Radial Behaviour of Harmonic Bloch Functions and Their Area Function

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ABSTRACT. Let u be a harmonic function in the upper half space $\mathbb{R}^{n+1}_+$ and $A(u)$ its (truncated) area function. Classical results of Calderón, Stein and Zygmund assert that the following two sets $\{x \in \mathbb{R}^n : u \text{ has non-tangential limit at } x\}$, $\{x \in \mathbb{R}^n : A(u)(x) < \infty\}$ can only differ in a set of zero Lebesgue measure. When these sets have zero Lebesgue measure, the Law of the Iterated Logarithm proved by Bañuelos, Klemeš and Moore, describes the maximal non-tangential growth of $u(x,y)$ in terms of its (doubly) truncated area function $A(u)(x,y)$, at almost evey point $x \in \mathbb{R}^n_+$. In this paper we show that if u is in the Bloch space and its area function diverges at almost every point, one can prescribe any "reasonable" radial behaviour of u in a set of rays of maximal Hausdorff dimension. More concretely, if $\gamma : [0,\infty) \to \mathbb{R}$ satisfies certain regularity conditions, the set $\{x \in \mathbb{R}^n : \lim_{y\to 0} \sup |u(x,y) - \gamma(A^2(u)(x,y))| < \infty\}$ has Hausdorff dimension n. A multiplicative version of this result is also proved.

Introduction. Let u be a harmonic function in the upper half space $\mathbb{R}^{n+1}_+ = \{(x,y) : x \in \mathbb{R}^n,\, y > 0\}$. For any $x_0 \in \mathbb{R}^n$, $\alpha > 0$, we let $\Gamma(x_0)$ denote the (truncated) cone

$$\Gamma(x_0) = \Gamma(x_0,\alpha) = \{(x,y) \in \mathbb{R}^{n+1}_+ : |x - x_0| < \alpha y,\, 0 < y < 1\}$$

and $A(u)(x_0)$ the (truncated) area function,

$$A^2(u)(x_0) = A^2_\alpha(u)(x_0) = \int_{\Gamma(x_0)} |\nabla u(x,y)|^2 y^{1-n}\, dm(x)\, dy.$$

At an individual point $x_0 \in \mathbb{R}^n$, the two conditions

$$\sup_{(x,y)\in\Gamma(x_0)} |u(x,y)| < \infty,$$

$$A(u)(x_0) < \infty,$$

are independent. However celebrated results of Calderón, Stein and Zygmund assert that the two sets

$$\{x \in \mathbb{R}^n : u \text{ has non-tangential limit at } x\},$$

$$\{x \in \mathbb{R}^n : A(u)(x) < \infty\},$$

can only differ in a set of Lebesgue measure 0 (see [16, p. 206, 238]).

The (doubly) truncated area function is

$$A^2(u)(x_0,t) = A^2_\alpha(u)(x_0,t) = \int_{\Gamma(x_0,t)} |\nabla u(x,y)|^2 y^{1-n}\, dm(x)\, dy,$$

where $\Gamma(x_0,t)$ is the (doubly) truncated cone

$$\Gamma(x_0,t) = \Gamma(x_0,t,\alpha) = \Gamma(x_0) \cap \{(x,y) \in \mathbb{R}^{n+1}_+ : t < y < 1\}.$$

When the area function diverges at a set of positive measure, the law of the iterated logarithm proved by Bañuelos, Klemĕs and Moore describes the non tangential growth of a harmonic function in terms of its truncated area function, at almost evey point of this set (see [1], [2], [3]).

A harmonic function u in $\mathbb{R}^{n+1}_+$ belongs to the Bloch space B if the quantity

$$\|u\|_B = \sup\{y|\nabla u(x,y)| : (x,y) \in \mathbb{R}^{n+1}_+\}$$

is finite. This condition has the following geometrical interpretation: Bloch functions map hyperbolic balls of a fixed radius in $\mathbb{R}^{n+1}_+$ into intervals of the real line of a fixed length. Also, by Harnack's inequality, any bounded harmonic function is in the Bloch space. The little Bloch space B_0 is the subspace of those $u \in B$ for which

$$\lim_{y\to 0} y|\nabla u(x,y)| = 0.$$

The law of the iterated logarithm for a Bloch harmonic function u asserts that,

$$0 < \limsup_{y\to 0} \frac{|u(x,y)|}{\sqrt{A^2(u)(x,y)\log\log A^2(u)(x,y)}} < C,$$

for almost every $x \in \{x \in \mathbb{R}^n : A(u)(x) = \infty\}$. Here C is a positive constant only depending on the dimension and the aperture α used to define the area function.

So, at almost every ray, the maximal growth of a harmonic Bloch function is determined by its truncated area function.

In this paper we are interested on rays along which the behaviour of the harmonic function is completely controlled by its truncated area function. The motivation comes from the following two recent results.

Let u be a bounded harmonic function in the half-plane $\mathbb{R}^2_+$. Then, Bourgain ([4], [5]) has proved that the set of points $x \in \mathbb{R}$ at which

$$\int_0^1 |\nabla u(x,y)|\, dy < \infty,$$

has Hausdorff dimension 1. This solved a question by W. Rudin ([15]) who exhibited bounded harmonic functions with infinite variation along almost every ray. Recently, Jones and Müller ([10]) have shown that if u is a Bloch harmonic function in the upper half plane $\mathbb{R}^2_+$, there exists a point $x \in \mathbb{R}$ and a constant $C = C(x) > 0$ such that

$$u(x,y) \geq C \int_y^1 |\nabla u(x,t)|\, dt - C^{-1},$$

for any $0 < y < 1$. So, at any point (x,y) of this ray, the harmonic function is controlled by the corresponding radial variation, that is, the length (counting multiplicities) of $u\{(x,t); y < t < 1\}$. Corresponding results when $n > 1$ seem to be open (see [7]).

In this paper we look for analogues of these results when the radial variation is replaced by the area function. Our first result asserts that when the area function diverges almost everywhere, one can prescribe any "reasonable" radial behaviour on a set of maximal Hausdorff dimension.

Theorem 1. *Let $\gamma : [0,\infty) \to \mathbb{R}$ be a continuous function satisfying*

$$\lim_{t\to\infty} \sup_{|h|\leq 1} |\gamma(t+h) - \gamma(t)| = 0.$$

Let u be a harmonic Bloch function in $\mathbb{R}^{n+1}_+$. Assume $A(u)(x) = \infty$ at almost every point $x \in \mathbb{R}^n$. Then, the set

$$E = \{x \in \mathbb{R}^n : \limsup_{y\to 0} |u(x,y) - \gamma(A^2(u)(x,y))| < \infty\}$$

has Hausdorff dimension n.

Also, when the area function diverges at almost every point, one can prescribe any "reasonable" growth along a set of rays of maximal Hausdorff dimension.

Theorem 2. *Let $\gamma : [0,\infty) \to [1,\infty)$ be an increasing function, satisfying*

$$\sup_{t>0} \gamma(t+1) - \gamma(t) < \infty.$$

Let u be a Bloch harmonic function in $\mathbb{R}^{n+1}_+$. Assume $A(u)(x) = \infty$ at almost every point $x \in \mathbb{R}^n$. Then, the set E of points $x \in \mathbb{R}^n$ which satisfy the following two conditions,

$$\liminf_{y\to 0} \frac{u(x,y)}{\gamma(A^2(u)(x,y))} > 0,$$

$$\limsup_{y\to 0} \frac{u(x,y)}{\gamma(A^2(u)(x,y))} < C,$$

has Hausdorff dimension n. Here $C = C(\gamma, u)$ is a constant only depending on γ and u.

When γ is bounded, both results tell that the function u is bounded on a set of rays of maximal Hausdorff dimension. This was first proved using martingale techniques by N.Makarov ([12]) when $n = 1$ (see also [14]) and by J. Llorente ([11]) for $n > 1$ (and in Lipschitz domains). Also, related results for analytic functions in the Bloch space can be found in [14].

When γ is unbounded, the constant C can be taken as small as desired, that is, given $\varepsilon > 0$, the set $E = E(\varepsilon)$ of points $x \in \mathbb{R}^n$ where the following two conditions are satisfied

$$\liminf_{y\to 0} \frac{u(x,y)}{\gamma(A^2(u)(x,y))} > 0$$

$$\limsup_{y\to 0} \frac{u(x,y)}{\gamma(A^2(u)(x,y))} < \varepsilon,$$

has Hausdorff dimension n.

The maximal order of growth of γ allowed by the conditions in Theorem 1 (Theorem 2) is

$$\lim_{t\to\infty} \frac{|\gamma(t)|}{t} = 0 \quad \left(\limsup_{t\to\infty} \frac{\gamma(t)}{t} < \infty \right).$$

Considering a class of Bloch functions constructed by P. W. Jones in [9], it is easy to see that the above orders of magnitude are best possible.

The corresponding results for the little Bloch space are the following.

Theorem 1'. *Let $\gamma : [0,\infty) \to \mathbb{R}$ be a continuous function satisfying*

$$\limsup_{t\to\infty} \sup_{|h|<1} |\gamma(t+h) - \gamma(t)| = 0.$$

Let u be a harmonic function in $\mathbb{R}^{n+1}_+$ in the little Bloch space. Assume $A(u)(x) = \infty$ at almost every point $x \in \mathbb{R}^n$. Then, the set

$$E = \{x \in \mathbb{R}^n : \lim_{y\to 0} u(x,y) - \gamma(A^2(u)(x,y)) = 0\}$$

has Hausdorff dimension n.

Theorem 2'. *Let $\gamma : [0,\infty) \to [1,\infty)$ be an increasing function, satisfying*

$$\sup_{t>0} \gamma(t+1) - \gamma(t) < \infty.$$

Let u be a harmonic function in $\mathbb{R}^{n+1}_+$ in the little Bloch space. Assume $A(u)(x) = \infty$ at almost every point $x \in \mathbb{R}^n$. Then for any number $a \in \mathbb{R}$, the set $E = E(a)$ of points $x \in \mathbb{R}^n$ such that

$$\lim_{y\to 0} \frac{u(x,y)}{\gamma(A^2(u)(x,y))} = a$$

has Hausdorff dimension n.

These results have local versions, that is, if one assumes $A(u)(x) = \infty$ at almost every point of a given cube $Q \subset \mathbb{R}^n$, then the corresponding set $E \cap Q$ has Hausdorff dimension n.

The proofs of these results consist of constructing a Cantor type set contained in E and evaluate its dimension. Stopping time arguments and Green's formula are used to choose nested collections of dyadic cubes in the upper half space, where the increment of the harmonic function can be controlled by the increment of its corresponding truncated area function. Similar arguments have been used in [3]. The projections of such collection of dyadic cubes give the generations of the Cantor set.

The paper is organized as follows. Section 2 contains some preliminary facts and the building block of the construction. Section 3 is devoted to the proofs of Theorems 1 and 2. Finally in Section 4, the conditions on the function γ are discussed.

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2. Preliminary facts. Given a cube Q in $\mathbb{R}^n$, we let $|Q|$ denote its volume, $\ell(Q)$ its side length and x_Q its center. Given $k = 1, 2, \dots$, the dyadic subcubes of Q of generation k are the 2^{kn} pairwise disjoint cubes $G_k = \{S_j : j = 1, \dots, 2^{kn}\}$ contained in Q, of equal side length $\ell(S_j) = 2^{-k}\ell(Q)$, $j = 1, \dots, 2^{kn}$. It is clear that

$$\sum_{j=1}^{2^{kn}} |S_j| = |Q|.$$

The generations are nested, that is $G_{k+1} \subset G_k$, $k = 1, 2 \dots$.

Given a cube Q in $\mathbb{R}^n$, the cube $\hat{Q} \subset \mathbb{R}^{n+1}_+$ is defined as

$$\hat{Q} = \{(x, y) : x \in Q,\, 0 \leq y \leq \ell(Q)\},$$

that is, $\hat{Q}$ is the cube in the upper half space whose projection is Q. The center of the top side of $\hat{Q}$ is denoted by z_Q, that is, $z_Q = (x_Q, \ell(Q)) \in \mathbb{R}^{n+1}_+$.

The hyperbolic distance between two points $z, w \in \mathbb{R}^{n+1}_+$ is

$$\rho(z, w) = \inf_{\gamma} \int_{\gamma} \frac{|ds|}{s_{n+1}},$$

where the infimum is taken over all arcs γ in $\mathbb{R}^{n+1}_+$ joining z to w. If $S \subset Q$ are dyadic cubes in $\mathbb{R}^n$ of correlative generations, it follows

$$\rho(z_S, z_Q) \leq C(n),$$

where $C(n)$ is a constant only depending on the dimension. Similarly, if S, Q are cubes in $\mathbb{R}^n$ of the same size and $\operatorname{dist}(S, Q) \leq C\ell(Q)$, then

$$\rho(z_S, z_Q) \leq K(C, n),$$

where $K(C, n)$ is a constant depending on C and the dimension.

The harmonic functions $u : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ which are Lipschitz when $\mathbb{R}^{n+1}_+$ is equipped with the hyperbolic metric, are the Bloch functions.

Lemma 2.1. *Let u be a harmonic function in $\mathbb{R}^{n+1}_+$. Then $u \in B$ if and only if there exists a constant $C > 0$ such that for any $z, w \in \mathbb{R}^{n+1}_+$, one has*

$$|u(z) - u(w)| \leq C\rho(z, w).$$

Moreover if $u \in B$, the infimum of such C is $\|u\|_B$.

Proof. Assume $u \in B$. If L is the hyperbolic geodesic joining z to w, one has

$$|u(z) - u(w)| \leq \int_L |\nabla u(\xi)||d\xi| \leq \|u\|_B \rho(z, w).$$

Conversely, when $|w - z| < m z_{n+1}$, $0 < m < 1$, one has

$$\rho(z, w) \leq (1 - m)^{-1} z_{n+1}^{-1} |z - w|.$$

So, we deduce

$$\|u\|_B \leq C. \qquad \square$$

Lemma 2.2. *Let $Q \subset \mathbb{R}^n$ be a cube and $\mathcal{F} = \{S\}$ a collection of pairwise disjoint subcubes of Q which cover almost every point of Q, that is,*

$$\sum_{S \in \mathcal{F}} |S| = |Q|.$$

Let u be a Bloch harmonic function in $\mathbb{R}^{n+1}_+$. Assume u is bounded in $\hat{Q} \setminus \bigcup_{\mathcal{F}} \hat{S}$. Then

$$u(z_Q) = \sum_{S \in \mathcal{F}} u(z_S) \frac{|S|}{|Q|} + O(1) \|u\|_B,$$

where $O(1)$ is a quantity bounded by a constant only depending on the dimension, that is, independent of u, Q and $\mathcal{F}$.

Proof. We assume that u is bounded in

$$R = \hat{Q} \setminus \bigcup_{\mathcal{F}} \hat{S}$$

for technical reasons. Actually, it implies that the series in the statement is absolutely convergent. Hence, one may assume that the collection $\mathcal{F}$ is finite. Then we apply Green's formula in the domain R to obtain,

$$\int_{\partial R} y \partial_n u = \int_{\partial R} u \partial_n y = \int_Q u(x, \ell(Q))\, dm(x) - \sum_{\mathcal{F}} \int_S u(x, \ell(S))\, dm(x).$$

Since $\sigma(\partial R) \leq C(n)|Q|$ using Lemma 2.1, one finishes the proof. $\square$

Given two cubes Q, S, in $\mathbb{R}^n$, $S \subset Q$, and $x \in S$, consider the truncated cone

$$\Gamma(x, S, Q) = \Gamma(x) \cap \{(t, y) : \ell(S) \leq y \leq \ell(Q)\}$$

and the truncated area function

$$A^2_{S,Q}(u)(x) = \int_{\Gamma(x,S,Q)} |\nabla u(t, y)|^2 y^{1-n}\, dm(t)\, dy, \quad x \in S.$$

Observe that if $x, x' \in S$ one has

$$|A^2_{S,Q}(u)(x) - A^2_{S,Q}(u)(x')| \leq C(n, \alpha) \|u\|^2_B.$$

So, when u is a Bloch harmonic function, the mean

$$A^2_{S,Q}(u) = \frac{1}{|S|} \int_S A^2_{S,Q}(u)(x)\, dm(x),$$

differs at most by a bounded amount from $A^2_{S,Q}(u)(x)$, for any $x \in S$.

The building block of the construction is given in the following result

Proposition 2.3. *Let u be a Bloch harmonic function in $\mathbb{R}^{n+1}_+$, $\|u\|_B = 1$. Let $Q \subset \mathbb{R}^n$ be a cube and assume $A(u)(x) = \infty$ at almost every point $x \in Q$. Then there exists a constant $K_0 = K_0(n, \alpha)$ such that for any number K, $K > K_0$, one can find a collection $\mathcal{F}$ of dyadic subcubes of Q satisfying the following properties:*

(a) *For any $S \in \mathcal{F}$, one has $\ell(S) \leq 2^{-K/C}\ell(Q)$.*

(b) $\sum_{\mathcal{F}} |S| \geq \frac{1}{3}|Q|$.

(c) *For any $S \in \mathcal{F}$, one has $K \leq u(z_S) - u(z_Q) \leq K + C$.*

(d) *If L is a dyadic subcube of Q which contains some cube of $\mathcal{F}$, one has $|u(z_L) - u(z_Q)| \leq 2K$.*

(e) *For any $S \in \mathcal{F}$, one has*

$$C^{-1}K^2 \leq A^2_{S,Q}(u) \leq CK^2.$$

Here $C = C(n, \alpha)$ is a constant only depending on the dimension n and the aperture α used to define the area function.

Remark. At the points associated to the cubes of $\mathcal{F}$ the function u has approximately increased K units. It is clear that one can also find cubes where u has approximately decreased K-units, that is, one can replace condition (c) by

(c') For any $S \in \mathcal{F}$, one has

$$-K - C \leq u(z_S) - u(z_Q) \leq -K.$$

Applying these two versions of this proposition alternately t times, one has the following result.

Corollary 2.4. *Let u be a harmonic Bloch function in $\mathbb{R}^{n+1}_+$, $\|u\|_B = 1$. Let Q be a cube in $\mathbb{R}^n$ and assume $A(u)(x) = \infty$ at almost every point $x \in Q$. Then there exists a constant $K_1 = K_1(n, \alpha)$ such that for any large numbers K, t, $K > K_1$, $t > K_1$, one can find a collection $\mathcal{F}$ of dyadic subcubes of Q satisfying the following properties:*

(a) *For any $S \in \mathcal{F}$, one has $\ell(S) \leq 2^{-Kt/C}\ell(Q)$.*

(b) $\sum_{\mathcal{F}} |S| \geq 3^{-t}|Q|$.

(c) *For any $S \in \mathcal{F}$, one has $K \leq u(z_S) - u(z_Q) \leq K + C$.*

(d) *For any dyadic cube $L \subset Q$ containing some cube of $\mathcal{F}$ one has $|u(z_L) - u(z_Q)| \leq 3K$.*

(e) *For any $S \in \mathcal{F}$, one has $C^{-1}K^2 t \leq A^2_{S,Q}(u) \leq CK^2 t$.*

Here $C = C(n, \alpha)$ is a constant depending only on the dimension n and the aperture α used to define the area function.

This result and Lemma 3.1 give that a harmonic Bloch function is bounded on a set of rays of Hausdorff dimension n. As mentioned in the introduction this was proved by Makarov ([12]) when $n = 1$ and Llorente for $n > 1$ ([11]).

As in Proposition 2.3, it is clear that one can also find cubes where u has approximately decreased K units, that is, one can replace condition (c) by

(c') For any $S \in \mathcal{F}$, one has

$$-K - C \leq u(z_S) - u(z_Q) \leq -K.$$

Proof. [Proof of Corollary 2.4] We will apply the two versions of Proposition 2.3 alternately. The main difficulty is to show that one can do it in such a way that the errors coming from estimates (c) and (c') do not add up.

Let $K_1 = 2K_0$ and $K > K_1$. One may assume that

$$-C \leq u(z_Q) \leq 0\,,$$

where C is the constant appearing in Proposition 2.3. Applying the Proposition 2.3 with the constant K replaced by $K - u(z_Q)$, one gets a collection $\mathcal{F}_1$ of dyadic subcubes of Q satisfying (a)-(e). In particular, (c) gives that

$$K \leq u(z_S) \leq K + C\,,$$

for any $S \in \mathcal{F}_1$. Next, in each $S \in \mathcal{F}_1$, we apply Proposition 2.3 again with the constant K replaced by $u(z_S)$ and with condition (c') instead of (c). Hence, one obtains a collection $\mathcal{F}_1(S)$ of dyadic subcubes of S satisfying (a), (b), (c'), (d) and (e). In particular, (c') gives

$$-C \leq u(z_L) \leq 0\,,$$

for any $L \in \mathcal{F}_1(S)$. Now, one repeats this procedure $[t/2] - 1$ times and, since one wants (c), after that, one repeats the first half of the construction above. The construction and (d) of Proposition 2.3 give (c) and (d) of the Corollary. Also, adding up the estimates (a), (b) and (e) from Proposition 2.3, one deduces the corresponding estimates in Corollary 2.4 □

Proof. [Proof of Proposition 2.3] We let $C_i = C_i(n, \alpha)$, $i = 1, 2, \ldots$, denote various positive constants only depending on n and α which may change from line to line. We will use a stopping time argument. Consider the collection $\mathcal{G}$ of maximal dyadic subcubes S of Q such that

$$|u(z_S) - u(z_Q)| \geq K. \tag{2.1}$$

The maximality and Lemma 2.1 give that for any dyadic cube $L \subset Q$ containing some cube of $\mathcal{G}$, one has

$$|u(z_L) - u(z_Q)| \le K + C. \tag{2.2}$$

Another application of Lemma 2.1 gives $\ell(S) \le 2^{-K/C}\ell(Q)$, if $S \in \mathcal{G}$. We will show that at a fixed portion of cubes S in $\mathcal{G}$, the truncated area function $A_{S,Q}(u)$ is comparable to K.

Since u is non-tangentially bounded at almost no point of Q, one has

$$\sum_{S \in \mathcal{G}} |S| = |Q|.$$

Replacing, if necessary, some small portion of cubes in $\mathcal{G}$ by bigger ones, one may assume that the collection $\mathcal{G}$ is finite. Then, of course, (2.1) will be satisfied only for a large portion of the cubes in $\mathcal{G}$, but this is all that is needed.

We will apply Green's formula in a hyperbolic neighbourhood R of $\hat{Q} \setminus \bigcup_{\mathcal{G}} \hat{S}$,

$$R = \left\{ z \in \mathbb{R}^{n+1}_{+} : \rho\Big(z, \hat{Q} \setminus \bigcup_{\mathcal{G}} \hat{S}\Big) < C_0 \right\},$$

where $C_0 = C_0(n, \alpha)$ is chosen so that the truncated cones

$$\Gamma(x, S, Q), \quad x \in S,$$

are contained in R for any $S \in \mathcal{G}$. Then, Green's formula applied to the functions $(u - u(z_Q))^2$ and y gives

$$2 \int_R y |\nabla u|^2 = \int_{\partial R} (u - u(z_Q))^2 \partial_{\vec{n}} y - \int_{\partial R} y \partial_{\vec{n}} (u - u(z_Q))^2.$$

Here, $\vec{n}$ denotes the inward normal. If K is sufficiently large, Lemma 2.1 and (2.2) give that $|u - u(z_Q)| \le 2K$ in R and thus

$$\int_R y |\nabla u|^2 \le CK^2 |Q|.$$

On the other hand, Fubini's theorem gives

$$\sum_{S \in \mathcal{G}} \int_S A^2_{S,Q}(u)(x)\, dm(x) \le C_1 \int_R y |\nabla u|^2$$

and thus

$$\sum_{S \in \mathcal{G}} A^2_{S,Q}(u) |S| \le C_2 K^2 |Q|.$$

Therefore, one has

$$\sum_{S\in\mathcal{G}_1} |S| \le \frac{1}{100}|Q|,$$

where $\mathcal{G}_1$ is the subcollection of $\mathcal{G}$ formed by those cubes S satisfying

$$A^2_{S,Q}(u) \ge 100 C_2 K^2.$$

Denote by $\mathcal{G}_2$ the subcollection of $\mathcal{G}$ formed by those cubes S satisfying

$$A^2_{S,Q}(u) < rK^2.$$

We will show that for sufficiently small r, one has

$$\sum_{S\in\mathcal{G}_2} |S| \le \frac{1}{100}|Q|. \tag{2.3}$$

Assume (2.3) does not hold. In order to find a contradiction, we construct a subregion of $\hat{Q}\setminus\bigcup_{\mathcal{G}}\hat{S}$, so that from any point of it, the cubes in $\mathcal{G}_2$ are "visible". Consider the collection $\mathcal{L}$ of the maximal dyadic cubes L in Q containing some cube of the family $\mathcal{G}$, such that

$$\sum_{S\in\mathcal{G}_2,\, S\subset L} |S| < \frac{1}{1000}|L|. \tag{2.4}$$

The maximality gives

$$\sum_{S\in\mathcal{G}_2,\, S\subset 2L} |S| \ge \frac{2}{1000}|L|,$$

where $2L$ is the predecessor of L in the dyadic decomposition of Q. Observe that the family $\mathcal{G}_2\cup\mathcal{L}$ covers Q. Also, if $T\subset Q$ is a dyadic subcube of Q which contains a cube in $\mathcal{G}_2\cup\mathcal{L}$, then its predecessor $2T$ contains at least a fixed portion of cubes in $\mathcal{G}_2$, that is,

$$\sum_{S\in\mathcal{G}_2,\, S\subset 2T} |S| \ge \frac{1}{1000}|T|.$$

Consider the region

$$\mathcal{R} = \hat{Q}\setminus \bigcup_{S\in\mathcal{G}_2} \hat{S} \cup \bigcup_{\mathcal{L}} 2^{\hat{C}}L\,,$$

where $C = C(\alpha)$ is chosen so that the tent $T(z) = \{x\in\mathbb{R}^n : \|x-z\| < \sqrt{2}\alpha z_{n+1}\}$

over any point $z \in \mathcal{R}$ contains a fixed portion of cubes in $\mathcal{G}_2$. Thus,

$$\begin{aligned}
&\int_{\mathcal{R}} y|\nabla u(t,y)|^2 \, dm(t)\, dy \\
&\leq C_4 \int_{\mathcal{R}} y^{1-n} |\nabla u(t,y)|^2 \Big(\sum_{S \in \mathcal{G}_2} \int_S \chi_{\Gamma(x,S,Q)}(t,y)\, dm(x) \Big)\, dm(t)\, dy \\
&\leq C_5 \sum_{S \in \mathcal{G}_2} \int_S A^2_{S,Q}(u)(x)\, dm(x) \\
&\leq C_6 r K^2,
\end{aligned} \tag{2.5}$$

where C_2, C_3, C_4 are constants depending on α and the dimension.

On the other hand, if (2.3) does not hold, (2.4) gives

$$\sum_{\mathcal{G}_3} |S| \geq \frac{9}{1000} |Q|,$$

where $\mathcal{G}_3$ is the collection of those cubes in $\mathcal{G}_2$ which are not contained in any cube of $\mathcal{L}$. Then, Green's formula gives

$$2\int_{\mathcal{R}} y|\nabla u|^2 = \int_{\partial\mathcal{R}} (u - u(z_Q))^2 \partial_{\vec{n}} y - \int_{\partial\mathcal{R}} y \partial_{\vec{n}} (u - u(z_Q))^2.$$

Since, by (2.1), $|u(z_S) - u(z_Q)| \geq K$ for any $S \in \mathcal{G}_2$, we deduce

$$\int_{\mathcal{R}} y|\nabla u|^2 \geq CK^2 |Q|.$$

This contradicts (2.5) if r is small enough and (2.3) is proved. So, if C is sufficiently large (depending only on α and the dimension), one has

$$\sum_{\mathcal{G}_4} |S| \geq \frac{49}{50} |Q|,$$

where $\mathcal{G}_4$ is the subcollection of $\mathcal{G}$ formed by those cubes S satisfying

$$C^{-1} K^2 \leq A_{S,Q}(u) \leq CK^2,$$

where $C = C(n, \alpha)$. We will consider the subcollection $\mathcal{F}$ of $\mathcal{G}_4$, formed by those cubes S for which $u(z_S) - u(z_Q) \geq K$. Observe that Lemma 2.2 gives

$$\left| \sum_{S \in \mathcal{G}} (u(z_S) - u(z_Q)) \frac{|S|}{|Q|} \right| \leq C \|u\|_B.$$

Since for any $S \in \mathcal{F}$ one has $K + C\|u\|_B \geq u(z_S) - u(z_Q) \geq K$, applying (2.2) we deduce

$$\sum_{\mathcal{F}} |S| \geq \frac{2}{5}|Q|,$$

if K is sufficiently large. $\square$

3. Proofs. The proofs consist of constructing a Cantor type set contained in the set E and evaluating its dimension according to the following result.

Lemma 3.1. [[8], [12]] *Let $0 < \varepsilon < C < 1$ be two constants and let $(\mathcal{F}_j)$ be families of pairwise disjoint cubes of $\mathbb{R}^n$ satisfying*

(a) *For any $Q \in \mathcal{F}_j$ there exists a unique $R \in \mathcal{F}_{j-1}$ such that $Q \subset R$. Moreover, one has $\ell(Q) < \varepsilon \ell(R)$.*

(b) *If $R \in \mathcal{F}_{j-1}$, one has*

$$\sum_{Q \in \mathcal{F}_j,\, Q \subset R} |Q| \geq C|R|.$$

Then,

$$\operatorname{Dim}\left(\bigcap_j \bigcup_{Q \in \mathcal{F}_j} Q\right) \geq n\left(1 - \frac{\log C}{\log \varepsilon}\right).$$

The construction of such Cantor sets will be made inductively, that is, by generations. These generations will be the projection (into $\mathbb{R}^n$) of nested collections of dyadic cubes of the upper half space $\mathbb{R}^{n+1}_+$. Such dyadic cubes are chosen so that the increment of the harmonic function can be controlled by the increment of its truncated area function. The main step in proving Theorems 1 and 2 is given in Lemmas 3.2 and 3.3.

Lemma 3.2. *Let γ and u be as in Theorem 1. Assume γ is unbounded. Then, there exists a constant $M_0 = M_0(\gamma, u)$ such that whenever $M > M_0$ and Q is a cube in $\mathbb{R}^n$ such that*

$$|u(z_Q) - \gamma(A^2(u)(x_Q, \ell(Q)))| \leq M,$$

there exist constants $t = t(M)$, $r = r(M)$, tending to ∞ as $M \to \infty$, and a collection $\mathcal{F}$ of dyadic subcubes of Q satisfying

(a) *If $S \in \mathcal{F}$, one has $\ell(S) \leq 2^{-rt}\ell(Q)$.*

(b) $\sum_{S \in \mathcal{F}} |S| \geq 3^{-t}|Q|$.

(c) *If $S \in \mathcal{F}$, one has $|u(z_S) - \gamma(A^2(u)(x_S, \ell(S)))| \leq M$.*

(d) *If L is a dyadic subcube of Q which contains some cube of $\mathcal{F}$, one has*

$$|u(z_L) - \gamma(A^2(u)(x_L, \ell(L)))| \leq 20M.$$

Proof. [Proof of Theorem 1] As mentioned in the introduction, when γ is bounded, one has to find a set of dimension n of rays along which the function u is bounded. This was proved by Llorente ([11]). As mentioned before, Corollary 2.4 also gives this result. So, one may assume γ is unbounded. We will show that for sufficiently large numbers a, the set E_a of points $x \in \mathbb{R}^n$, $A(u)(x) = \infty$, where

$$\overline{\lim_{y \to 0}} |u(x,y) - \gamma(A^2(u)(x,y))| < a,$$

contains a Cantor set whose Hausdorff dimension tends to n as $a \to \infty$.

Fix a cube $Q \subset \mathbb{R}^n$ and $M > 0$ sufficiently large so that

$$|u(z_Q) - \gamma(A^2(u)(x_Q, \ell(Q)))| \leq M.$$

The first generation $G_1 = G_1(Q)$ of the Cantor set is the subcollection $\mathcal{F}$ given by Lemma 3.2. Given any cube $S \in G_1$ we may use Lemma 3.2 again to obtain $G_1(S)$. Then,

$$G_2 = \bigcup_{S \in G_1} G_1(S).$$

Next generations are defined recursively,

$$G_n = \bigcup_{S \in G_{n-1}} G_1(S).$$

Now, estimates (a), (b) and Lemma 3.1, give

$$\mathrm{Dim}\left(\bigcap_k G_k\right) \geq n\left(1 - \frac{\log 3}{r(M)\log 2}\right),$$

while if $S \in G_n$ for some n, one has

$$|u(z_S) - \gamma(A^2(x_S, \ell(S)))| \leq M.$$

Also, if $x \in G_n$, estimate (d), Lemma 2.1 and the hypothesis on γ give that

$$|u(x,y) - \gamma(A^2(u)(x,y))| \leq 22M, \quad 0 < y < 1,$$

and the proof is completed. $\square$

Proof. [Proof of Lemma 3.2] We let $C_i = C_i(n, \alpha)$, $i = 1, 2, \dots$, denote various positive constants which depend only on n and α but which may change from line to line. Assume $\|u\|_B = 1$. Given a sufficiently large number M and a cube Q, let $\ell = \ell(M)$ be the smallest positive number such that

$$|\gamma(A^2(Q) + \ell^2) - \gamma(A^2(Q))| = 10M,$$

where $A^2(Q) = A^2(u)(x_Q, \ell(Q))$. The hypothesis on γ gives that

$$\lim_{M\to\infty} \ell^2/M = \infty.$$

Assume $\gamma(A^2(Q) + \ell^2) - \gamma(A^2(Q)) \geq 0$. Let $K(M) = K < M$, $K \to \infty$ as $M \to \infty$, $K^2 < M$, $KM < \ell^2$, be a large number to be fixed later and apply Proposition 2.3 to obtain a collection $\mathcal{F}_1$ of dyadic cubes in Q with properties (a)-(d). In particular, if $S \in \mathcal{F}_1$ one has

$$K \leq u(z_S) - u(z_Q) \leq K + C_0,$$

where $C_0 = C_0(n, \alpha)$. We repeat this procedure in each $S \in \mathcal{F}_1$ with the constant K replaced by

$$u(z_Q) - u(z_S) + 2K\,,$$

to obtain the family $\mathcal{F}_1(S)$. Thus,

$$2K \leq u(z_L) - u(z_Q) \leq 2(K + C_0)\,,$$

for any $L \in \mathcal{F}_1(S)$. Then $\mathcal{F}_2 = \bigcup_{\mathcal{F}_1} \mathcal{F}_1(S)$. In this way one obtains collections $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \dots \supset \mathcal{F}_n$ of dyadic subcubes of Q satisfying

$$\sum_{\mathcal{F}_n} |S| \geq 3^{-n}|Q|$$

and if $S \in \mathcal{F}_n$, one has

$$\ell(S) \leq 2^{-Kn/C_0}\ell(Q),$$

$$C_0^{-1}K^2 n \leq A^2_{S,Q}(u) \leq C_0 K^2 n$$

$$Kn \leq u(z_S) - u(z_Q) \leq (K + C_0)n.$$

Moreover if L is a dyadic subcube of Q containing some cube of $\mathcal{F}_n$, one has

$$|u(z_L) - u(z_Q)| \leq 2Kn.$$

We now let n to be the integer part of

$$\frac{1}{K}(\gamma(A^2(Q) + \ell^2) - u(z_Q)) + 1.$$

Thus, if $S \in \mathcal{F}_n$ one has

$$0 \leq u(z_S) - \gamma(A^2(Q) + \ell^2) \leq \frac{C_1 M}{K} + (K + C_0),$$

$$A^2_{S,Q}(u) \leq C_1 K M < \frac{1}{2}\ell^2.$$

Now, in each $S \in \mathcal{F}_n$, Corollary 2.4 will be applied several times, till the truncated area function has increased ℓ units. Given $S \in \mathcal{F}_n$, let $K_0 = K_0(M, S)$, $t_0 = t_0(M, S)$, $K_0, t_0 \to \infty$ as $M \to \infty$, be two large numbers to be fixed later. We will apply Corollary 2.4 with the parameters K_0 and t_0 several times, alternating the conditions (c) and (c'). In each iteration, the corresponding truncated area function will increase an amount comparable to $K_0^2 t_0$, while the variation of u is controlled by $3K_0$. As in the proof of Corollary 2.4, alternating conditions (c) and (c') gives that the corresponding errors do not add up.

So, we apply Corollary 2.4 repeatedly in each $S \in \mathcal{F}_n$, alternating conditions (c) and (c') and stop the first time we get a cube S for which

$$0 \leq A^2_{S,Q}(u) - \ell^2 \, .$$

Since $A^2_{S,Q}(u) \leq \ell^2/2$ for any cube $S \in \mathcal{F}_n$ and in each iteration the square of the corresponding truncated area function is comparable to $K_0^2 t_0$, one needs to apply Corollary 2.4 an amount of times comparable to

$$m = \ell^2 / K_0^2 t_0 \, .$$

In this way, one obtains a collection $\mathcal{F}$ of dyadic subcubes of the cubes of $\mathcal{F}_n$ satisfying

$$\sum_{\mathcal{F}} |S| \geq 3^{-n - C_2 m t_0} |Q| \tag{3.1}$$

and if $S \in \mathcal{F}$, one has the following estimates:

$$\ell(S) \leq 2^{-K_0 t_0 C_3 m - Kn/C_0} \ell(Q), \tag{3.2}$$

$$|u(z_S) - \gamma(A^2(Q) + \ell^2)| \leq 3K_0 + \frac{C_1 M}{K} + (K + C_0), \tag{3.3}$$

$$0 \leq A^2_{S,Q}(u) - \ell^2 \leq C K_0^2 t_0. \tag{3.4}$$

Properties (3.1) and (3.2) follow from (a) and (b) in Corollary 2.4 and the corresponding properties of the cubes in the family $\mathcal{F}_n$. Property (3.3) holds because we are alternating (c) and (c') and hence, we never move far away from the original value $u(z_S)$, $S \in \mathcal{F}_n$, which satisfied

$$|u(z_S) - \gamma(A^2(Q) + \ell^2)| \leq C_1 M / K + (K + C_0) \, .$$

Property (3.4) follows from the maximality of the cubes in $\mathcal{F}$.

Moreover, if L is a dyadic subcube of Q containing some cube of $\mathcal{F}$, one has

$$|u(z_L) - u(z_Q)| \leq 3K_0 + \frac{C_1 M}{K} + 12M. \tag{3.5}$$

We now choose $t = n + C_2 m t_0$, $r = (K_0 t_0 C_3 m + K C_0^{-1} n)(n + m t_0 C_2)^{-1}$. It is clear that $t, r \to \infty$ as $M \to \infty$ and (3.1), (3.2) give (a) and (b) in the statement.

If $K_0^2 t_0 / M$ is sufficiently small, the hypothesis on γ and (3.4) give that for any $S \in \mathcal{F}$ one has

$$|\gamma(A^2(u)(x_S, \ell(S))) - \gamma(A^2(Q) + \ell^2)| \leq \frac{1}{2} M.$$

Thus, if $K = K(M)$, $K_0 = K_0(M)$, $K \to \infty$, $K_0 \to \infty$ as $M \to \infty$, is chosen so that

$$3K_0 + \frac{C_1 M}{K} + 2K < \frac{M}{2},$$

the estimate (3.3) gives that

$$|u(z_S) - \gamma(A^2(u)(x_S, \ell(S)))| \leq M,$$

which is (c). Also, if L is a dyadic subcube of Q which contains a cube of $\mathcal{F}$, (3.5) gives

$$\begin{aligned} |u(z_L) - \gamma(A^2(u)(x_L, \ell(L)))| &\leq |u(z_L) - u(z_Q)| \\ &+ |u(z_Q) - \gamma(A^2(Q))| + |\gamma(A^2(Q)) - \gamma(A^2(u)(x_L, \ell(L)))| \leq 20M, \end{aligned}$$

which is (d). This finishes the proof. $\square$

Lemma 3.3. *Let γ, u be as in Theorem 2. Assume γ is unbounded, that is, $\lim_{t\to\infty} \gamma(t) = \infty$. Then, given $\varepsilon > 0$, there exists a constant $M_0 = M_0(\gamma, u, \varepsilon) > 0$ such that whenever $M > M_0$ and Q is a cube in $\mathbb{R}^n$, there exist constants $t = t(M)$, $r = r(M)$, $t, r \to \infty$ as $M \to \infty$, and a collection $\mathcal{F}$ of dyadic subcubes of Q satisfying the following properties:*

(a) *If $S \in \mathcal{F}$, one has $\ell(S) \leq 2^{-rt} \ell(Q)$.*

(b) $\sum_{\mathcal{F}} |S| \geq 3^{-t} |Q|$.

(c) *If $S \in \mathcal{F}$ one has*

$$\frac{M}{2} \leq \gamma(A^2(u)(x_S, \ell(S))) - \gamma(A^2(u)(x_Q, \ell(Q))) \leq 2M.$$

(d) *If $S \in \mathcal{F}$, one has*

$$M^{-1} \leq \frac{u(z_S) - u(z_Q)}{\gamma(A^2(u)(x_S, \ell(S))) - \gamma(A^2(u)(x_Q, \ell(Q)))} \leq \varepsilon.$$

(e) *If L is a dyadic subcube of Q which contains some cube of $\mathcal{F}$, one has*

$$|u(z_L) - u(z_Q)| \leq \varepsilon M.$$

Proof. [Proof of Theorem 2] As mentioned in the introduction, when γ is bounded, one has to find a set of dimension n of rays along which the function u is bounded. This was proved by Llorente ([11]). As mentioned before, Corollary 2.4 also gives this result.

So, one may assume γ is unbounded. Given $\varepsilon > 0$ and a large number t, we will show that the set E_t of points $x \in \mathbb{R}^n$ where the following two estimates hold,

$$\liminf_{y \to 0} \frac{u(x,y)}{\gamma(A^2(u)(x,y))} > t^{-1}$$

$$\limsup_{y \to 0} \frac{u(x,y)}{\gamma(A^2(u)(x,y))} < \varepsilon,$$

contains a Cantor set whose Hausdorff dimension tends to n as $t \to \infty$.

Let Q be a cube in $\mathbb{R}^n$. One may assume $u(z_Q) = 0$. Fix a large number $M > 0$. Lemma 3.3 provides a finite collection $G_1(Q)$ of dyadic subcubes of Q which satisfy

$$M^{-1}\gamma(A^2(u)(x_S, \ell(S))) \leq u(z_S) \leq \varepsilon\gamma(A^2(u)(x_S, \ell(S))),$$

for any $S \in G_1(Q)$. Given any cube $S \in G_1(Q)$, another application of Lemma 3.3 provides a finite collection $G_1(S)$ of dyadic subcubes of S. Then, the second generation is

$$G_2 = \bigcup_{S \in G_1} G_1(S).$$

Next generations are defined recursively,

$$G_k = \bigcup_{S \in G_{k-1}} G_1(S).$$

Observe that estimates (a), (b) of Lemma 3.3 and Lemma 3.1 give

$$\mathrm{Dim}\left(\bigcap_k G_k\right) \geq n\left(1 - \frac{\log 3}{r \log 2}\right),$$

which tends to n, because $r = r(M) \to \infty$ as $M \to \infty$.

Also, if $S \in G_k$ for some k, adding condition (d) one gets

$$M^{-1} \le \frac{u(z_S)}{\gamma(A^2(u)(x_S, \ell(S)))} \le \varepsilon.$$

Moreover, if L is a dyadic cube, $S' \subset L \subset S$, where $S \in G_k$ and $S' \in G_{k+1}$, condition (e) gives

$$|u(z_L) - u(z_S)| \le \varepsilon M,$$

while condition (c) gives

$$0 \le \gamma(A^2(u)(x_L, \ell(L))) - \gamma(A^2(u)(x_S, \ell(S))) \le 2M.$$

Also, applying (c) k times one gets

$$\gamma(A^2(u)(x_S, \ell(S))) \ge Mk/2.$$

So, if k is sufficiently large one has

$$\frac{1}{2M} \le \frac{u(z_L)}{\gamma(A^2(u)(x_L, \ell(L)))} \le 2\varepsilon. \qquad \square$$

Proof. [Proof of Lemma 3.3] As before, we let $C_i = C_i(n, \alpha)$, $i = 1, 2, \ldots$, denote various positive constants which depend only on n and α but which may change from line to line. We may assume $\|u\|_B = 1$. Let $\ell = \ell(M)$ be the smallest positive number such that

$$\gamma(A^2(u)(x_Q, \ell(Q)) + \ell^2) - \gamma(A^2(u)(x_Q, \ell(Q))) = M.$$

The assumption on γ gives that there exists an absolute constant C_0 such that $\ell^2 \ge 4C_0 M$.

Let $K = K(M) < M$, $K \to \infty$ as $M \to \infty$, $K^{-1} < \varepsilon$, be a large number to be fixed later. Apply Proposition 2.3 to get a collection $\mathcal{F}_1$ of dyadic cubes of Q satisfying (a)-(e) (in Proposition 2.3). In particular if $S \in \mathcal{F}_1$, one has

$$K \le u(z_S) - u(z_Q) \le K + C.$$

We repeat this procedure in each $S \in \mathcal{F}_1$ with the constant K replaced by

$$u(z_Q) - u(z_S) + 2K\,.$$

Thus,

$$2K \le u(z_L) - u(z_Q) \le 2(K + C)\,,$$

for any $L \in \mathcal{F}_1(S)$. Then $\mathcal{F}_2 = \bigcup_{\mathcal{F}_1} \mathcal{F}_1(S)$. In this way, one obtains collections $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots \supset \mathcal{F}_n$ of dyadic subcubes of Q satisfying

$$\sum_{\mathcal{F}_n} |S| \geq 3^{-n}|Q|$$

and if $S \in \mathcal{F}_n$, one has

$$\ell(S) \leq 2^{-Kn/C}\ell(Q),$$

$$C^{-1}nK^2 \leq A^2_{S,Q}(u) \leq CnK^2,$$

$$Kn \leq u(z_S) - u(z_Q) \leq (K+C)n.$$

Moreover, if L is a dyadic subcube of Q containing some cube of $\mathcal{F}_n$, one has

$$|u(z_L) - u(z_Q)| \leq 2Kn.$$

We now choose n so that $CnK^2 \sim C_0M$, more precisely, $n+1$ is the integer part of $C_0MC^{-1}K^{-2}$. Then, if $S \in \mathcal{F}_n$ one has

$$C_1M \leq A^2_{S,Q}(u) \leq C_0M < \frac{1}{4}\ell^2,$$

$$C_3\frac{M}{K} \leq u(z_S) - u(z_Q) \leq C_2\frac{M}{K}.$$

Now, in each $S \in \mathcal{F}_n$, Corollary 2.4 will be applied several times, till the truncated area function has increased ℓ units. Given $S \in \mathcal{F}_n$, let $K_0 = K_0(M,S)$, $t_0 = t_0(M,S)$, $K_0, t_0 \to \infty$ as $M \to \infty$, $K_0/M < \varepsilon$, be two large numbers to be fixed later. We will apply Corollary 2.4 with the parameters K_0 and t_0 several times, alternating the conditions (c) and (c'). In each iteration, the square of the corresponding truncated area function will increase an amount comparable to $K_0^2t_0$, while the variation of u is controlled by $3K_0$. As in the proof of Corollary 2.4, the fact that one alternates conditions (c) and (c') imply that the corresponding errors do not add up.

So, we apply Corollary 2.4 repeatedly in each $S \in \mathcal{F}_n$, alternating conditions (c) and (c'), and stop the first time we get a cube S for which

$$0 \leq A^2_{S,Q}(u) - \ell^2 .$$

Since $A^2_{S,Q}(u) \leq \ell^2/4$ for $S \in \mathcal{F}_n$ and in each iteration the square of the corresponding truncated area function is comparable to ${K_0}^2t_0$, one needs to apply Corollary 2.4 an amount of times comparable to

$$m = \frac{\ell^2}{{K_0}^2t_0} .$$

In this way, one obtains a collection $\mathcal{F}$ of dyadic subcubes of the cubes of $\mathcal{F}_n$ satisfying

$$\sum_{\mathcal{F}} |S| \geq 3^{-n-Cmt_0}|Q| \tag{3.6}$$

and if $S \in \mathcal{F}$ one has

$$\ell(S) \leq 2^{-K_0t_0mC_4-Kn/C}\ell(Q), \tag{3.7}$$

$$0 \leq A^2_{S,Q}(u) - \ell^2 \leq CK_0^2t_0, \tag{3.8}$$

$$C_3\frac{M}{K} - 3K_0 \leq u(z_S) - u(z_Q) \leq 3K_0 + C_2\frac{M}{K}. \tag{3.9}$$

Moreover, if L is a dyadic subcube of Q containing some cube of $\mathcal{F}$, one has

$$|u(z_L) - u(z_Q)| \leq C_5\frac{M}{K} + 3K_0. \tag{3.10}$$

We now choose $t = n + Cmt_0$ and $r = (K_0t_0mC_4 + Kn/C)t^{-1}$. It is clear that t, $r \to \infty$ as $M \to \infty$. Also (3.6) and (3.7) give (a) and (b) of the statement of the lemma.

Also, if $S \in \mathcal{F}$, (3.8) gives

$$\begin{aligned} \gamma(A^2_{S,Q}(u) + A^2(Q)) - \gamma(A^2(Q)) &= o(M) + \gamma(A^2(Q) + \ell^2) - \gamma(A^2(Q)) \\ &= o(M) + M, \end{aligned}$$

where $o(M)$ is a quantity that divided by M tends to 0, as $M \to \infty$. So, (c) is proved. Now (c) and (3.9) give

$$C_6\frac{1}{K} - 3\frac{K_0}{M} \leq \frac{u(z_S) - u(z_Q)}{\gamma(A^2(u)(x_S, \ell(S))) - \gamma(A^2(u)(x_Q, \ell(Q)))} \leq C_7\left(\frac{K_0}{M} + \frac{1}{K}\right)$$

which gives (d). Finally (e) follows from (3.10). □

The proof of Theorems 1' and 2' follow from an elaboration of the previous ones and are not presented here. We just mention that when the function u is in the little Bloch space and Q is a cube in $\mathbb{R}^n$, the constant K_0 in Proposition 2.3 and Corollary 2.4 can be taken as small as desired if $\ell(Q)$ is sufficiently small.

4. A class of Bloch functions. In [9], P. W. Jones constructed analytic functions b in the upper half plane, which belong to the Bloch space and such that there exist constants $C, \varepsilon > 0$ satisfying

$$\sup\{(\operatorname{Im} z)|b'(z)| : \rho(z, w) < C\} > \varepsilon \tag{4.1}$$

for any point $w \in \mathbb{R}^2_+$. Observe that by subharmonicity, one gets

$$A^2(b)(x,y) = \int_{\Gamma(x,y)} |b'(w)|^2 \, dm(w) \geq K \log y^{-1},$$

where $K = K(\varepsilon, C)$ and the aperture of the cone has been chosen sufficiently large, depending on the constant C in (4.1) . Observe that

$$|b(x+iy) - b(x+i)| \leq 2\|b\|_B \log y^{-1}, \quad y > 0.$$

Similar estimates hold for $u = \operatorname{Re} b$. Actually for any $x \in \mathbb{R}$, $0 < y < 1$, one has

$$\frac{|u(x+iy) - u(x+i)|}{A^2(u)(x,y)} \leq \frac{2\|u\|_B}{K}.$$

Hence in Theorem 2 the condition

$$\varlimsup_{t \to \infty} \frac{\gamma(t)}{t} < \infty,$$

is necessary. This is the maximal order of growth allowed by the assumption on γ. Also, for the harmonic function u described above, one has

$$A^2(u)(x,y) - A^2(u)(x,2y) \geq A(C,\varepsilon) > 0.$$

Then in Theorem 1 the condition

$$\sup_{|h|<1,\, t>0} |\gamma(t^2+h) - \gamma(t^2)| < \infty$$

is necessary. On the other hand, considering mu, for large $m > 0$, one shows that

$$\lim_{t \to \infty} \frac{\gamma(t)}{t} = 0$$

is also necessary in Theorem 1.

So, in both results the hypothesis on γ are not necessary but imply the maximal possible order of growth.

When $n = 1$, the result of Jones and Müller mentioned in the introduction was first proved by M. O'Neill, when dealing with analytic functions b satisfying (4.1) ([13]). The following result describes the situation in more variables.

Proposition 4.1. *Let u be a Bloch harmonic function in $\mathbb{R}^{n+1}_+$ such that there exist two constants $C, \varepsilon > 0$ satisfying*

$$\sup\{y|\nabla u(x,y)| : \rho((x,y),z) \leq C\} > \varepsilon,$$

for any $z \in \mathbb{R}^{n+1}_+$*. Then, the set* E *of points* $x \in \mathbb{R}^n$ *for which there exists a constant* $C = C(x) > 0$ *satisfying*

$$u(x,y) \geq C \int_y^1 |\nabla u(x,t)|\, dt,$$

for any $0 < y < 1$*, has Hausdorff dimension* n*.*

Proof. Theorem 2 applied when γ is the identity map provides a set $E \subset \mathbb{R}^n$, $\dim E = n$, such that for any $x \in E$, there exists a constant $C = C(x) > 0$ such that

$$u(x,y) \geq CA^2(u)(x,y).$$

Since

$$\int_y^1 |\nabla u(x,y)|\, dt \leq \|u\|_B \log y^{-1} \leq \|u\|_B K(\varepsilon, C)^{-1} A^2(u)(x,y),$$

the result follows. □

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