

$$|g'(z_k)|(1 - |z_k|) > \varepsilon,$$

and the following lemma, which is a slight variation of a result in [Su].

Lemma 8. *Let f be a function in BMOA, that is, an analytic function f in $\mathbb{D}$ for which*

$$\mu = |f'(w)|^2(1 - |w|)dm(w)$$

is a Carleson measure; and let $||f||_*^2$ *be the infimum of the positive numbers* $C > 0$ *such that* $\mu(Q) \leq C\, l(Q)$, *for any square* Q. *Given* $\eta > 0$, *let* $A = A(f, \eta)$ *be the set of points* $z \in \mathbb{D}$ *for which there exists* w, $d(w, z) \leq 1/2$ *such that*

$$|f'(w)|(1 - |w|) \geq \eta .$$

Then the measure $\sigma = (1 - |z|)^{-1}\chi_A(z)dm(z)$ *is a Carleson measure and, in fact,*

$$\sigma(Q) \leq C\frac{||f||_*^2}{\eta^2} l(Q) ,$$

for any "square" $Q := \{re^{i\theta} : 0 < 1 - r < l(Q), |\theta - \theta_0| < l(Q)\}$. *Here* C *is a numerical constant.*

We accept Lemma 8 (and thus (5.2)) temporarily. Then Lemma 8 and (5.1) also give that for j sufficiently large, there exists $n(j)$, $1 \leq n(j) \leq m(j)$, such that

$$\sum_{\alpha(j)} (1 - |a_k|) \geq \frac{\eta}{2}(1 - |b_j|) ,$$

where $\alpha(j)$ is the set of points $a_k \in \mathcal{G}_{n(j)}(b_j)$ which are also in the family $\mathcal{G}$. Now, denoting by $\mathcal{M} = \mathcal{M}(j)$ the set of indices k corresponding to points $a_k \in \alpha(j)$, we have

$$\omega(b_j, \bigcup_{\mathcal{M}} \partial D_k , \mathbb{D} \setminus \bigcup_{\mathcal{M}} D_k) \geq C(\eta) > 0 ,$$

where $C(\eta)$ is a constant depending only on η. Here $\omega(z, E, \Omega)$, $E \subset \partial\Omega$, denotes the harmonic measure at the point $z \in \Omega$ of the set E in the domain Ω. Since $|f(\phi(z))| < \varepsilon$ for any $z \in D_k$ if $a_k \in \mathcal{G}$ and $\log|f \circ \phi|$ is a negative subharmonic function, one obtains

$$\log|f(\phi(z))| < (\log \varepsilon)\omega(z, \bigcup_{\mathcal{M}} \partial D_k, \mathbb{D} \setminus \bigcup_{\mathcal{M}} D_k)$$

for any $z \in \mathbb{D} \setminus \bigcup_{\mathcal{M}} D_k$. Taking $z = b_j$, we have

$$|f(0)| < \varepsilon^{C(\eta)} ,$$

which leads to a contradiction.

Proof of Lemma 8. For each $z \in A$, consider $D(z) = \{w \in \mathbb{D}\colon d(w, z) \leq 3/4\}$. Observe that for $z \notin A$, one has by subharmonicity

$$\frac{C\eta^2}{(1 - |z|)^2} \leq \frac{1}{m(D(z))} \int_{D(z)} |f'(w)|^2\, dm(w) ,$$

where C is a numerical constant. So, for any $z \in A$, one has

$$C\eta^2(1-|z|) \le \int_{D(z)} |f'(w)|^2(1-|w|)\,dm(w)\,. \tag{5.3}$$

Now we apply the Besicovich covering lemma to obtain a family of discs $D(z_j)$, $z_j \in A$, $A \subset \bigcup_j D(z_j)$ such that

$$\sum_j \chi_{D(z_j)} \le N\,,$$

where N is a fixed constant. Then if Q is a "square", we have

$$\sigma(Q) \le C_1 \sum_{z_j \in 4Q} (1-|z_j|)\,.$$

Let $\mathcal{A}$ be the family of $z_j \in 4Q$. Then, using (5.3), one gets

$$\begin{aligned}\sigma(Q) &\le \frac{C_1}{C\eta^2}\sum_{\mathcal{A}}\int_{D(z_j)} |f'(w)|^2(1-|w|)dm(w)\\ &\le \frac{C_1 N}{C\eta^2}\int_{4Q} |f'(w)|^2(1-|w|)dm(w);\end{aligned}$$

and this finishes the proof. □

6 Essential minorants on thick sets

We begin with an easy observation.

Lemma 9. *Assume that there exists $\alpha > 0$ such that*

$$\int_0^1 \frac{g(r)^\alpha dr}{1-r} < \infty. \tag{6.1}$$

Then g is an essential minorant on thick sets.

Proof of Lemma 9. First notice that if g is an essential minorant on thick sets and $\alpha > 0$, then g^α is too. Indeed, when $\alpha \ge 1$, $g(r)^\alpha \le g(r)$ for r close enough to 1, so the property is immediate. Suppose $\alpha < 1$ and $|f(z)| \le g(|z|)^\alpha$ for all z in some thick set, where $f \in H^\infty$. Then for any integer $m \ge 1/\alpha$, $f^m \in H^\infty$, $|f(z)|^m \le g(|z|)$, so $f^m = 0$; therefore $f = 0$.

This means that we may as well assume

$$\int_0^1 \frac{g(r)dr}{1-r} < \infty.$$

Now let $\{a_k\}$ be a thick separated sequence and $f \in H^\infty$, $|f(a_k)| \le g(|a_k|)$, $k = 1, 2, \ldots$. Then

$$\sum_k (1 - |a_k|)|f(a_k)| \le \sum_k (1 - |a_k|)\, g(|a_k|)\,.$$

Let D_k be the hyperbolic disc of center a_k and radius $\delta > 0$. Taking $\delta > 0$ sufficiently small, we may assume that the discs $\{D_k\}$ are pairwise disjoint. Then the last sum can be bounded by a fixed multiple of

$$\sum_k \int_{D_k} \frac{g(|z|)}{1 - |z|}\, dm(z) \le \int_{\mathbb{D}} \frac{g(|z|)}{1 - |z|}\, dm(z)\,.$$

So the integral condition on g gives that f must vanish identically. This proves the Lemma. □

Proof of Theorem 3. Now given g satisfying the condition in the theorem, it is elementary to see that it satisfies the condition in the Lemma as soon as

$$\frac{1}{\alpha} < \frac{1}{2} \liminf_{r \to 1} \frac{\log \frac{1}{g(r)}}{\log |\log \frac{1}{1-r}|}\,.$$

Therefore, it is an essential minorant on thick sets.

To prove the converse, we work in the upper half plane, with the dyadic partition given in Section 2. Assume that g satisfies

$$\liminf_{r \to 1} \frac{\log \frac{1}{g(r)}}{\log |\log \frac{1}{1-r}|} = 0\,.$$

It will be convenient to assume that the function g measuring the decrease of f depends on $y = \operatorname{Im} z$ rather than being radial, and that it be constant on dyadic cubes. Specifically, define

$$\beta_n := -\inf\{\log g(\Psi^{-1}(x + iy)) : -1 \le x \le 1, 2^{-n-1} < y \le 2^{-n}\},$$

where Ψ is the Cayley map (see Section 2). We then have that β_n increases to ∞ and that for some $C_1 > C_2 > 0$,

$$g(1 - C_1 2^{-n}) \ge e^{-\beta_n} \ge g(1 - C_2 2^{-n})\,;$$

therefore

$$\liminf_{n \to \infty} \frac{\beta_n}{\log n} = 0\,.$$

If we can find f bounded and holomorphic in the upper half plane and a thick sequence of points $a_{n,j}$, where $\operatorname{Im} a_{n,j} = 2^{-n}$, such that $|f(a_{n,j})| \le \exp(-\beta_n)$, then g will not be an essential minorant.

The sequence $a_{n,j}$ will be constructed (and its thickness proved) along the general lines of the proof of part (b) of Theorem 6. We choose an increasing sequence of integers n_k such that

$$n_{k+1} > n_k^3 \quad \text{and} \quad \frac{\beta_{n_k}}{\log n_k} \le 2^{-k} .$$

Define the lengths l_k as multiples of $2^{-[\sqrt{n_k}]-1}$ which are close to $(\log n_k)^{-1}$, that is,

$$l_k := 2^{-[\sqrt{n_k}]-1} \left[\frac{2^{[\sqrt{n_k}]+1}}{\log n_k} \right] .$$

Here $[\cdot]$ stands for the integer part of a real number. Notice that, for k large enough, $0 < l_{k+1} < l_k/2$. Set

$$a_{n,j} = 2^{-n} j + 2^{-n} i, \quad 0 \le j < 2^n l_k, \quad \text{for } n \in \mathcal{J}_k := \{[\sqrt{n_k}] + 1, \dots, n_k\} ;$$

this guarantees that the union of the dyadic intervals $I_{n,j}$ corresponding to a single "level" $n \in \mathcal{J}_k$ will be the interval $[0, l_k)$. Note that here the points at the level n, i.e., having imaginary part equal to 2^{-n}, do not constitute the n-th generation, because of the gaps we have introduced. So the proof of Theorem 6(b) must be adapted to show that the sequence is thick.

We now choose a holomorphic function f which will be bounded in modulus by $g \circ \Psi^{-1}$ on the sequence $a_{n,j}$ by requiring $f(i) > 0$ and $|f| = e^{-H}$, where H is the Poisson integral of the boundary values

$$H^*(x) := c_0 \sum_k \beta_{n_k} \chi_{(l_{k+1}, l_k]}(x) ,$$

for $c_0 > 0$ some constant to be chosen. The function H^* is integrable on the real line because

$$\sum_k \beta_{n_k}(l_k - l_{k+1}) \le \sum_k \beta_{n_k} l_k \le \sum_k \frac{\beta_{n_k}}{\log n_k} < \infty ,$$

by the choice of n_k. Since $H^*(x) \ge c_0 \beta_{n_k}$ for $x \in (0, l_k]$, and for $n \in \mathcal{J}_k$ we have $\operatorname{Im} a_{n,j} = 2^{-n} \le l_k$, an easy harmonic measure estimate yields, for $0 \le j < 2^n l_k$,

$$H(a_{n,j}) \ge c c_0 \beta_{n_k} > \beta_{n_k} \ge \beta_n ,$$

for c_0 well chosen. Therefore $|f(a_{n,j})| \le e^{-\beta_n}$, as required.

To prove that $\{a_{n,j}\}$ is thick, as in the proof of Theorem 6(b), it will be enough to show that there is no finite positive measure μ on the real line such that if

$$S(z) := \int_{\mathbb{R}} P_z(t)d\mu(t)\,,$$

then

$$\sum_n 2^{-n} \sum_j \exp\left(-S(a_{n,j})\right) < \infty\,.$$

Suppose S satisfies the last two properties. For $n \in \mathcal{J}_k$, set

$$\sigma_n := 2^{-n} \sum_{j=0}^{2^n l_k - 1} \exp\left(-S(a_{n,j})\right)\,.$$

Consider the set of indices

$$K := \{k : \sigma_n \geq l_k/n \quad \forall n \in \mathcal{J}_k\}\,.$$

Claim. There exists a constant $C > 0$ such that, for all large enough $k \notin K$,

$$\int_{l_k/2}^{l_k} d\mu(t) \geq C l_k \log n_k.$$

Accepting the claim, we can finish the proof. Indeed, by our assumptions and the definition of K,

$$\infty > \sum_n \sigma_n \geq \sum_{k \in K} \sum_{n \in \mathcal{J}_k} \frac{l_k}{n} \geq c \sum_{k \in K} l_k \log n_k,$$

where c is a numerical constant. Since the series $\sum_k l_k \log n_k$ diverges, we must have

$$\infty = C \sum_{k \notin K} l_k \log n_k \leq \int_0^1 d\mu(t)\,,$$

a contradiction. □

Proof of the Claim. The fact that $k \notin K$ means that we can fix some $n \in \mathcal{J}_k$ such that $\sigma_n < l_k/n$. Let

$$S_n := \{j \in \{0, \ldots, 2^n l_k - 1\} : \exp\left(-S(a_{n,j})\right) \leq 4/n\}\,.$$

Then by Chebyshev's inequality,

$$\#\left(\{0,\ldots,2^n l_k - 1\} \setminus S_n\right) \le \frac{l_k/n}{2^{-n}4/n} = \frac{2^n l_k}{4};$$

thus $\# S_n \ge (3/4)2^n l_k$. Furthermore, for k large enough, and any $n \in \mathcal{J}_k$,

$$l_k \ge 8 \cdot 2^{-[\sqrt{n_k}]/2} \ge 8 \cdot 2^{-n/2}.$$

Therefore, the "boundary effects" do not matter much:

$$\# S_n' := \#\left(S_n \cap [\tfrac{1}{2}2^n l_k; 2^n l_k - 2^{n/2}]\right) \ge \tfrac{1}{8}2^n l_k .$$

We now finish the proof of the claim as for its counterpart in the proof of Theorem 6(b): for $j \in S_n'$,

$$\int_{l_k/2}^{l_k} P_{2^{-n}(i+j)}(t)\, d\mu(t) \ge \int_{j2^{-n}-2^{-n/2}}^{j2^{-n}+2^{-n/2}} P_{2^{-n}(i+j)}(t)\, d\mu(t) \ge S(2^{-n}(i+j)) - \frac{1}{\pi}\mu(\mathbb{R}).$$

Summing over $j \in S_n'$, and recalling that for $j \in S_n$, $S(2^{-n}(i+j)) \ge \log(n/4)$, we get

$$C2^n \int_{l_k/2}^{l_k} d\mu(t) \ge \frac{1}{8}2^n l_k \left(\log\frac{n}{4} - \frac{1}{\pi}\mu(\mathbb{R})\right).$$

Thus

$$\int_{l_k/2}^{l_k} d\mu(t) \ge C_1 l_k \log n \ge C_1 \frac{1}{2} l_k \log n_k ,$$

where C_1 is a numerical constant. □

7 Remarks and questions

The main open problem that remains is to obtain a geometrical characterization of separated thin sequences. Theorem 6 tells that such a description cannot be written in terms of the length of generations. However, our arguments, applied to the following subalgebra of H^∞, give a satisfactory result.

Let A be the set of bounded analytic functions in $\mathbb{D}$ which can be written as $f = Bh$, where B is a Blaschke product, h has no zeros and $\mathcal{M}(\log h) \in L^1(\partial\mathbb{D})$, where $\mathcal{M}$ is the nontangential maximal function.

Theorem 10. *Let $\{a_k\}$ be a separated sequence of points in the unit disk and let $\{G_n\}$ be the corresponding generations. Then there exists a nonidentically zero function $f \in A$ such that*

$$\sum_{k=1}^{\infty}(1-|a_k|)|f(a_k)| < \infty$$

if and only if

$$\sum_{n=1}^{\infty} \frac{|G_n|}{n} < \infty .$$

The sufficiency follows from (a) of Theorem 6, and a variation of the proof in (c) gives the necessity.

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