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OSCILLATION OF HÖLDER CONTINUOUS FUNCTIONS

Abstract

Local oscillation of a function satisfying a Hölder condition is considered, and it is proved that its growth is governed by a version of the Law of the Iterated Logarithm.

1 Introduction

For $0 < \alpha < 1$, let $\Lambda_\alpha(\mathbb{R})$ be the class of functions $f\colon \mathbb{R} \to \mathbb{R}$ for which there exists a constant $C = C(f) > 0$ such that $|f(x) - f(y)| \le C|x-y|^\alpha$ for any $x, y \in \mathbb{R}$. The infimum of such constants C is denoted by $\|f\|_\alpha$. For $b > 1$, G. H. Hardy proved in [7] that the Weierstrass function

$$f_b(x) = \sum_{j=1}^{\infty} b^{-j\alpha} \cos(b^j x), \quad x \in \mathbb{R},$$

is in $\Lambda_\alpha(\mathbb{R})$ and exhibits the extreme behaviour

$$\limsup_{h\to 0} \frac{|f_b(x+h) - f_b(x)|}{|h|^\alpha} > 0$$

for any $x \in \mathbb{R}$. However, for fixed $x \in \mathbb{R}$, one may expect many changes of sign of $f_b(x+h) - f_b(x)$ as $h \to 0$. The next definition provides a way of

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quantifying it. Given a function $f \in \Lambda_\alpha(\mathbb{R})$ and $0 < \varepsilon < 1/2$, consider

$$\Theta_\varepsilon(f)(x) = \int_\varepsilon^1 \frac{f(x+h) - f(x-h)}{h^\alpha} \frac{dh}{h}, \quad x \in \mathbb{R}. \tag{1.1}$$

It is clear that $\|\Theta_\varepsilon(f)\|_\infty \leq 2^\alpha \|f\|_\alpha \log(1/\varepsilon)$. Moreover, this uniform estimate cannot be improved as the elementary example $f(x) = |x|^\alpha \operatorname{sign}(x)$ shows. However, at almost every point x, the uniform estimate can be substantially improved. The main result of the paper is the following:

Theorem 1. *Fix $0 < \alpha < 1$. For $f \in \Lambda_\alpha(\mathbb{R})$ and $0 < \varepsilon < 1/2$, let $\Theta_\varepsilon(f)(x)$ be given by* (1.1)*. Then, there exists a constant $c(\alpha) > 0$, independent of ε and f, such that*

(a) For any interval $I \subset \mathbb{R}$, $|I| = 1$, one has

$$\int_I |\Theta_\varepsilon(f)(x)|^2 \, dx \leq c(\alpha)(\log 1/\varepsilon)\|f\|_\alpha^2.$$

(b) At almost every point $x \in \mathbb{R}$, one has

$$\limsup_{\varepsilon \to 0^+} \frac{|\Theta_\varepsilon(f)(x)|}{\sqrt{\log(1/\varepsilon) \log\log\log(1/\varepsilon)}} \leq c(\alpha)\|f\|_\alpha.$$

The main technical step in the proof is the following estimate, which provides the right subgaussian decay: there exists a constant $c = c(\alpha) > 0$ such that for any $t > 0$ one has

$$|\{x \in [0,1] : |\Theta_\varepsilon^*(f)(x)| > t\sqrt{\log(1/\varepsilon)}\|f\|_\alpha\}| \leq c e^{-t^2/c}. \tag{1.2}$$

Here $\Theta_\varepsilon^*(f)$ is the maximal function given by $\Theta_\varepsilon^*(f)(x) = \sup\{|\Theta_\delta(f)(x)| : 1/2 \geq \delta \geq \varepsilon\}$. Theorem 1 follows from this subgaussian estimate by standard arguments. Our proof of (1.2) is organized in two steps. First, we state and prove a dyadic version of (1.2), and later we use an averaging procedure due to J. Garnett and P. Jones ([6]). Theorem 1 is sharp up to the value of the constant $c(\alpha)$. Moreover, there exists $f \in \Lambda_\alpha(\mathbb{R})$ for which there exists a constant $c = c(f) > 0$ such that for any $0 < \varepsilon < 1/2$ one has

$$\int_\varepsilon^1 \frac{|f(x+h) - f(x-h)|}{h^\alpha} \frac{dh}{h} > c \log(1/\varepsilon)$$

at almost every $x \in \mathbb{R}$. So, Theorem 1 holds due to certain cancellations which occur in the integral defining $\Theta_\varepsilon(f)(x)$.

Subgaussian estimates and Law's of the Iterated Logarithm play a central role in the boundary behavior of martingales and have also appeared in function theory. For instance, in the relation between the boundary behaviour of a harmonic function in an upper half space and the size of its area function ([12], [3], [2]), or in differentiability properties of functions defined in the euclidean space ([1], [11]). Our result is inspired by the nice work of Y. Lyubarskii and E. Malinnikova ([9]) who studied the oscillation of harmonic functions in the Koremblum class. Related results can be found in [4] and [5].

The paper is organized as follows. Section 2 is devoted to the dyadic version of Theorem 1. The averaging procedure, which is used to prove the results in the continuous setting from their dyadic counterparts, is given in Section 3. Section 4 contains the proof of the subgaussian estimate (1.2) as well as the proof of Theorem 1. In Section 5, the sharpness of the results is discussed. Finally, Section 6 provides a higher dimensional analogue of Theorem 1.

The letters c and $c(\alpha)$ will denote a constant and a constant depending on the parameter α whose value may change from line to line.

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2 Dyadic model

For $1 \leq \rho \leq 2$, let $\mathcal{D} = \mathcal{D}(\rho)$ be the collection of intervals of the form $[j2^{-k}\rho, (j+1)2^{-k}\rho)$, where $j \in \mathbb{Z}$ and $k = 0, 1, 2, \dots$. Let $\mathcal{D}_k = \mathcal{D}_k(\rho)$ be the collection of intervals of $\mathcal{D}$ of length $2^{-k}\rho$ and let $\mathcal{F}_k = \mathcal{F}_k(\rho)$ be the σ-algebra generated by the intervals of $\mathcal{D}_k$. In the rest of this section, the number $1 \leq \rho \leq 2$ is fixed. A dyadic martingale is a sequence of functions $\{S_k\}$ defined in $[0, \rho]$ such that for any $k = 0, 1, 2, \dots$ the following two conditions hold: (a) S_k is adaptated to $\mathcal{F}_k$; (b) the conditional expectation of S_{k+1} with respect to $\mathcal{F}_k$ is S_k. In other words: S_k is constant in each interval of $\mathcal{D}_k$ and

$$\frac{1}{|I|} \int_I (S_{k+1}(x) - S_k(x))\, dx = 0$$

for any $I \in \mathcal{D}_k$, $k = 0, 1, 2, \dots$. Given a dyadic martingale $\{S_n\}$, its quadratic variation $\langle S \rangle_n$ is defined as

$$\langle S \rangle_n^2(x) = \sum_{k=1}^{n} (S_k(x) - S_{k-1}(x))^2, \quad n = 1, 2, \dots .$$

It is well known that the quadratic variation governs the boundary behaviour of the martingale. More concretely, the sets $\{x \in [0, \rho] : \lim_{n\to\infty} S_n(x) \text{ exists}\}$

and $\{x \in [0,\rho] : \langle S\rangle_\infty(x) < \infty\}$ coincide except at most for a set of Lebesgue measure 0. Moreover, there exits a universal constant $c > 0$ such that

$$\limsup_{n\to\infty} \frac{|S_n(x)|}{\sqrt{\langle S\rangle_n^2(x)\log\log\langle S\rangle_n(x)}} \leq c,$$

at almost every point x where $\langle S\rangle_\infty(x) = \infty$. See [2, p. 64]. We also mention that an elementary orthogonality argument gives that

$$\int_0^\rho |S_n(x)|^2\,dx = \int_0^\rho \langle S\rangle_n^2(x)\,dx, \quad n = 1, 2, \dots .$$

Fix $0 < \beta < 1$. Let $\{S_n\}$ be a dyadic martingale satisfying $\|S_n\|_\infty \leq 2^{n\beta}$, $n = 0, 1, 2, \dots$. For $N = 1, 2, \dots$, consider

$$\Gamma_N(x) = \Gamma_N(\{S_n\})(x) = \sum_{k=1}^N 2^{-k\beta} S_k(x).$$

It is clear that $\|\Gamma_N\|_\infty \leq N$. Moreover, this uniform estimate is best possible. Actually, if the initial martingale $\{S_n\}$ satisfies $S_0 \equiv 0$, $\|S_n\|_\infty = 2^{n\beta}$ and $S_k(x) = 2^{k\beta}$ for some $x \in \mathbb{R}$ and any $k \leq N$, then $\|\Gamma_N(\{S_n\})\|_\infty = N$. However, as next result shows, this uniform estimate can be substantially improved at almost every point. Parts (b) and (c) are the discrete analogues of Theorem 1.

Theorem 2. *Fix $0 < \beta < 1$ and $C > 0$. Let $\{S_n\}$ be a dyadic margingale with respect $\mathcal{D}(\rho)$ with $S_0 \equiv 0$ and $\|S_n\|_\infty \leq C2^{n\beta}$, $n = 1, 2, \dots$. For $N = 1, 2, \dots$, consider*

$$\Gamma_N(x) = \sum_{k=1}^N 2^{-k\beta} S_k(x),$$

$$\Gamma_N^*(x) = \sup_{k\leq N} |\Gamma_k(x)|.$$

Then, there exists a constant $c = c(\beta, C) > 0$ such that

(a) For any $\lambda > 0$ and any $N = 1, 2, \dots$, one has

$$\int_0^\rho \exp\left(\lambda\Gamma_N^*(x)\right)\,dx \leq ce^{c\lambda^2 N}.$$

(b) For any $N = 1, 2, \dots$, one has

$$\int_0^\rho |\Gamma_N^*(x)|^2\,dx \leq cN.$$

(c) For almost every $x \in [0, \rho]$ one has

$$\limsup_{n\to\infty} \frac{|\Gamma_N(x)|}{\sqrt{N \log\log N}} \leq c.$$

Proof. We can assume $C = 1$. Although $\{\Gamma_N\}$ is not a dyadic martingale, we will show that its size is comparable to the size of a dyadic martingale with bounded differences. Actually, consider the dyadic martingale $\{T_n\}$ defined by $T_0 \equiv 0$ and

$$T_n = \sum_{k=1}^{n} \frac{S_k - S_{k-1}}{2^{k\beta}}, \quad n = 1, 2, \dots .$$

The subgaussian estimate (see [2, p. 69]) gives that

$$|\{x \in [0, \rho] : T_n^*(x) > t\}| \leq 2\exp(-t^2/2\|\langle T\rangle_n^2\|_\infty),$$

for any $t > 0$. Here $T_n^*(x) = \sup\{|T_k(x)| : 1 \leq k \leq n\}$. Hence,

$$\int_0^\rho \exp\left(T_n^*(x)\right)\, dx = \int_0^\infty e^t |\{x \in [0,\rho] : T_n^*(x) > t\}|\, dt$$
$$\leq 2\int_0^\infty \exp\left(t - t^2/2\|\langle T\rangle_n^2\|_\infty\right) dt.$$

We deduce that

$$\int_0^\rho \exp\left(T_n^*(x)\right)\, dx \leq 2\sqrt{2\pi}\|\langle T\rangle_n\|_\infty \exp\left(\|\langle T\rangle_n^2\|_\infty/2\right), \quad n = 1, 2, \dots$$

Since $\|T_{n+1} - T_n\|_\infty \leq 1 + 2^{-\beta}$ for any n, one has $\|\langle T\rangle_n^2\|_\infty \leq n(1 + 2^{-\beta})^2$ for $n = 1, 2, \dots$. We deduce that for any $\lambda > 0$, one has

$$\int_0^\rho \exp\left(\lambda T_n^*(x)\right)\, dx \leq 2(1+2^{-\beta})\sqrt{2\pi n}\lambda \exp\left(\frac{\lambda^2}{2} n(1 + 2^{-\beta})^2\right), \quad n = 1, 2, \dots$$

On the other hand, summation by parts gives that

$$T_n = (1 - 2^{-\beta})\Gamma_{n-1} + 2^{-n\beta} S_n.$$

Hence,

$$\Gamma_n^* \leq (1 - 2^{-\beta})^{-1}(T_{n+1}^* + 1). \tag{2.1}$$

We deduce that for any $n = 1, 2, \dots$, and any $\lambda > 0$, one has

$$\int_0^\rho \exp\left(\lambda \Gamma_n^*(x)\right)\, dx$$

$$2\frac{1 + 2^{-\beta}}{1 - 2^{-\beta}}\sqrt{2\pi(n+1)}\lambda \exp\left(\lambda(1 - 2^{-\beta})^{-1}\right) \exp\left(\frac{1}{2}\left(\frac{1 + 2^{-\beta}}{1 - 2^{-\beta}}\right)^2 \lambda^2(n+1)\right).$$

Hence, the trivial estimate $\lambda(1-2^{-\beta})^{-1} \le \lambda^2/2 + (1-2^{-\beta})^{-2}/2$ finishes the proof of (a).

The estimate (2.1) gives

$$\int_0^\rho |\Gamma_n^*(x)|^2\,dx \le 2(1-2^{-\beta})^{-2} \int_0^\rho |T_{n+1}^*(x)|^2\,dx + 2\rho(1-2^{-\beta})^{-2}.$$

Since by Doob's maximal inequality ([10, p.493])

$$\int_0^\rho |T_{n+1}^*(x)|^2\,dx \le c\int_0^\rho \langle T\rangle_{n+1}^2(x)\,dx \le c(n+1),$$

(b) follows. As mentioned before, we have $\|\langle T\rangle_n^2\|_\infty \le n(1+2^{-\beta})^2$ for $n = 1, 2, \ldots$. Hence, the Law of the Iterated Logarithm applied to $\{T_n\}$ gives

$$\limsup_{n\to\infty} \frac{|T_n(x)|}{\sqrt{n\log\log n}} \le c \text{ a.e. } x.$$

We deduce

$$\limsup_{n\to\infty} \frac{|\Gamma_n(x)|}{\sqrt{n\log\log n}} \le c(1-2^{-\beta})^{-1} \text{ a.e. } x,$$

which finishes the proof. □

3 Averaging

An averaging procedure due to J. Garnett and P. Jones ([6]) will be used to go from the discrete situation of Theorem 2 to the continuous one of Theorem 1.

Given $x \in \mathbb{R}$, let $I_k^\rho(x)$ be the unique interval of $\mathcal{D}(\rho)$ of length $2^{-k}\rho$ which contains x. Given a function $f\colon \mathbb{R}\to\mathbb{R}$ and an interval $I = [a, b)$, we denote $\Delta f(I) = f(b) - f(a)$ and consider the dyadic martingale with respect to the filtration $\mathcal{D}(\rho)$ given by

$$S_k^{(\rho)}(f)(x) = \frac{\Delta f(I_k^{(\rho)}(x))}{2^{-k}\rho}, \quad k = 0, 1, 2, \ldots .$$

If $f \in \Lambda_\alpha(\mathbb{R})$, we have $\|S_k^{(\rho)}(f)\|_\infty \le (2^k/\rho)^\beta \|f\|_\alpha$, $k = 0, 1, \ldots$ where $\beta = 1-\alpha$. As in Section 2, consider

$$\begin{aligned}\Gamma_n^{(\rho)}(f)(x) &= \Gamma_n^{(\rho)}(\{S_k^{(\rho)}\})(x)\\ &= \sum_{k=1}^n 2^{-k\beta}\rho^\beta S_k^{(\rho)}(f)(x) = \sum_{k=1}^n \frac{\Delta f(I_k^{(\rho)}(x))}{(2^{-k}\rho)^\alpha}.\end{aligned} \tag{3.1}$$

The main purpose of this section is to describe an averaging argument with respect to both $\rho \in [1,2]$ and translates of the dyadic net $\mathcal{D}(\rho)$. We start with a preliminary result.

Lemma 3. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a locally integrable function. For $s \in \mathbb{R}$ let f_s be the function defined by $f_s(x) = f(x-s)$, $x \in \mathbb{R}$. Then for any $x \in \mathbb{R}$ and any $k = 1, 2, \dots,$ one has*

$$\int_0^\rho \Delta f_s(I_k^{(\rho)}(x+s))\,ds = 2^k \int_0^{2^{-k}\rho} (f(x+t) - f(x-t))\,dt.$$

PROOF. Fix $x \in \mathbb{R}$ and $k = 1, 2, \dots$. Let $I_k^{(\rho)}(x) = [a, b)$. Fix an integer j with $0 \le j \le 2^k - 1$ and consider $[2^{-k}j\rho, 2^{-k}(j+1)\rho) = J \cup K$ where $J = J(x) = [2^{-k}j\rho, 2^{-k}j\rho + b - x)$ and $K = K(x) = [2^{-k}j\rho + b - x, 2^{-k}(j+1)\rho)$. Note that for $s \in J$ one has $I_k^{(\rho)}(x+s) = [a + 2^{-k}j\rho, b + 2^{-k}j\rho)$ and

$$\begin{aligned}\int_J \Delta f_s(I_k^{(\rho)}(x+s))\,ds &= \int_J (f(b + 2^{-k}j\rho - s) - f(a + 2^{-k}j\rho - s))\,ds \\ &= \int_0^{b-x} (f(x+t) - f(x+t-2^{-k}\rho))\,dt.\end{aligned}$$

For $s \in K$ one has $I_k^{(\rho)}(x+s) = [a + 2^{-k}(j+1)\rho, b + 2^{-k}(j+1)\rho)$ and

$$\begin{aligned}\int_K \Delta f_s(I_k^{(\rho)}(x+s))\,ds &= \int_K (f(b + 2^{-k}(j+1)\rho - s) - f(a + 2^{-k}(j+1)\rho - s))\,ds \\ &= \int_{b-x}^{2^{-k}\rho} (f(x+t) - f(x+t-2^{-k}\rho))\,dt.\end{aligned}$$

Thus

$$\begin{aligned}\int_{2^{-k}j\rho}^{2^{-k}(j+1)\rho} \Delta f_s(I_k^{(\rho)}(x+s))\,ds &= \int_0^{2^{-k}\rho} (f(x+t) - f(x+t-2^{-k}\rho))\,dt \\ &= \int_0^{2^{-k}\rho} (f(x+t) - f(x-t))\,dt.\end{aligned}$$

Adding on $j = 0, \dots, 2^k - 1$, one finishes the proof. □

We now state the main result of this section.

Proposition 4. *Fix $0 < \alpha \leq 1$. Let f be a locally integrable function. For $s \in \mathbb{R}$, let f_s be the function defined by $f_s(x) = f(x-s)$, $x \in \mathbb{R}$. For $n = 1, 2, \ldots,$ consider $\Gamma_n^{(\rho)}(f_s)$ as defined in (3.1). Then for any $x \in \mathbb{R}$, one has*

$$\int_1^2 \int_0^\rho \Gamma_n^{(\rho)}(f_s)(x+s)\, ds \frac{d\rho}{\rho^2} = \frac{1}{1+\alpha} \int_{2^{-n}}^1 \frac{f(x+t)-f(x-t)}{t^{1+\alpha}}\, dt + A_n(f)(x),$$

where

$$|A_n(f)(x)| \leq c(\alpha) \int_{2^{-n}}^1 |f(x+t) - f(x-t)|\, dt + c(\alpha) 2^{n(1+\alpha)} \int_0^{2^{-n}} |f(x+t)-f(x-t)|\, dt.$$

In particular if $f \in \Lambda_\alpha(\mathbb{R})$, one has $\sup_{n,x} |A_n(f)(x)| < C(\alpha) \|f\|_\alpha$.

PROOF. For $k = 1, 2, \ldots,$ consider

$$B_k = \int_1^2 \int_0^\rho \frac{\Delta(f_s)(I_k^{(\rho)}(x+s))}{(2^{-k}\rho)^\alpha}\, ds \frac{d\rho}{\rho^2}.$$

Lemma 3 gives that

$$B_k = \int_1^2 2^k \int_0^{2^{-k}\rho} \frac{f(x+t)-f(x-t)}{(2^{-k}\rho)^\alpha}\, dt \frac{d\rho}{\rho^2}.$$

The change of variables $h = 2^{-k}\rho$ gives

$$B_k = \int_{2^{-k}}^{2^{-k+1}} \frac{1}{h^{2+\alpha}} \int_0^h (f(x+t)-f(x-t))\, dt\, dh.$$

Adding on $k = 1, \ldots, n$, one deduces

$$\int_1^2 \int_0^\rho \Gamma_n^{(\rho)}(f_s)(x+s)\, ds \frac{d\rho}{\rho^2} = \int_{2^{-n}}^1 \frac{1}{h^{2+\alpha}} \int_0^h (f(x+t)-f(x-t))\, dt\, dh.$$

Applying Fubini's Theorem one deduces

$$\begin{aligned} \int_1^2 \int_0^\rho \Gamma_n^{(\rho)}(f_s)(x+s)\, ds \frac{d\rho}{\rho^2} &= \frac{1}{1+\alpha} \int_{2^{-n}}^1 \frac{f(x+t)-f(x-t)}{t^{1+\alpha}}\, dt \\ &\quad - \frac{1}{1+\alpha} \int_{2^{-n}}^1 (f(x+t)-f(x-t))\, dt \\ &\quad + \frac{2^{n(1+\alpha)}-1}{1+\alpha} \int_0^{2^{-n}} (f(x+t)-f(x-t))\, dt, \end{aligned}$$

which finishes the proof. □

4 Continuous setting

In this section, the results of the dyadic model of Section 2 and the averaging procedure of Section 3 will be used to prove Theorem 1.

Given $f \in \Lambda_\alpha(\mathbb{R})$ and $0 < \varepsilon < 1$, pick an integer N such that $2^{-N-1} \leq \varepsilon < 2^{-N}$. Observe that $|\Theta_\varepsilon(f)(x) - \Theta_{2^{-N}}(f)(x)| \leq 2\|f\|_\alpha$. Hence, the estimates of $\Theta_{2^{-N}}(f)(x)$ can be easily transferred to $\Theta_\varepsilon(f)(x)$. The main technical step in proving the relevant subgaussian estimate of $\Theta_{2^{-N}}(f)(x)$ is stated in next result.

Proposition 5. *Let $f \in \Lambda_\alpha([-1,2])$ with $\|f\|_\alpha \leq 1$. For $x \in [0,1]$ and $N = 1, 2, \dots,$ consider*

$$\Theta_{2^{-N}}(f)(x) = \int_{2^{-N}}^{1} \frac{f(x+h) - f(x-h)}{h^\alpha} \frac{dh}{h},$$
$$\Theta^*_{2^{-N}}(f)(x) = \sup_{k \leq N} |\Theta_{2^{-k}}(f)(x)|.$$

Then, there exists a constant $c(\alpha) > 0$ such that for any $\lambda > 0$ and any $N = 1, 2, \dots,$ one has

$$\int_0^1 \exp\left(\lambda \Theta^*_{2^{-N}}(f)(x)\right) dx \leq c(\alpha) \exp\left(c(\alpha)\lambda^2 N\right).$$

Proof. Consider the set $A = \{(\rho, s) : 1 \leq \rho \leq 2,\, 0 \leq s \leq \rho\}$ and the measure $d\mu$ defined as

$$\mu(E) = \int_{E \bigcap A} ds \frac{d\rho}{\rho^2}\,, E \subset \mathbb{R}^2.$$

For any $k = 1, 2, \dots,$ Proposition 4 gives that

$$\Theta_{2^{-k}}(f)(x) = (1+\alpha) \int_A \Gamma_k^{(\rho)}(f_s)(x+s)\, d\mu(\rho, s) + A_k(f)(x)\,.$$

Moreover, there exists a constant $C = C(\alpha)$ such that $\sup_{k,x} |A_k(x)| \leq C$. Here is where the normalization $\|f\|_\alpha \leq 1$ is used. Hence, if k and N are integers with $k \leq N$, we deduce

$$|\Theta_{2^{-k}}(f)(x)| \leq (1+\alpha) \int_A (\Gamma_N^{(\rho)})^*(f_s)(x+s)\, d\mu(\rho, s) + C.$$

Here $(\Gamma_N^{(\rho)})^*(f_s)(x) = \sup\{|\Gamma_k^{(\rho)}(f_s)(x)| : k \leq N\}$. Hence, for any $N = 1, 2, \dots$, one has

$$\Theta^*_{2^{-N}}(f)(x) \leq (1+\alpha) \int_A (\Gamma_N^{(\rho)})^*(f_s)(x+s)\, d\mu(\rho, s) + C.$$

Now, Jensen's inequality and Fubini's Theorem give that

$$\begin{aligned} &\int_0^1 \exp\left(\lambda \Theta^*_{2^{-N}}(f)(x)\right) dx \\ &\leq \exp(\lambda C) \int_A \int_0^1 \exp\Big(\lambda(\alpha+1)(\Gamma_N^{(\rho)})^*(f_s)(x+s)\Big)\, dx\, d\mu(\rho, s). \end{aligned} \tag{4.1}$$

Recall that $\Gamma_N^{(\rho)}(f_s)$ is defined via the formula (3.1) from the martingale $S_k^{(\rho)}(f_s)$, which is given by $S_k^{(\rho)}(f_s)(x) = (f_s(b) - f_s(a))/(b-a)$, where $x \in I_k^{(\rho)}(x) = [a, b) \in \mathcal{D}(\rho)$. The normalization $\|f\|_\alpha \leq 1$ gives that there exists an absolute constant $c_1 > 0$ such that $|S_0^{(\rho)}(f_s)| \leq c_0$ for any $(\rho, s) \in A$. Recall that if $\|f\|_\alpha \leq 1$, the martingale $S_k^{(\rho)}$ satisfies $\|S_k^{(\rho)}\|_\infty \leq (2^k/\rho)^{1-\alpha}$. Applying (a) of Theorem 2 to the martingale $S_k^{(\rho)}(f_s) - S_0^{(\rho)}(f_s)$, there exists a constant $c_1(\alpha) > 0$ such that

$$\int_0^1 \exp\Big(\lambda(1+\alpha)(\Gamma_N^{(\rho)})^*(f_s)(x+s)\Big)\, dx \leq c_1(\alpha) \exp\left(c_1(\alpha)(c_0\lambda + \lambda^2 N)\right).$$

The trivial estimate $2\lambda \leq \lambda^2 + 1$ shows that there exists a constant $c_2(\alpha) > c_1(\alpha)$ such that

$$\int_0^1 \exp\Big(\lambda(1+\alpha)(\Gamma_N^{(\rho)})^*(f_s)(x+s)\Big)\, dx \leq c_2(\alpha) e^{c_2(\alpha)\lambda^2 N}.$$

By (4.1), one deduces

$$\int_0^1 \exp\left(\lambda \Theta^*_{2^{-N}}(f)(x)\right) dx \leq c_2(\alpha) \exp(C\lambda) \exp\left(c_2(\alpha)\lambda^2 N\right).$$

Again the trivial estimate $2\lambda \leq \lambda^2 + 1$ finishes the proof. □

Now the subgaussian estimate follows easily.

Corollary 6. *Let $f \in \Lambda_\alpha([-1, 2])$ with $\|f\|_\alpha \leq 1$. Then there exists a constant $c(\alpha) > 0$ such that for any $N > 0$ and any $t > 0$ one has*

$$|\{x \in [0,1] : \Theta^*_{2^{-N}}(f)(x) > \sqrt{N} t\}| \leq c(\alpha) \exp\left(-t^2/c(\alpha)\right).$$

Proof. Let $E = \{x \in [0,1] : \Theta^*_{2^{-N}}(f)(x) > \sqrt{N}t\}$. Previous Proposition 5 and Chebyshev inequality gives that for any $\lambda > 0$, one has $\exp\left(\lambda\sqrt{N}t\right)|E| \leq c(\alpha)\exp\left(c(\alpha)\lambda^2 N\right)$, that is,

$$|E| \leq c(\alpha)\exp(c(\alpha)\lambda^2 N - \lambda\sqrt{N}t)\,.$$

We take $\lambda = t/2c(\alpha)\sqrt{N}$ and deduce $|E| \leq c(\alpha)\exp(-t^2/4c(\alpha))$ which finishes the proof. □

We can now prove Theorem 1.

Proof of Theorem 1. In the proof of part (a) we can assume that I is the unit interval and $\|f\|_\alpha = 1$. Given $0 < \varepsilon < 1/2$, pick an integer N such that $2^{-N-1} \leq \varepsilon < 2^{-N}$. Since $|\Theta_\varepsilon(f)(x) - \Theta_{2^{-N}}(f)(x)| \leq 2$, Corollary 6 gives that

$$|\{x \in [0,1] : |\Theta_\varepsilon(f)(x)| > \sqrt{N}t\}| \leq c(\alpha)e^{-t^2/c(\alpha)},$$

for any $t > 0$ such that $t\sqrt{N} > 4$. Now (a) follows easily from

$$\int_0^1 |\Theta_\varepsilon(f)(x)|^2\,dx = 2\int_0^\infty \lambda|\{x \in [0,1] : |\Theta_\varepsilon(f)(x)| > \lambda\}|\,d\lambda.$$

The Law of the Iterated Logarithm of part (b) follows from the subgaussian estimate of Corollary 6 via a standard Borel-Cantelli argument. Consider the set A of points $x \in [0,1]$ for which

$$\Theta^*_{2^{-N}}(f)(x) > 2c\sqrt{N\log\log N}$$

for infinitely many $N \geq 0$. Here $c = c_0(\alpha)$ is a constant which will be chosen later. Let $N_m = 2^m$. If $\Theta^*_{2^{-N}}(f)(x) > 2c\sqrt{N\log\log N}$ and $N_{m-1} < N \leq N_m$, then

$$\Theta^*_{2^{-N_m}}(f)(x) \geq \Theta^*_{2^{-N}}(f)(x) > 2c\sqrt{N\log\log N} \geq c\sqrt{N_m\log\log N_m}.$$

Thus $A \subset \cap_k \cup_{m\geq k} A_{N_m}$ where

$$A_{N_m} = \{x : \Theta^*_{2^{-N_m}}f(x) > c\sqrt{N_m\log\log N_m}\}.$$

Now Corollary 4.2 with $t = c\sqrt{\log\log N_m} = c(\log(m\log 2))^{1/2}$ gives $|A_{N_m}| \leq c(\alpha)(m\log 2)^{-c^2/c(\alpha)}$, and for $c^2 > c(\alpha)$, the Borel-Cantelli lemma gives $|A| = 0$. Thus, for almost every $x \in [0,1]$, one has

$$\limsup_{N\to\infty} \frac{|\Theta^*_{2^{-N}}(f)(x)|}{\sqrt{N\log\log N}} \leq 2c,$$

and the proof is completed. □

5 Sharpness

In this section, the sharpness of our results is discussed.

5.1 Sharpness of Theorem 1

Both parts (a) and (b) in Theorem 1, as well as Proposition 5 and its Corollary 6, are sharp up to the value of the constants $c(\alpha)$. Since Theorem 1 follows from Corollary 6, it is sufficient to construct a function $f \in \Lambda_\alpha(\mathbb{R})$ for which

$$\int_0^1 |\Theta_\varepsilon(f)(x)|^2 \, dx \geq c(\log(1/\varepsilon)), \quad 0 < \varepsilon < 1/2 \tag{5.1}$$

and

$$\limsup_{\varepsilon \to 0} \frac{\Theta_\varepsilon(f)(x)}{\sqrt{\log(1/\varepsilon) \log\log\log(1/\varepsilon)}} > c, \quad \text{a.e. } x \in [0,1] \tag{5.2}$$

for a certain constant $c = c(\alpha) > 0$. Fix $0 < \alpha < 1$. As it is usual in this kind of questions, the function f will be given by a lacunary series. More concretely, consider

$$f(x) = \sum_{j=0}^{\infty} 2^{-j\alpha} \sin(2\pi 2^j x).$$

Then,

$$\begin{aligned} \Theta_{2^{-N}}(f)(x) &= \int_{2^{-N}}^1 \frac{f(x+h) - f(x-h)}{h^\alpha} \frac{dh}{h} \\ &= 2 \sum_{j=0}^{\infty} 2^{-j\alpha} \left(\int_{2^{-N}}^1 \frac{\sin(2^j 2\pi h)}{h^{\alpha+1}} \, dh \right) \cos(2^j 2\pi x) \\ &= 2 \sum_{j=0}^{\infty} c_{j,N} \cos(2^j 2\pi x), \end{aligned}$$

where

$$c_{j,N} = \int_{2^{j-N}}^{2^j} \frac{\sin(2\pi t)}{t^{\alpha+1}} \, dt.$$

Integrating by parts, one shows that there exists a constant $c_1(\alpha) > 0$ such that

$$|c_{j,N}| \leq c_1(\alpha) 2^{-(j-N)(\alpha+1)}, \quad j = 1, 2, \ldots, \quad N = 1, 2, \ldots .$$

Hence,

$$\sum_{j\geq N} |c_{j,N}| \leq 2c_1(\alpha). \tag{5.3}$$

On the other hand, using the estimate $|\sin t| \leq t$, we have

$$\sum_{j=0}^{N} \left| \int_0^{2^{j-N}} \frac{\sin 2\pi t}{t^{\alpha+1}}\, dt \right| \leq \frac{2\pi}{1-\alpha} \sum_{j=0}^{N} 2^{(j-N)(1-\alpha)} \leq c_2(\alpha). \tag{5.4}$$

Using (5.3) and (5.4), one deduces that

$$\Theta_{2^{-N}}(f)(x) = \sum_{j=0}^{N} b_j \cos(2^j 2\pi x) + E_N(x), \tag{5.5}$$

where $|E_N(x)| \leq c_3(\alpha) = 2c_1(\alpha) + c_2(\alpha)$ for any $x \in \mathbb{R}$ and any $N = 1, 2, \ldots$ and

$$b_j = 2\int_0^{2^j} \frac{\sin 2\pi t}{t^{\alpha+1}}\, dt.$$

Consider

$$A(\alpha) = \lim_{j\to\infty} b_j = 2\int_0^{\infty} \frac{\sin 2\pi t}{t^{\alpha+1}}\, dt$$

and observe that $A(\alpha) > 0$. By orthogonality for N sufficiently large one has

$$\|\Theta_{2^{-N}}(f)\|^2_{L^2[0,1]} \geq \frac{1}{4} A(\alpha)^2 N,$$

which gives (5.1).

A classical result by M. Weiss ([12]) gives that

$$\limsup_{N\to\infty} \frac{\left|\sum\limits_{j=0}^{N} b_j \cos(2^j 2\pi x)\right|}{\sqrt{N \log\log N}} = A(\alpha).$$

Thus, from (5.5), one deduces

$$\limsup_{N\to\infty} \frac{|\Theta_{2^{-N}}(f)(x)|}{\sqrt{N \log\log N}} = A(\alpha),$$

which gives (5.2).

5.2 Cancellation

Theorem 1 says that the uniform estimate $\|\Theta_\varepsilon(f)\|_\infty \le c(\log 1/\varepsilon)\|f\|_\alpha$, $0 < \varepsilon < 1/2$, can be substantially improved at almost every point. This is due to certain cancellations which occur in the integral defining $\Theta_\varepsilon(f)(x)$. Actually, there exist $f \in \Lambda_\alpha(\mathbb{R})$ and $c = c(f) > 0$ such that for any $0 < \varepsilon < 1/2$ one has

$$\int_\varepsilon^1 \frac{|f(x+h) - f(x-h)|}{h^\alpha} \frac{dh}{h} \ge c\log(1/\varepsilon) \tag{5.6}$$

for almost every $x \in \mathbb{R}$. Let $b > 1$ be a large positive integer to be fixed later. Consider

$$f(x) = \sum_{j=0}^{\infty} b^{-j\alpha}\cos(b^j x), \quad x \in \mathbb{R}.$$

Fix $k \ge 0$ and h such that $b^{-k}/2 \le h \le 2b^{-k}$. Observe that

$$\frac{2}{h^\alpha}\sum_{j>k} b^{-j\alpha} \le c(\alpha)b^{-\alpha}$$

and

$$\frac{1}{h^\alpha}\sum_{j<k} b^{-j\alpha}|\cos(b^j x + b^j h) - \cos(b^j x - b^j h)| \le c(\alpha)b^{\alpha-1}.$$

On the other hand,

$$\cos(b^k x + b^k h) - \cos(b^k x - b^k h) = -2\sin(b^k x)\sin(b^k h).$$

Hence,

$$\int_{b^{-k}/2}^{2b^{-k}} \frac{|f(x+h) - f(x-h)|}{h^\alpha}\frac{dh}{h} \ge c|\sin(b^k x)| - c(\alpha, b),$$

where $c(\alpha, b) = c(\alpha)(b^{-\alpha} + b^{\alpha-1})$ and $c > 0$. Thus, if b is taken sufficiently large so that $c(\alpha, b) < c/4$, one has

$$\int_\varepsilon^1 \frac{|f(x+h) - f(x-h)|}{h^\alpha}\frac{dh}{h} > ct(\varepsilon, x)/4,$$

where $t(\varepsilon, x)$ is the number of positive integers k such that $b^{-k} \ge 2\varepsilon$ which satisfy $|\sin(b^k x)| \ge 1/2$. The uniform distribution of $\{b^k x\}$ (see Corollary 4.3 of [8]) gives that there exists a constant $c_1 > 0$ such that $t(\varepsilon, x) \ge c_1 \ln(2\varepsilon)^{-1}/\ln b$ almost every $x \in \mathbb{R}$. So (5.6) follows.

6 Higher dimensions

Theorem 1 can be easily extended to higher dimensions. For $0 < \alpha < 1$, let $\Lambda_\alpha(\mathbb{R}^d)$ be the class of functions $f\colon \mathbb{R}^d \to \mathbb{R}$ for which there exists a constant $c = c(f) > 0$ such that $|f(x) - f(y)| \le c\|x - y\|^\alpha$ for any $x, y \in \mathbb{R}^d$. The infimum of the constants $c > 0$ verifying this estimate is denoted by $\|f\|_\alpha$. Lebesgue measure in $\mathbb{R}^d$ is denoted by dm. Let σ be a probability measure in the unit sphere of $\mathbb{R}^d$. For $0 < \varepsilon < 1/2$, consider

$$\Theta_\varepsilon(f)(x) = \int_{\{|\xi|=1\}} \Theta_{\varepsilon,\xi}(f)(x)\, d\sigma(\xi), \quad x \in \mathbb{R}^d,$$

where

$$\Theta_{\varepsilon,\xi}(f)(x) = \int_\varepsilon^1 \frac{f(x+\rho\xi) - f(x-\rho\xi)}{\rho^\alpha}\frac{d\rho}{\rho}, \quad x \in \mathbb{R}^d,$$

for $\xi \in \mathbb{R}^d$, $|\xi| = 1$. The next result is the higher dimensional analogue of Theorem 1

Theorem 7. *Let $0 < \alpha < 1$ and $f \in \Lambda_\alpha(\mathbb{R}^d)$. Then, there exists a constant $c(\alpha, d) > 0$ such that*

(a) For any cube $Q \subset \mathbb{R}^d$ with $m(Q) = 1$, one has

$$\int_Q |\Theta_\varepsilon(f)(x)|^2 dm(x) \le c(\alpha, d)(\log 1/\varepsilon)\|f\|_\alpha^2.$$

(b) At almost every $x \in \mathbb{R}^d$, one has

$$\limsup_{\varepsilon\to 0} \frac{|\Theta_\varepsilon(f)(x)|}{\sqrt{\log(1/\varepsilon)\log\log\log(1/\varepsilon)}} \le c(\alpha, d)\|f\|_\alpha.$$

Proof. We will take $\varepsilon = 2^{-N}$ and will write $\Theta_{N,\xi}$ and Θ_N instead of $\Theta_{2^{-N},\varepsilon}$ and $\Theta_{2^{-N}}$. Also, $\Theta^*_{N,\varepsilon}$, Θ^*_N will denote the maximal functions defined as

$$\Theta^*_{N,\xi}(f)(x) = \sup\{|\Theta_{k,\xi}(f)(x)| : k \le N\},$$
$$\Theta^*_N(f)(x) = \sup\{|\Theta_k(f)(x)| : k \le N\}.$$

Then

$$\Theta^*_N(f)(x) \le \int_{\{|\xi|=1\}} \Theta^*_{N,\xi}(f)(x)\, d\sigma(\xi), \quad x \in \mathbb{R}^d, \quad N = 1, 2 \ldots .$$

To prove (a) we can assume that Q is the unit cube and $\|f\|_\alpha \leq 1$. Jensen's inequality and Fubini's Theorem give

$$\int_Q \exp\left(\lambda \Theta_N^*(f)(x)\right)\, dm(x) \leq \int_{\{|\xi|=1\}} \int_Q \exp\left(\lambda \Theta_{N,\xi}^*(f)(x)\right)\, dm(x)\, d\sigma(\xi).$$

Fixed $\xi \in \mathbb{R}^d$ with $|\xi| = 1$, consider the lines in the direction of ξ intersecting Q. Since the length of these intersections is uniformly bounded by a constant only depending on d, we can apply Proposition 5 with the interval $[0,1]$ replaced by some fixed interval depending on d. Hence for any $\xi \in \mathbb{R}^d$, $|\xi| = 1$, we obtain

$$\int_Q \exp\left(\lambda \Theta_{N,\xi}^*(f)(x)\right)\, dm(x) \leq c(\alpha, d) \exp\left(c(\alpha)\lambda^2 N\right),$$

for any $\lambda > 0$ and any $N = 1, 2, \ldots$. We deduce

$$\int_Q \exp\left(\lambda \Theta_N^*(f)(x)\right)\, dm(x) \leq c(\alpha, d) e^{c(\alpha)\lambda^2 N}.$$

Now arguing as in Corollary 6, one deduces the subgaussian estimate

$$m\{x \in Q : |\Theta_N^*(f)(x)| > \sqrt{N} t\} \leq c_1(\alpha, d) \exp\left(-t^2/c_1(\alpha, d)\right).$$

Arguing as in the proof of Theorem 1, one finishes the proof. □

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