The Nevanlinna-Pick Interpolation Problem

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1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane $\mathbb{C}$. The Nevanlinna-Pick problem can be stated as follows: *Given a sequence of distinct points $\{z_n\} \subset \mathbb{D}$ and a sequence of values $\{w_n\} \subset \mathbb{C}$, is there an analytic function $f : \mathbb{D} \to \overline{\mathbb{D}}$ such that $f(z_n) = w_n, n = 1, 2, \dots$?* We will discuss three types of questions:

1. When the problem has a solution?
2. How can the set of all solutions be described?
3. How can one find solutions with certain extremal properties?

First session is devoted to questions 1 and 2. There are two classical approaches due respectively, to Nevanlinna and Pick. We will follow Nevanlinna's ideas which also lead to a solution of 2. At the end of the session we will mention a modern approach to Pick's result which leads to a different set of problems and results. Next sessions are devoted to 3. Second session is devoted to prove Nevanlinna's main result, which states that if the Nevanlinna-Pick problem has more than one solution, then all extremal solutions are inner functions. In the third section we will prove a refinement of this result due to A. Stray which states that actually most extremal solutions are Blaschke products. Last session is devoted to extremal solutions of scaled problems.

The Nevanlinna-Pick problem has been considered in many different spaces and extended in many different directions. In 1968, Adamyan, Arov and Krein extended Nevanlinna's parametrization. See [1] or [19, p.146] or [47]. D. Sarason found deep relations between the Nevanlinna-Pick problem and several results in operator theory, see [42] and [43, p.68]. His work has been extremely influential and has been extended by many authors. See the monography [2]. The Nevanlinna-Pick problem has also been considered in other spaces of analytic functions, see [2], [30], and the monography [44]. In these lectures we will not try to review these results; instead, we will follow a geodesic which will bring us from the classical ideas of Nevanlinna to some modern results, mainly due to Arne Stray.

This paper collects the material of the four lectures I gave at the Summerschool in Mekrijärvi, Finland, in June 2014. It is a pleasure to thank the organizers for their kind invitation and to all participants for the nice atmosphere, modulo mosquitoes, during those days at this wonderful research facility.

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2 Nevanlinna's and Pick's Approaches

Nevanlinna published his results a few years later than Pick but was unaware of the latter's work due probably to the poor communication during the First World War. The approaches of Nevanlinna and Pick are quite different. We will follow Nevanlinna's ideas, which are based on Schur's algorithm. This is a beautiful technique which can also be used in other problems such as the Caratheodory problem, where one assigns Taylor coefficients instead of function values.

2.1 Nevanlinna's Approach

We start with some notation. The space of bounded analytic funtions in the unit disc is denoted by H^∞. Given $f \in H^\infty$, consider $||f||_\infty = \sup_{z\in\mathbb{D}} |f(z)|$. Any function $f \in H^\infty$ has radial limit $f\left(\mathrm{e}^{i\theta}\right) = \lim_{r\to 1} f\left(r\mathrm{e}^{i\theta}\right)$ at almost every point $\mathrm{e}^{i\theta}$ of the unit circle and $||f||_\infty = ||f||_{L^\infty(\partial\mathbb{D})}$.

Bilinear transformations $T(z) = \frac{az+b}{cz+d}$, with $ad-bc \neq 0$, will be represented by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z$. The main advantage is the following fact: if $\tilde{T}(z) = \frac{\tilde{a}z+\tilde{b}}{\tilde{c}z+\tilde{d}}$, $\tilde{a}\tilde{d} - \tilde{b}\tilde{c} \neq 0$, the composition $T \circ \tilde{T}$ is represented by the product of these two matrices: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$.

We use the notation b_a for the automorphism of the unit disc given by $b_a(z) = \frac{|a|}{a} \frac{a-z}{1-\bar{a}z}$, where $a \in \mathbb{D}\setminus\{0\}$ is fixed. Also, $b_0(z) = z$. We will use the following two elementary facts.

Fact 1. If $f : \mathbb{D} \to \mathbb{D}$ is analytic and $f(a) = 0$, then $f(z) = b_a(z) f_1(z)$ where $f_1 : \mathbb{D} \to \overline{\mathbb{D}}$ is an analytic function. This follows by considering the analytic function $\frac{f(z)}{b_a(z)} = f_1(z)$ for $z \in \mathbb{D}$ and applying Maximum Modulus Principle.

Fact 2. If $\sum(1 - |a_n|) < \infty$, then $B(z) = \prod b_{a_n}(z)$ converges uniformly on compacts of $\mathbb{D}$. The function $B(z)$ is analytic, $||B||_\infty = \sup_{z\in\mathbb{D}} |B(z)| = 1$ and $B(a_n) = 0, n = 1, 2, \ldots$ $B(z)$ is called the Blaschke product with zeros $\{a_n\}$. See, for instance, [19, p. 51].

We first consider Nevanlinna-Pick problems with finitely many points.

Finite Case. Assume we have a finite set of points $\{z_1, \ldots, z_N\}$ and values $\{w_1, \ldots, w_N\}$; in other words, given $\{z_1, \ldots, z_N\} \subset \mathbb{D}$ and $\{w_1, \ldots, w_N\} \subset \mathbb{C}$, the problem can be stated as follows:

$(*)_N$: Find $f \in H^\infty$, $||f||_\infty \leq 1$ with $f(z_i) = w_i,\ i = 1, \ldots, N$.

We will consider simultaneously questions 1 and 2:

- Case $N = 1$, that is, if we have a single point z_1 and a single value w_1. There are three cases:

– If $|w_1| > 1$, we have no solution.

– If $|w_1| = 1$, we have a unique solution.

– If $|w_1| < 1$, we have infinitely many solutions. Moreover, assume f is a solution. Then by Fact 1,

$$\frac{f - w_1}{1 - \overline{w_1} f} = b_{z_1} f_1, \qquad ||f_1||_\infty \leq 1,$$

that is,

$$f = \frac{b_{z_1} f_1 + w_1}{1 + \overline{w_1} b_{z_1} f_1}, \qquad ||f_1||_\infty \leq 1\,.$$

Hence, the set of all solutions is

$$\{f \in H^\infty : ||f||_\infty \leq 1, f(z_1) = w_1\} = \left\{ f = \frac{b_{z_1} f_1 + w_1}{1 + \overline{w_1} b_{z_1} f_1} : f_1 \in H^\infty, ||f_1||_\infty \leq 1 \right\} =$$

$$= \left\{ f = \frac{1}{\sqrt{1 - |w_1|^2}} \begin{pmatrix} b_{z_1} & w_1 \\ \overline{w_1} b_{z_1} & 1 \end{pmatrix} f_1 : f_1 \in H^\infty, ||f_1||_\infty \leq 1 \right\}.$$

It will be useful to denote
$$U_1 = \frac{1}{\sqrt{1-|w_1|^2}} \begin{pmatrix} b_{z_1} & w_1 \\ \overline{w_1} b_{z_1} & 1 \end{pmatrix}.$$
The factor in front of the matrix is chosen so that $\det U_1 = b_{z_1}$.

- Case $N > 1$. We will argue inductively: f is a solution of the Nevanlinna-Pick problem with N points $z_1, \dots, z_N$ if and only if $f = \frac{b_{z_1} f_1 + w_1}{1+\overline{w_1} b_{z_1} f_1}$ and $f_1(z_j) = \frac{1}{b_{z_1}(z_j)} \frac{w_j - w_1}{1-\overline{w_1} w_j}$ for $j = 2, \dots, N$. Writing $w_2^{(1)} = \frac{1}{b_{z_1}(z_2)} \frac{w_2 - w_1}{1-\overline{w_1} w_2}$, we have three possibilities:

– If $|w_2^{(1)}| > 1$ we have no solution.

– If $|w_2^{(1)}| = 1$ we have a unique f_1 and therefore a unique solution of the Nevanlinna-Pick problem $(*)_2$ with two points z_1, z_2.

– If $|w_2^{(1)}| < 1$, then the previous argument gives
$$f_1 = \frac{b_{z_2} f_2 + w_2^{(1)}}{1+\overline{w_2^{(1)}} b_{z_2} f_2} = U_2 f_2$$
for some $f_2 \in H^\infty$ with $||f_2||_\infty \leq 1$, where
$$U_2 = \frac{1}{\sqrt{1-|w_2^{(1)}|^2}} \begin{pmatrix} b_{z_2} & w_2^{(1)} \\ \overline{w_2^{(1)}} b_{z_2} & 1 \end{pmatrix}.$$
Then $f = U_1 U_2(f_2), f_2 \in H^\infty, ||f_2||_\infty \leq 1$.

Iterating this procedure we have that the problem $(*)_N$ has more than one solution if and only if $|w_i^{(i-1)}| < 1,\ i = 1, \dots, N$. If $|w_i^{(i-1)}| > 1$ for some i, then the problem has no solution, while if $|w_i^{(i-1)}| = 1$ and $|w_k^{(k-1)}| < 1$ for $k \leq i$, then the problem
$$(*)_i \quad \text{Find } f \in H^\infty,\ ||f||_\infty \leq 1,\ f(z_k) = w_k,\ k = 1, \dots, i.$$
has a unique solution.

We will restrict attention to Nevanlinna-Pick problems with more than one solution. If the problem $(*)_N$ has more than one solution, the set of all solutions to problem $(*)_N$ is given by $f = U_1 \cdots U_N(f_N)$, where $f_N \in H^\infty,\ ||f_N||_\infty \leq 1$ and
$$U_i = \frac{1}{\sqrt{1-|w_i^{(i-1)}|^2}} \begin{pmatrix} b_{z_i} & w_i^{(i-1)} \\ \overline{w_i^{(i-1)}} b_{z_i} & 1 \end{pmatrix}.$$

In other words, denoting $\begin{pmatrix} P_N & Q_N \\ R_N & S_N \end{pmatrix} = U_1 \cdots U_N$, we have
$$f = \begin{pmatrix} P_N & Q_N \\ R_N & S_N \end{pmatrix} f_N.$$

The functions P_N, Q_N, R_N, S_N are called Nevanlinna coefficients. Let us mention some of their properties:

– P_N, Q_N, R_N, S_N are rational functions with poles contained in the set $\{\frac{1}{\overline{z_i}} : i = 1, \dots, N\}$. This is clear because the components of the matrices U_i satisfy it.

– $P_N S_N - Q_N R_N = B_N$, the Blaschke product with zeros $z_1, \dots, z_N$. This is clear because $\det U_j = b_{z_j},\ j = 1, \dots, N$.

– Consider $\Delta_N(z) = \{f(z) : f \text{ solves problem } (*)_N\}$. Then $\Delta_N(z)$ is a Euclidean disc of center $c_N(z)$ and radius $\rho_N(z)$ given by

$$c_N(z) = \frac{P_N(z)\overline{\left(-\frac{R_N}{S_N}(z)\right)} + Q_N(z)}{R_N(z)\overline{\left(-\frac{R_N}{S_N}(z)\right)} + S_N(z)},$$
$$\rho_N(z) = \frac{|B_N(z)|}{|S_N(z)|^2 - |R_N(z)|^2}.$$

Let's prove this third property:

Proof. Fix $z \in \overline{\mathbb{D}}$. Since $|w_i^{(i-1)}| < 1$, the map $U_{i,z} = U_i : \mathbb{D} \to \mathbb{D}$ defined by

$$U_i(w) = \frac{b_{z_i}(z)w + w_i^{(i-1)}}{1 + \overline{w_i^{(i-1)}} b_{z_i}(z)w}$$

maps the unit disc into itself and if $|z| = 1$, U_i is onto. The same holds for $\begin{pmatrix} P_N & Q_N \\ R_N & S_N \end{pmatrix} = U_1 \cdots U_N$, that is, fixed $z \in \overline{\mathbb{D}}$, the map

$$\begin{aligned} T_{N,z} &: \mathbb{D} \to \mathbb{D} \\ w &\mapsto \frac{P_N(z)w + Q_N(z)}{R_N(z)w + S_N(z)} \end{aligned}$$

is into and if $|z| = 1$ it is onto. Consider $\Delta_N(z) = \{f(z) : f \text{ solves problem } (*)_N\}$. Then,

$$\Delta_N(z) = \left\{ \frac{P_N(z)w + Q_N(z)}{R_N(z)w + S_N(z)} : w \in \overline{\mathbb{D}} \right\} = T_{N,z}(\overline{\mathbb{D}}).$$

Hence $\Delta_N(z)$ is a disc. Since $T_{N,z}\left(-\frac{S_N}{R_N}(z)\right) = \infty$, by reflection, $T_{N,z}$ maps the point $\overline{-\frac{R_N}{S_N}(z)}$ to the center of $\Delta_N(z)$. Since $\rho_N(z) = \left| T_{N,z}(\mathrm{e}^{i\theta}) - T_{N,z}\left(-\overline{\frac{R_N(z)}{S_N}}(z)\right) \right|$ for any $e^{i\theta} \in \partial\,\mathbb{D}$, a calculation shows that the radius $\rho_N(z)$ of $\Delta_N(z)$ is

$$\rho_N(z) = \frac{|B_N(z)|}{|S_N(z)|^2 - |R_N(z)|^2}.$$

$\square$

– On $\partial\,\mathbb{D}$, the following identities hold: $|S_N|^2 - |R_N|^2 = 1$, $P_N\overline{R_N} = Q_N\overline{S_N}$, $P_N = B_N\overline{S_N}$, $Q_N = B_N\overline{R_N}$.

Proof. If $|z| = 1$, $\Delta_N(z) = \overline{\mathbb{D}}$, hence $\rho_N(z) = 1$, that is, $|S_N(z)|^2 - |R_N(z)|^2 = 1$, $C_N(z) = 0$, that is, $P_N(z)\overline{R_N(z)} = Q_N(z)\overline{S_N(z)}$. Since $P_N(z)S_N(z) - Q_N(z)R_N(z) = B_N(z)$, we have that $P_N(z)|S_N(z)|^2 - Q_N(z)\overline{S_N(z)}R_N(z) = B_N(z)\overline{S_N(z)}$, and since $Q_N(z)\overline{S_N(z)} = P_N(z)\overline{R_N(z)}$ we deduce that $P_N(z)|S_N(z)|^2 - P_N(z)|R_N(z)|^2 = B_N(z)\overline{S_N(z)}$ and we obtain that $P_N(z) = B_N(z)\overline{S_N(z)}$. A similar argument proves last identity. $\square$

– For $z \in \mathbb{D}$, the following identities hold:

$$|S_N(z)| \geq 1,$$
$$P_N(z) = B_N(z)\overline{S_N(1/\overline{z})},$$
$$Q_N(z) = B_N(z)\overline{R_N(1/\overline{z})},$$
$$\max\{|P_N(z)|, |Q_N(z)|, |R_N(z)|\} \leq |S_N(z)|.$$

Proof. Since $|S_N(z)|^2 - |R_N(z)|^2 \geq 0$ we deduce that $S_N(z) \neq 0$ because otherwise $R_N(z) = 0$ and then $P_N(z) = Q_N(z) = 0$, but $B_N = P_N S_N - Q_N R_N$, which has no double zeros. Hence S_N does not vanish at $\mathbb{D}$, so $1/S_N \in H^\infty$ is continuous in $\overline{\mathbb{D}}$ and $|1/S_N| \leq 1$ on $\partial\mathbb{D}$, because on $\partial\mathbb{D}$ we have $|S_N|^2 = 1 + |R_N|^2 \geq 1$. By the maximum principle we deduce that $|1/S_N| \leq 1$ on $\mathbb{D}$. The formula $P_N(z) = B_N(z)\overline{S_N(1/\overline{z})}$ follows from $P_N = B_N \overline{S_N}$ on $\partial\mathbb{D}$ by analytic continuation. The formula $Q_N(z) = B_N(z)\overline{R_N(1/\overline{z})}$ follows similarly. Since $|P_N| = |S_N|$ and $|Q_N| = |R_N| < |S_N|$ on $\partial\mathbb{D}$ and S_N has no zeros on $\mathbb{D}$, we deduce that

$$\max\{|P_N(z)|, |Q_N(z)|, |R_N(z)|\} \leq |S_N(z)|$$

for all $z \in \mathbb{D}$. □

Infinite Case. Assume we have an infinite sequence of points $\{z_i\}$ and values $\{w_i\}$ and consider the Nevanlinna-Pick problem

$$(*) \ \textit{Find } f \in H^\infty,\ ||f||_\infty \leq 1,\ f(z_i) = w_i,\ i = 1, 2, \ldots.$$

Consider the Nevanlinna-Pick problem with the first N points. We know that if the problem has more than one solution, then all solutions can be parametrized as

$$\begin{aligned} \{f \in H^\infty : ||f||_\infty \leq 1,\ f(z_i) = w_i,\ i = 1, \ldots, N\} = \\ = \left\{ f = \frac{P_N\varphi + Q_N}{R_N\varphi + S_N} : \varphi \in H^\infty,\ ||\varphi||_\infty \leq 1 \right\}. \end{aligned}$$

We can assume $R_N(0) = 0$. This is just a normalization one can achieve replacing φ in Nevanlinna's formula by $\frac{\varphi - \alpha}{1 - \overline{\alpha}\varphi}$, where $\alpha = \overline{\left(\frac{R_N(0)}{S_N(0)}\right)}$. In order to consider Nevanlinna's parametrization of the set of solutions in the infinite case we need to make sure that the infinite case problem has more than one solution. One of the nice features of Schur's algorithm is that it is reversible. In our approach, given the values $\{w_i\}$, we have obtained the points $\{w_i^{(i-1)}\}$. Conversely, given points $\{w_i^{(i-1)} : i = 1, \ldots, N\} \subset \mathbb{D}$, one can obtain the values $\{w_i : i = 1, \ldots, N\}$ and consider the correspondent Nevanlinna-Pick problem . We shall mention the following fact, proved by Denjoy: *The Nevanlinna-Pick problem with infinite points has more than one solution if and only if* $\sum(1 - |z_i|)/(1 - |w_i^{(i-1)}|) < \infty$, see [52, p. 300]. Nonetheless, we will not use (and we will not prove) this result. The first main Theorem is the following.

Theorem 1. (Nevanlinna, 1919) *Assume the Nevanlinna-Pick problem*

$$(*) \ \textit{Find } f \in H^\infty,\ ||f||_\infty \leq 1,\ f(z_n) = w_n,\ n = 1, 2, \ldots$$

has more than one solution. Then the set of all solutions can be parametrized as

$$\begin{aligned} \{f \in H^\infty : ||f||_\infty \leq 1,\ f(z_n) = w_n,\ n = 1, 2, \ldots\} = \\ = \left\{ f = \frac{P\varphi + Q}{R\varphi + S} : \varphi \in H^\infty,\ ||\varphi||_\infty \leq 1 \right\}, \end{aligned} \tag{1}$$

where P, Q, R, S are analytic functions in $\mathbb{D}$ which satisfy

1 *Let B be the Blaschke product with zeros $\{z_n\}$. Then, $PS - QR = B$.*

2 *The Nevanlinna's coefficients P, S, Q, R belong to the Nevanlinna class $N(\mathbb{D})$.*

3 *The set*

$$\Delta(z) = \{f(z) : f \text{ solves } (*)\} = \left\{ \frac{P(z)w + Q(z)}{R(z)w + S(z)} : w \in \overline{\mathbb{D}} \right\}$$

is a Euclidean disc of center $c(z)$ and radius $\rho(z)$, given by

$$c(z) = \frac{P(z)\overline{\left(-\frac{R}{S}(z)\right)} + Q(z)}{R(z)\overline{\left(-\frac{R}{S}(z)\right)} + S(z)}, \qquad \rho(z) = \frac{|B(z)|}{|S(z)|^2 - |R(z)|^2}.$$

Moreover, A. Stray proved that P, S, Q and R are meromorphic in $\mathbb{C} \backslash \overline{\{1/\overline{z_j}\}}$. See [46].

4 At almost every point of $\partial\mathbb{D}$ *we have* $|S|^2 - |R|^2 = 1$, $P = B\overline{S}$, $Q = B\overline{R}$ *and* $P\overline{R} - Q\overline{S} = 0$.

5 For all $z \in \mathbb{D}$ *we have* $\max\{|P(z)|, |Q(z)|, |R(z)|\} \leq |S(z)|$. *Moreover, S is an outer function and* $|S(z)| \geq 1$.

Proof. Consider the truncated Nevanlinna-Pick problem $(*)_N$ and the corresponding Nevanlinna's coefficients P_N, Q_N, R_N, S_N. Since $R_N(0) = 0$, by Schwarz's lemma, we have $\left|\frac{R_N(z)}{S_N(z)}\right| \leq |z|$. Hence,

$$\frac{|B_N(z)|}{|S_N(z)|^2(1-|z|^2)} \geq \rho_N(z) = \frac{|B_N(z)|}{|S_N(z)|^2 - |R_N(z)|^2} \geq \frac{|B_N(z)|}{|S_N(z)|^2}.$$

Since the problem has more than one solution, there exists $z_0 \in \mathbb{D}$ such that $\lim_{N\to\infty} \rho_N(z_0) \neq 0$. Hence, $|S_N(z_0)| \not\to \infty$. Since $\log|S_N|$ are positive harmonic functions, considering a subsequence if necessary, Harnack's principle gives that $\{S_N\}$ is uniformly bounded on compact subsets of $\mathbb{D}$. Since $\max\{|P_N|, |Q_N|, |R_N|\} \leq |S_N|$, we deduce that there exist subsequences $P_{N_k}, Q_{N_k}, R_{N_k}, S_{N_k}$ which converge uniformly on compacts of $\mathbb{D}$. The limit functions are called P, Q, R and S. Then, the parametrization (1) follows from the finite case. Moreover, $\Delta(z) = \left\{\frac{P(z)w+Q(z)}{R(z)w+S(z)} : w \in \overline{\mathbb{D}}\right\}$ and property 3 follows as in the finite case. Also, since $P_N S_N - Q_N R_N = B_N$ and $\max\{|P_N|, |Q_N|, |R_N|\} \leq |S_N|$ on $\mathbb{D}$, we deduce 1 and $\max\{|P|, |Q|, |R|\} \leq |S|$ on $\mathbb{D}$. Since $|S(z)| \geq 1$ for all $z \in \mathbb{D}$, we have $1/S \in H^\infty$ and then $S \in N(\mathbb{D})$. Since S has no zeros and $\max\{|P|, |Q|, |R|\} \leq |S|$, we deduce that $P, Q, R \in N(\mathbb{D})$.
Property 4 will be proven as a consequence of a theorem of Nevanlinna which will be the main topic of the next session. It remains to prove that S is outer. A function $g \in H^\infty$, $||g||_\infty \leq 1$ is an extreme point of the unit ball of H^∞ if it can not be written as $g = \frac{g_1+g_2}{2}$ where $g_i \in H^\infty$ and $||g_i||_\infty \leq 1$, $i = 1, 2$. A result by de Leeuw-Rudin, see [27], tells us that $g \in H^\infty$, $||g||_\infty \leq 1$ is an extreme point of the unit ball of H^∞ if and only if $\int_0^{2\pi} \log(1 - |g(\mathrm{e}^{i\theta})|)^{-1}\,\mathrm{d}\theta = +\infty$. This is related to the Nevanlinna-Pick problem because the set of solutions is clearly convex. Hence, if there are two solutions to $(*)$, there exists a solution $f_0 \in H^\infty$, $||f_0||_\infty \leq 1$ such that $\int_0^{2\pi} \log(1 - |f_0(\mathrm{e}^{i\theta})|)^{-1}\,\mathrm{d}\theta < +\infty$. Write $f_0 = \frac{P\varphi_0+Q}{R\varphi_0+S}$ for some $\varphi_0 \in H^\infty$, $||\varphi_0||_\infty \leq 1$. Consider $f_1 = f_0 + BE$ where E is the outer function whose boundary values have modulus $1 - |f_0|$, that is,

$$E(z) = \exp\left(\int_0^{2\pi} \frac{\mathrm{e}^{i\theta}+z}{e^{i\theta}-z} \log\left(1 - \left|f_0\left(\mathrm{e}^{i\theta}\right)\right|\right)\,\mathrm{d}\theta\right), \; z \in \mathbb{D}.$$

Then f_1 is a solution of $(*)$. Hence, $f_1 = \frac{P\varphi_1+Q}{R\varphi_1+S}$ for some $\varphi_1 \in H^\infty$, $||\varphi_1||_\infty \leq 1$. Therefore,

$$BE = f_1 - f_0 = \frac{P\varphi_1+Q}{R\varphi_1+S} - \frac{P\varphi_0+Q}{R\varphi_0+S} = \frac{B}{S^2}\frac{\varphi_1 - \varphi_0}{\left(\frac{R}{S}\varphi_1+1\right)\left(\frac{R}{S}\varphi_0+1\right)}.$$

We know $|S| \geq 1$ on $\mathbb{D}$. Assume $\frac{1}{S}$ has a singular inner factor. Then, since $1 + \frac{R}{S}\varphi_i$, $i = 0, 1$, are outer because they have positive real part, we deduce that BE would be divisible by a singular inner function, which leads to contradiction. $\square$

In classical moment problems one tries to find positive measures in a half line with prescribed moments. It is worth mentioning that under suitable conditions, one can parametrize all solutions of the problem by a formula which is analogue to Nevanlinna's parametrization. See [29].

2.2 Pick's Approach

In this section, we shall describe an idea due to Sarason which leads to Pick's classical result on the existence of solutions to the Nevanlinna-Pick problems , see [42]. A different proof can be found in [19, p. 7].

Theorem 2. (Pick, 1916) *The Nevanlinna-Pick problem $(*)$ has a solution if and only if the matrices*

$$\left(\frac{1 - w_i\overline{w_j}}{1 - z_i\overline{z_j}}\right)_{i,j=1,\ldots,N}$$

are positive semidefinite for any $N = 1, 2, \ldots$.

Proof. By normal families it is enough to prove the result for Nevanlinna-Pick problem with finitely many points. We will only prove the necessity. Let H be a Hilbert space of analytic functions in $\mathbb{D}$. For instance, H could be the Hardy space

$$H = \mathbb{H}^2 = \left\{ f : \mathbb{D} \to \mathbb{C} \text{ analytic} : ||f||_2^2 = \sup_{r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 \, \mathrm{d}t < \infty \right\}$$

and

$$< f, g >_H = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \overline{g(e^{it})} \, \mathrm{d}t.$$

Let M_H be its multiplier space, that is, $M_H = \{\varphi : \mathbb{D} \to \mathbb{C} \text{ analytic} : \varphi f \in H \text{ for any } f \in H\}$. The norm in M_H is given by

$$||\varphi||_{M_H} = \sup_{f \in H \setminus \{0\}} \frac{||\varphi f||_H}{||f||_H}.$$

For instance, $M_{\mathbb{H}^2} = \mathbb{H}^\infty$ and $||\varphi||_{M_{\mathbb{H}^2}} = ||\varphi||_\infty$. Assume that the evaluation at a point $z \in \mathbb{D}$ given by

$$\begin{aligned} H &\to \mathbb{C} \\ f &\mapsto f(z) \end{aligned}$$

is continuous. Then, there exists $k_z \in H$, called reproducing kernel, such that $f(z) =< f, k_z >_H$ for any $f \in H$. For instance, if $f \in H^2$, by Cauchy's formula,

$$f(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(\xi)}{\xi - z} \, \mathrm{d}\xi = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta})}{1 - z\mathrm{e}^{-i\theta}} \, \mathrm{d}\theta =< f, k_z >,$$

where

$$k_z \left(\mathrm{e}^{i\theta}\right) = \frac{1}{1 - \overline{z}\mathrm{e}^{i\theta}}.$$

The main idea is the following. Pick $\varphi \in M_H$ and consider the multiplication operator $M_\varphi : H \to H$ defined by $M_\varphi(f) = \varphi f$. Then, the adjoint operator M_φ^* satisfies $M_\varphi^*(k_z) = \overline{\varphi(z)} k_z$ because $< f, M_\varphi^*(k_z) > = < M_\varphi(f), k_z > = < \varphi f, k_z > = \varphi(z) < f, k_z > = < f, \overline{\varphi(z)} k_z >$ for any $f \in H$. Hence, if $||\varphi||_{M_H} \le 1$, then $||M_\varphi^*(\sum \lambda_i k_{z_i})||_H \le ||\sum \lambda_i k_{z_i}||_H$, that is,

$$\begin{aligned} \sum_{i,j} \lambda_i \overline{\lambda_j} \overline{\varphi(z_i)} \varphi(z_j) k_{z_i}(z_j) &= ||\sum \lambda_i \overline{\varphi(z_i)} k_{z_i}||_H^2 \le ||\sum \lambda_i k_{z_i}||_H^2 = \\ &= < \sum \lambda_i k_{z_i}, \sum \lambda_j k_{z_j} > = \sum_{i,j} \lambda_i \overline{\lambda_j} k_{z_i}(z_j). \end{aligned}$$

In the case $H = \mathbb{H}^2$, $M_H = \mathbb{H}^\infty$ and $k_{z_i}(z) = \frac{1}{1 - \overline{z_i} z}$, if φ is a solution to the Nevanlinna-Pick problem $(*)$, then

$$\sum \lambda_i \overline{\lambda_j} \overline{w_i} w_j \frac{1}{1 - \overline{z_i} z_j} \le \sum \lambda_i \overline{\lambda_j} \frac{1}{1 - \overline{z_i} z_j},$$

that is, the matrix

$$\left(\frac{1 - \overline{w_i} w_j}{1 - \overline{z_i} z_j} \right)_{i,j=1,\ldots,N},$$

is positive semidefinite. This proves the necessity in Pick's result. □

A Hilbert space H of analytic functions in the disc with reproducing kernel k_z has the Pick property if for any sequence $\{w_j\} \subset \mathbb{D}$ such that the matrices $((1 - w_i \overline{w_j}) k_{z_i}(z_j))_{i,j=1,\ldots,N}$ are positive semidefinite for any N, there exists $\varphi \in M_H$, $||\varphi||_{M_H} \le 1$ such that $\varphi(z_i) = w_i$, $i = 1, 2, \ldots$. Pick's theorem tells us that $H = \mathbb{H}^2$ has the Pick's property. Agler proved that the Dirichlet space has the Pick property. The Bergman space does not have Pick's property. Pick's property is closely related to many other important notions as interpolating sequences and Carleson Measures. See the books by Seip [44] and by Agler and McCarthy, [2].

3 Extremal Solutions

Given a Nevanlinna-Pick problem with more than one solution, Theorem 1 provides a parametrization of the set of all solutions. If one chooses φ to be a unimodular constant $\lambda \in \partial\mathbb{D}$, the corresponding solution $\frac{P\lambda+Q}{R\lambda+S}$ is called an extremal solution. This section is devoted to present Nevanlinna's classical result and its refinement due to Stray.

3.1 Extremal Solutions for Finite Problems

In this section we will show the following elementary fact. If we have a Nevanlinna-Pick problem with N points and with more than one solution, then, for any $\lambda \in \partial\mathbb{D}$, $\frac{P_N\lambda+Q_N}{R_N\lambda+S_N}$ is a Blaschke product of degree less or equal to N. This is clear because $\frac{P_N\lambda+Q_N}{R_N\lambda+S_N} = U_1\cdots U_N(\lambda)$ and $U_i(f)$ is a Blaschke product with i zeros whenever f is a Blaschke product with $i-1$ zeros.
An inner function is a bounded analytic function in $\mathbb{D}$ whose radial limits are of modulus one at almost every point of the unit circle. If I is an inner function, then $\frac{P_N I+Q_N}{R_N I+S_N}$ is also inner because on $\partial\mathbb{D}$ we have $P_N = B_N\overline{S_N}$, $Q_N = B_N\overline{R_N}$, and hence

$$\frac{P_N I + Q_N}{R_N I + S_N} = B_N \frac{\overline{S_N} I + \overline{R_N}}{R_N I + S_N},$$

which is unimodular at each point where $|I| = 1$.

3.2 Extremal Solutions for Infinite Problems

We now state the second main result by Nevanlinna:

Theorem 3. (Nevanlinna, 1929) *Given a Nevanlinna-Pick problem with more than one solution, consider its Nevanlinna's coefficients P, Q, R, S. Then for any $\lambda \in \partial\mathbb{D}$, the function $I_\lambda = \frac{P\lambda+Q}{R\lambda+S}$ is inner.*

Proof. Fix $\lambda \in \partial\mathbb{D}$ and $I = \frac{P\lambda+Q}{R\lambda+S}$. Since I solves the problem with finitely many points, for any $N = 1, 2, \ldots$, we have

$$I = \frac{P_N\varphi_N + Q_N}{R_N\varphi_N + S_N}$$

for some $\varphi_N \in H^\infty$, $||\varphi_N||_\infty \leq 1$. Taking convenient subsequences, we may assure that $P_{N_j}, Q_{N_j}, R_{N_j}, S_{N_j} \longrightarrow P, Q, R, S$ and $\varphi_{N_j} \longrightarrow \varphi$ uniformly on compacts of $\mathbb{D}$. Hence,

$$I = \frac{P_{N_j}\varphi_{N_j} + Q_{N_j}}{R_{N_j}\varphi_{N_j} + S_{N_j}} \longrightarrow \frac{P\varphi + Q}{R\varphi + S}.$$

Thus, $\varphi \equiv \lambda$, that is, $\varphi_{N_j} \longrightarrow \lambda$ uniformly on compacts of $\mathbb{D}$. Assume I is not inner, that is, there exists $K \subset \partial\mathbb{D}$ with $|K| > 0$ and $|I| \leq m < 1$ on K, where $|K|$ denotes the Lebesgue measure of K. Recall that if $|z| = 1$, the mapping $T_{N,z}$ is onto from $\mathbb{D}$ to $\mathbb{D}$, hence it preserves the pseudohyperbolic distances, that is, $\rho(a,b) = \rho(T_{N,z}(a), T_{N,z}(b))$. Here, $\rho(z,w) = \left|\frac{z-w}{1-\overline{w}z}\right|$ for $z, w \in \mathbb{D}$. Using that $T_{N,z}(\varphi_N) = I$, $T_{N,z}(0) = Q_N/S_N$, at almost every point of K, one has that

$$|\varphi_{N_j}| = \rho(\varphi_{N_j}, 0) = \rho\left(I, \frac{Q_{N_j}}{S_{N_j}}\right) \leq \frac{|I| + \left|\frac{Q_{N_j}}{S_{N_j}}\right|}{1 + \left|\frac{Q_{N_j}}{S_{N_j}}\right||I|} \leq \frac{m + \left|\frac{Q_{N_j}}{S_{N_j}}\right|}{1 + \left|\frac{Q_{N_j}}{S_{N_j}}\right|m}.$$

Since

$$C_0 \geq \log|S_{N_j}(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|S_{N_j}\left(e^{i\theta}\right)\right| d\theta \geq \frac{1}{2\pi}\int_K \log\left|S_{N_j}\left(e^{i\theta}\right)\right| d\theta,$$

there exists $K_1 \subset K$ with $|K_1| \geq |K|/2$ such that $|S_{N_j}| \leq C_1 = C_1(K)$ on K_1. Hence,

$$\frac{|Q_{N_j}|^2}{|S_{N_j}|^2} = 1 - \frac{1}{|S_{N_j}|^2} \leq 1 - \frac{1}{C_1^2} = C_2^2 \text{ on } K_1,$$

and we deduce

$$|\varphi_{N_j}| \le \frac{m + C_2}{1 + C_2 m} = m_1 < 1 \text{ on } K_1.$$

Then,

$$|\varphi_{N_j}(0)| \le \frac{1}{2\pi} \int_0^{2\pi} \left|\varphi_{N_j}\left(\mathrm{e}^{i\theta}\right)\right| \mathrm{d}\theta \le m_1|K_1| + (1 - |K_1|) < 1,$$

which leads to contradiction. □

Corollary 1. *If a Nevanlinna-Pick problem has more than one solution, then it has an inner solution.*

We can now deduce the identities stated in property 4 of Theorem 2. See [48].

Corollary 2. *At almost every point on $\partial\mathbb{D}$ we have $|S|^2 - |R|^2 = 1$, $P = B\overline{S}$, $Q = B\overline{R}$ and $P\overline{R} = Q\overline{S}$. Moreover, $\rho\left(r\mathrm{e}^{i\theta}\right) \to 1$ as $r \to 1$ at almost every point $\mathrm{e}^{i\theta} \in \partial\mathbb{D}$.*

Proof. Fix $\mathrm{e}^{i\theta} \in \partial\mathbb{D}$ such that $\left|I_\lambda\left(\mathrm{e}^{i\theta}\right)\right| = 1$ for three different values of $\lambda \in \partial\mathbb{D}$. Assume also $P\left(\mathrm{e}^{i\theta}\right), Q\left(\mathrm{e}^{i\theta}\right), R\left(\mathrm{e}^{i\theta}\right), S\left(\mathrm{e}^{i\theta}\right)$ exist and are finite. We have that

$$\begin{aligned} T\left(\mathrm{e}^{i\theta}\right) : \overline{\mathbb{D}} &\to \overline{\mathbb{D}} \\ w &\mapsto \frac{P\left(\mathrm{e}^{i\theta}\right) w + Q\left(\mathrm{e}^{i\theta}\right)}{R\left(\mathrm{e}^{i\theta}\right) w + S\left(\mathrm{e}^{i\theta}\right)} \end{aligned}$$

maps $\overline{\mathbb{D}}$ onto $\overline{\mathbb{D}}$. Since the center of $T(\mathbb{D})$ is the origin and the radius is 1, we have $P\left(\mathrm{e}^{i\theta}\right)\overline{R\left(\mathrm{e}^{i\theta}\right)} - Q\left(\mathrm{e}^{i\theta}\right)\overline{S\left(\mathrm{e}^{i\theta}\right)} = 0$ and $\left|S\left(\mathrm{e}^{i\theta}\right)\right|^2 - \left|R\left(\mathrm{e}^{i\theta}\right)\right|^2 = 1$. Hence, $\rho\left(r\mathrm{e}^{i\theta}\right) \to 1$ as $r \to 1$. Since $PS - QR = B$, we deduce $P\overline{R}S - Q|R|^2 = B\overline{R}$. Thus, $Q|S|^2 - Q|R|^2 = B\overline{R}$ and we deduce $Q = B\overline{R}$ on $\partial\mathbb{D}$. A similar argument shows that $P = B\overline{S}$. □

Arguing as in the finite case one can also deduce the following corollary:

Corollary 3. *Let I be an inner function. Then $\frac{PI+Q}{RI+S}$ is also inner.*

3.3 Blaschke Products Among Extremal Solutions

Let us first recall the following classical description of Blaschke products among the set of inner functions:

Lemma 1. *(Frostman) Let I be an inner function. Then I is a Blaschke product if and only if*

$$\lim_{r\to 1} \int_0^{2\pi} \log\left|I\left(r\mathrm{e}^{i\theta}\right)\right|^{-1} \mathrm{d}\theta = 0\,.$$

Proof. Assume $I = BS$, where S is a non trivial singular inner function. Then, $|I| \le |S|$ and

$$\int_0^{2\pi} \log\left|I\left(r\mathrm{e}^{i\theta}\right)\right|^{-1} \mathrm{d}\theta \ge \int_0^{2\pi} \log\left|S\left(r\mathrm{e}^{i\theta}\right)\right|^{-1} \mathrm{d}\theta = 2\pi \log|S(0)|^{-1}.$$

Conversely, assume I is a Blaschke product with zeros $\{z_n\}$. Then,

$$\begin{aligned} \int_0^{2\pi} \log\left|I\left(r\mathrm{e}^{i\theta}\right)\right|^{-1} \mathrm{d}\theta &= \sum_n \int_0^{2\pi} \log\left|\frac{1 - \overline{z_n} r\mathrm{e}^{i\theta}}{r\mathrm{e}^{i\theta} - z_n}\right| \mathrm{d}\theta = \\ &= -\sum_n \int_0^{2\pi} \log\left|r\mathrm{e}^{i\theta} - z_n\right| \mathrm{d}\theta. \end{aligned}$$

Now, $\int_0^{2\pi} \log\left|r\mathrm{e}^{i\theta} - z_n\right| \mathrm{d}\theta = 2\pi\log|z_n|$ if $|z_n| \ge r$ and $\int_0^{2\pi} \log\left|r\mathrm{e}^{i\theta} - z_n\right| \mathrm{d}\theta = 2\pi \log r$ if $|z_n| < r$. Hence,

$$\int_0^{2\pi} \log\left|I\left(r\mathrm{e}^{i\theta}\right)\right|^{-1} \mathrm{d}\theta = \sum_{|z_n|<r} \log(r) + \sum_{|z_n|\ge r} \log|z_n|,$$

which tends to 0 by the Blaschke condition, $\sum(1 - |z_n|) < \infty$. □

A compact set $K \subset \mathbb{C}$ has positive logarithmic capacity if there exists a probability measure μ supported on K such that the logarithmic potential

$$u(z) = \int\limits_K \log \frac{1}{|z-w|} \, \mathrm{d}\mu(w)$$

is uniformly bounded. A countable set has logarithmic capacity zero while a rectifiable curve has positive logarithmic capacity. Sets of logarithmic capacity zero are small in terms of size; for instance, they have Hausdorff dimension zero. Let I be an inner function. A classical result by Frostman says that for all $\alpha \in \mathbb{D}$, except possibly for a set of logarithmic capacity zero, the function $\frac{I-\alpha}{1-\bar{\alpha} I}$ is a Blaschke product. See [18] or [19, p.75]. We now state the main result of this section:

Theorem 4. (A. Stray, 1988, [49]) *Assume the Nevanlinna-Pick problem* $(*)$ *has more than one solution. Then, for all* $\lambda \in \partial\mathbb{D}$*, except possibly for a set of logarithmic capacity zero, the function* $I_\lambda = \frac{P\lambda+Q}{R\lambda+S}$ *is a Blaschke product.*

Proof. Let $E = \{\lambda \in \partial\mathbb{D} : I_\lambda = \frac{P\lambda+Q}{R\lambda+S}$ is not a Blaschke product$\}$. We want to show that the logarithmic capacity of E is 0. Let μ be a probability measure supported in E with

$$\sup_{z\in\mathbb{C}} \left| \int\limits_E \log|z-w|^{-1} \, \mathrm{d}\mu(w) \right| = M < \infty.$$

We want to show that $\mu(E) = 0$. By Lemma 1, it is enough to show that

$$\int\limits_E \lim_{r\to 1} \int_0^{2\pi} \log \left| I_\lambda \left(r\mathrm{e}^{i\theta} \right) \right|^{-1} \mathrm{d}\theta \, \mathrm{d}\mu(\lambda) = 0.$$

We know that $\rho\left(r\mathrm{e}^{i\theta}\right) \to 1$ at almost every $\mathrm{e}^{i\theta}$, hence, given $\varepsilon > 0$ and $\eta > 0$ there exists $K \subset \partial\mathbb{D}$, $|K| \geq 2\pi - \varepsilon$ with $\rho(r\mathrm{e}^{i\theta}) \geq 1-\eta$ for $\mathrm{e}^{i\theta} \in K$ if $1-r$ is sufficiently small. Since $I_\lambda\left(r\mathrm{e}^{i\theta}\right)$ is a boundary point of $\Delta\left(r\mathrm{e}^{i\theta}\right)$, we deduce that $\left|I_\lambda\left(r\mathrm{e}^{i\theta}\right)\right| \geq 1-2\eta$ for $\mathrm{e}^{i\theta} \in K$ and $\int\limits_K \log\left|I_\lambda\left(r\mathrm{e}^{i\theta}\right)\right|^{-1} \mathrm{d}\theta \leq C\eta$ for any $\lambda \in \partial\mathbb{D}$ if $1-r$ is sufficiently small. So, it is enough to show

$$\int\limits_E \lim_{r\to 1} \int\limits_{\partial\mathbb{D}\setminus K} \log \left| I_\lambda \left(r\mathrm{e}^{i\theta} \right) \right|^{-1} \mathrm{d}\theta \, \mathrm{d}\mu(\lambda) = 0.$$

Using that S is outer, one can show that

$$\int\limits_{\partial\mathbb{D}\setminus K} \log\left|S\left(r\mathrm{e}^{i\theta}\right)\right| \mathrm{d}\theta \xrightarrow[r\to 1]{} \int\limits_{\partial\mathbb{D}\setminus K} \log\left|S\left(\mathrm{e}^{i\theta}\right)\right| \mathrm{d}\theta.$$

Now, by Fatou's lemma and Fubini, one has

$$\int\limits_E \lim_{r\to 1} \int\limits_{\partial\mathbb{D}\setminus K} \log \left| I_\lambda \left(r\mathrm{e}^{i\theta} \right) \right|^{-1} \mathrm{d}\theta \, \mathrm{d}\mu(\lambda) \leq$$

$$\leq \liminf_{r\to 1} \int\limits_{\partial\mathbb{D}\setminus K} \int\limits_E \log \left| \frac{\frac{P}{S}\left(r\mathrm{e}^{i\theta}\right)\lambda + \frac{Q}{S}\left(r\mathrm{e}^{i\theta}\right)}{\frac{R}{S}\left(r\mathrm{e}^{i\theta}\right)\lambda + 1} \right|^{-1} \mathrm{d}\mu(\lambda)\mathrm{d}\theta.$$

Since

$$\sup_{z\in\mathbb{C}} \left| \int_{\partial\mathbb{D}} \log|z-\lambda|^{-1} \, \mathrm{d}\mu(\lambda) \right| = M < \infty,$$

we have

$$\int\limits_E \log\left|\frac{R}{S}\left(r\mathrm{e}^{i\theta}\right)\lambda + 1\right| \mathrm{d}\mu(\lambda) \leq M$$

and

$$\int\limits_E \log\left|\frac{P}{S}\left(re^{i\theta}\right)\lambda+\frac{Q}{S}\left(re^{i\theta}\right)\right|^{-1} d\mu(\lambda)\leq -\log\left(\max\left(\left|\frac{P}{S}\left(re^{i\theta}\right)\right|,\left|\frac{Q}{S}\left(re^{i\theta}\right)\right|\right)\right)+M.$$

Since $\frac{P}{S}-\frac{Q}{S}\frac{R}{S}=\frac{B}{S^2}$, we deduce that

$$\max\left(\left|\frac{P}{S}\left(re^{i\theta}\right)\right|,\left|\frac{Q}{S}\left(re^{i\theta}\right)\right|\right)\geq\frac{1}{2}\frac{\left|B\left(re^{i\theta}\right)\right|}{\left|S\left(re^{i\theta}\right)\right|^2}.$$

Hence,

$$\int\limits_E \lim_{r\to1}\int\limits_{\partial\mathbb{D}\setminus K}\log\left|I_\lambda\left(re^{i\theta}\right)\right|^{-1} d\theta\, d\mu(\lambda)\leq$$

$$\leq CM|\partial\mathbb{D}\setminus K|+C\lim_{r\to1}\inf\left(\int\limits_{\partial\mathbb{D}\setminus K}\log\left|B\left(re^{i\theta}\right)\right|^{-1} d\theta+\int\limits_{\partial\mathbb{D}\setminus K}\log\left|S\left(re^{i\theta}\right)\right|^{2} d\theta\right).$$

The first integral tends to zero as $r\to1$ because of Lemma 1 and the second tends to $\int\limits_{\partial\mathbb{D}\setminus K}\log\left|S\left(e^{i\theta}\right)\right|^2 d\theta$, which is arbitrarily small if $|\partial\mathbb{D}\setminus K|$ is small. □

We now mention an application of last theorem. Let $K\subset\partial\mathbb{D}$ be a compact set of zero length and let $\varphi: K\to\overline{\mathbb{D}}$ be a continuous function. In [34] it is proved that there exists a Blaschke product I such that for any $e^{i\theta}\in K$, one has $\lim_{r\to1} I(re^{i\theta})=\varphi(e^{i\theta})$. The proof proceeds as follows. One first finds a non extremal function f_0 of the unit ball of H^∞ whose radial limit at each point of $e^{i\theta}\in K$ is $\varphi(e^{i\theta})$. R. Berman constructed a Blaschke product B with zeros $\{z_n\}$ such that the radial limit of B vanishes at each point of K. One considers the Nevanlinna-Pick problem with points $\{z_n\}$ and values $\{f_0(z_n)\}$. Then applying Theorem 4, one can choose I to be a convenient extremal solution.

4 Scaled Nevanlinna-Pick problems

A. Stray has found relations between the classical Nevanlinna-Pick problem and more modern topics in function theory, see [46], [47] and [48]. We will show that a certain refinement of the Corona Theorem provides convenient estimates of the radius of $\Delta(z)$.

4.1 The Corona Theorem

The Corona Theorem appeared as part of an effort to understand the Banach algebra properties of H^∞. It is actually equivalent to the non existence of a corona in the maximal ideal space M, that is, that $\mathbb{D}$ is dense in M. The techniques introduced by Carleson in his solution had a huge impact in both Complex and Harmonic Analysis.

Theorem 5. (Carleson, 1962, [10] or [19, Chapter VIII]*). Let $f_1,\dots,f_n\in H^\infty$ such that*

$$\inf_{z\in\mathbb{D}}\sum_{j=1}^n|f_j(z)|\geq\delta>0.$$

Then there exist $g_1,\dots,g_n\in H^\infty$ with $\sum_{j=1}^n f_jg_j\equiv1$.

This famous result has been extremely influential. For instance, in his proof, Carleson invented a geometric construction known as the Corona construction that has led to many deep results in the theory of H^∞ as well as in harmonic analysis and many other areas. Another simpler proof based on Littlewood-Paley integrals was obtained by T. Wolff in the eighties, see [19, p.315]. Among other important concepts, Carleson introduced the notion of what we know today as Carleson measure (for the Hardy space). It is not known if the Corona Theorem holds for any domain in the complex plane. It is also open in the unit ball of $\mathbb{C}^N$, $N>1$ or in the polydisc. On the negative direction, it is known that the Corona Theorem fails in Riemann surfaces. The proof by Carleson is quite technical but contains also the notion of Carleson contour which will appear later, so let us describe it. In general, the level set of a bounded analytic function may be unrectifiable. Actually, P. Jones constructed $f \in H^\infty$, $||f||_\infty \le 1$ such that for any $\varepsilon \in (0,1)$ the level set $\{z \in \mathbb{D} : |f(z)| = \varepsilon\}$ has infinite lenght; see [24]. Carleson constructed a variant of a level set which is rectifiable.

Lemma 2. (Carleson, [10],[19, p.333]*). Let $f \in H^\infty$, $||f||_\infty = 1$ and $0<\varepsilon<1$. Then, there exists $\delta = \delta(\varepsilon) > 0$ and $\Gamma = \Gamma(\varepsilon) = \cup_j \Gamma_j$, where $\Gamma_j = \Gamma_j(\varepsilon)$ are piecewise $\mathcal{C}^1$ closed curves with interior $\mathring{\Gamma}_j$ such that*

(a) $|f(z)| \ge \varepsilon$ *if* $z \in \mathbb{D} \setminus \cup \mathring{\Gamma}_j$.

(b) $|f(z)| \le \delta$ *if* $z \in \cup \mathring{\Gamma}_j$.

(c) Lenght on Γ *is a Carleson measure, that is, there exists* $C = C(\varepsilon) > 0$ *such that* lenght$(D \cap \Gamma) \le Cr$ *for any disc* D *of radius* r.

We will use the following refinement of the Corona Theorem:

Theorem 6. (P. Jones, [25]) *Let* $f_1, f_2 \in H^\infty$, $||f_i||_\infty \le 1$, $i = 1,2$. *Assume* $1/2 > \eta > 0$ *satisfies*

$$\inf_{z \in \mathbb{D}} |f_1(z)| + |f_2(z)| \ge 1 - \eta.$$

Then there exist $g_1, g_2 \in H^\infty$ *with* $f_1 g_1 + f_2 g_2 \equiv 1$ *and*

$$\sup_{z \in \mathbb{D}} |f_1(z) g_1(z)| + |f_2(z) g_2(z)| \le 1 + \frac{A}{\log\left(\frac{1}{\eta}\right)},$$

where A *is an absolute constant.*

4.2 The Radius of a Scaled Problem

A Nevanlinna-Pick problem is called scaled if it has a solution f_0 such that $||f_0||_\infty < 1$. The crucial idea in this section is due to A. Stray, see [50], who, using the result of P. Jones stated above, observed that one can estimate the radius of $\Delta(z)$ of an scaled problem.

Lemma 3. ([50]) *Assume* $(*)$ *is an scaled Nevanlinna-Pick problem . Then,* $\rho(z) \to 1$ *as* $|B(z)| \to 1$.

Proof. Take $\varepsilon > 0$ small, to be fixed later. Fix $z \in \mathbb{D}$ with $|B(z)| \ge 1 - \varepsilon$. Consider the functions $B(w)$ and $\tau_z(w) = \frac{w-z}{1-\bar{z}w}$. Then, since by Schwarz's lemma

$$\left| \frac{B(w) - B(z)}{1 - \overline{B(z)} B(w)} \right| \le \left| \frac{w-z}{1-\bar{z}w} \right| = |\tau_z(w)|,$$

we deduce that $|B(w)| + |\tau_z(w)| \ge 1 - C(\varepsilon)$, where $C(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then, by Theorem 6, there exist $g_1, g_2 \in H^\infty$ such that $Bg_1 + \tau_z g_2 \equiv 1$ and $|B(w)g_1(w)| + |\tau_z(w) g_2(w)| \le 1 + A(\varepsilon)$ for any $w \in \mathbb{D}$. Here, $A(\varepsilon) \to 0$ as $\varepsilon \to 0$. Let f_0 be a solution of $(*)$ with $||f_0||_\infty < 1$. Then, for any $s \in \overline{\mathbb{D}}$, the function $f_s = f_0 \tau_z g_2 + \frac{sBg_1}{1+A(\varepsilon)}$ is a solution of $(*)$ if $\varepsilon > 0$ is chosen small enough so that $||f_0||_\infty \le \frac{1}{1+A(\varepsilon)}$. Now,

$$f_s(z) = \frac{s}{1+A(\varepsilon)} B(z) g_1(z) = \frac{s}{1+A(\varepsilon)}.$$

Hence, $\Delta(z)$ contains the disc $\{w \in \mathbb{C} : |w| \le \frac{1}{1+A(\varepsilon)}\}$. Since $\varepsilon > 0$ can be taken arbitrarily small, this finishes the proof of the lemma. □

Lemma 3 was used in [36] to study the Nevanlinna coefficients of a scaled problem.

In the last lecture we will discuss the following question: *If we know an additional information of the sequence of points* $\{z_n\}$*, what can be deduced about the behaviour of the extremal solutions of the*

Nevanlinna-Pick problem $(*)$*?* Let us first consider several classes of inner functions. For $0<\alpha<1$, the class $\mathcal{B}_\alpha$ consists of the Blaschke products B for which its zero sequence satisfies

$$\sum_{z:B(z)=0} (1-|z|)^{1-\alpha} < \infty.$$

The following result can be found in the dissertation of L. Carleson, [9]. Let B be a Blaschke product. Then, $B \in \mathcal{B}_\alpha$ if and only if

$$\int_{\mathbb{D}} \frac{\log |B(z)|^{-1}}{(1-|z|^2)^{1+\alpha}} \, \mathrm{d}A(z) < \infty. \tag{2}$$

This fact follows easily applying second Green's formula to the functions $\log|B(z)|^{-1}$ and $(1-|z|^2)^{1-\alpha}$. Our second class is defined as follows. An inner function I is in (the Hardy-Sobolev space) $H^{1,\alpha}$ if I' is in the Hardy space H^α, that is,

$$\sup_{r<1} \int_0^{2\pi} |I'\left(r\mathrm{e}^{i\theta}\right)|^\alpha \, \mathrm{d}\theta < \infty. \tag{3}$$

Let $\mathrm{d}A(z)$ be the two dimensional Lebesgue measure. For $\frac{1}{2}<\alpha<1$ and $1 \le p \le 2$, it is well known that $I \in H^{1,\alpha}$ if and only if

$$\int_{\mathbb{D}} |I'(z)|^p (1-|z|)^{p-\alpha-1} \, \mathrm{d}A(z) < \infty \tag{4}$$

See Theorem 13 of [14]. So, in this sense, inner functions do not distinguish between these Hardy-Sobolev and Besov spaces. The class of inner functions in $H^{1,\alpha}$ have been extensively studied by many authors but there is still no description of the inner functions I in $H^{1,\alpha}$ in terms of the geometry of its zero set and the behaviour of its associated singular measure. In 1973, Protas proved that for $\frac{1}{2}<\alpha<1$, $\mathcal{B}_\alpha \subset H^{1,\alpha}$, see [39]. The converse is not true, but Ahern proved that if an inner function $I \in H^{1,\alpha}$ then there exists $w \in \mathbb{D}$ such that $\frac{I-w}{1-\overline{w}I} \in \mathcal{B}_\alpha$ (see [3]). The paper [3] contains also many interesting related results. There is also a beautiful description of inner functions I in $H^{1,\alpha}$, $\frac{1}{2}<\alpha<1$, in terms of Carleson contours given by Cohn, see [12], which reads as follows: *Fix* $\frac{1}{2}<\alpha<1$*. An inner function* I *is in* $H^{1,\alpha}$ *if and only if*

$$\int_\Gamma \frac{|\mathrm{d}z|}{(1-|z|)^\alpha} < \infty,$$

where Γ *is a Carleson contour of* I*.*

Smoothness properties of inner functions have attracted the attention of many researchers. See [4], [5], [6], [7], [51], [13], [15], [16], [21], [20], [23], [37], [38], [40], [41], [8], and the monography [28]. Observe that in the case $\alpha=1$, either condition (3) or (4) implies that I is a finite Blaschke product. For instance, if (3) holds, then I extends continuously to the closed unit disc and hence it is a finite Blaschke product. If (4) holds and $p=2$, condition (4) tells that the area of the image $I(\mathbb{D})$, counting multiplicities, must be finite. Then, I is a finite Blaschke product. The general case $1<p<\infty$, $p\neq 2$ was considered in [26]. So, in the case $\alpha=1$, it is natural to consider weak spaces. An inner function I is in the class $H^{1,1}_\infty$ if there exists a constant $C>0$ such that for any $0<r<1$ and any $\lambda>0$ one has

$$\left|\{\mathrm{e}^{i\theta} : \left|I'\left(r\mathrm{e}^{i\theta}\right)\right| > \lambda\}\right| \le \frac{C}{\lambda}.$$

Here, $|E|$ denotes the length of the measurable set $E \subset \partial\mathbb{D}$. It is well known that this last condition holds if and only if the non-tangential maximal function $M(I')$ of I' satisfies the weak estimate

$$\left|\{\mathrm{e}^{i\theta} : M\left(I'\right)\left(\mathrm{e}^{i\theta}\right) > \lambda\}\right| \le \frac{C_1}{\lambda}$$

for any $\lambda>0$. For $1<p<\infty$ consider the measure $\mathrm{d}\mu_p(z)=(1-|z|)^{p-2}\mathrm{d}A(z)$. The weak analogue of condition (4) would read as follows: *An inner function* I *is in the weak Besov space* B^p_∞ *if there exists a constant* $C>0$ *such that for any* $\lambda>0$ *one has* $\mu_p\{z\in\mathbb{D} : |I'(z)|>\lambda\} \le C\lambda^{-p}$*.* In the papers [11] and [22], written in collaboration with J. Cima and J. Grohn, it was proved that for any

$1 < p < \infty$, $I \in H^{1,1}_\infty$ if and only if $I \in B^p_\infty$, and this holds if and only if I is a Blaschke product for which there exists a constant $C = C(I) > 0$ such that for any $n = 1, 2, \ldots$ one has

$$\#\{z : I(z) = 0,\, 2^{-n-1} < 1 - |z| \leq 2^{-n}\} \leq C. \tag{5}$$

It is easy to show that the sequence of zeros of I satisfies condition (5) if and only if it is the union of finitely many sequences which approach the unit circle exponentially fast. Let us denote $\mathcal{B}_1$ the class of Blaschke products satisfying (5). It is obvious that $\mathcal{B}_1 \subset \mathcal{B}_\alpha$ for any $0 < \alpha < 1$. We can now state the result on the Nevanlinna-Pick problem .

Theorem 7. [31] *Let $(*)$ be a scaled Nevanlinna-Pick problem and let B be the Blaschke product with zeros $\{z_n\}$. Let I_λ, $\lambda \in \partial\mathbb{D}$, be the extremal solutions of $(*)$.*

(a) Fix $0 < \alpha < 1$ and assume $B \in H^{1,\alpha}$. Then, $I_\lambda \in H^{1,\alpha}$ for any $\lambda \in \partial\mathbb{D}$.

(b) Assume $B \in \mathcal{B}_1$. Then, $I_\lambda \in \mathcal{B}_1$ for any $\lambda \in \partial\mathbb{D}$.

(c) Fix $0 < \alpha < 1$. Assume

$$\sum_n (1 - |z_n|)^{1-\alpha} \log\left(\frac{1}{1 - |z_n|}\right) < \infty.$$

Then, for all $\lambda \in \partial\mathbb{D}$, except possibly for a set of logarithmic capacity zero, $I_\lambda \in \mathcal{B}_\alpha$.

We shall not prove this result but we will make a few comments about it. We do not know if the assumption that $(*)$ be scaled is essential. The main obstacle is that there is no analogue to Lemma 3 for non-scaled problems, see [35]. We do not know wether condition (c) holds under the weaker assumption that $B \in \mathcal{B}_\alpha$. The essential tool in the proof of (a) and (b) is Lemma 3, where the assumption $(*)$ scaled is used. The proof of (c) uses the description (2) and arguments similar to the proof of Stray's theorem. We state an easy consequence Theorem 7.

Corollary 4. *Let $\{z_n\}$ be a Blaschke sequence and let B be the corresponding Blaschke product. Let $\{w_n\}$ be a bounded sequence of complex numbers such that*

$$M = \sup\{||f||_\infty \,:\, f \in H^\infty,\, f(z_n) = w_n,\, n = 1, 2, \ldots\} < \infty.$$

Fix $\varepsilon > 0$.

(a) There exists a Blaschke product I with $(M + \varepsilon)I(z_n) = w_n$, $n = 1, 2, \ldots$.

(b) Fix $0 < \alpha < 1$. Assume $B \in H^{1,\alpha}$. Then, there exists a Blaschke product $I \in H^{1,\alpha}$ with $(M + \varepsilon)I(z_n) = w_n$, $n = 1, 2, \ldots$.

(c) Assume $B \in \mathcal{B}_1$. Then, there exists a Blaschke product $I \in \mathcal{B}_1$ with $(M+\varepsilon)I(z_n) = w_n$, $n = 1, 2, \ldots$.

(d) Fix $0 < \alpha < 1$. Assume $\sum(1 - |z_n|)^{1-\alpha}|\log(1 - |z_n|)| < \infty$, then there exists $I \in \mathcal{B}_\alpha$ with $(M + \varepsilon)I(z_n) = w_n$, $n = 1, 2, \ldots$.

This result follows from Theorems 4 and 7 because for $\varepsilon > 0$ the Nevanlinna-Pick problem

$$(*) \text{ Find } f \in H^\infty,\, ||f||_\infty \leq 1,\, f(z_n) = \tfrac{w_n}{M+\varepsilon},\, n = 1, 2, \ldots$$

is scaled. It is worth mentioning that the result does not hold when $\varepsilon = 0$, see [45].

We finally state an open question which could have applications: *Let δ_z be the Dirac mass at the point $z \in \mathbb{D}$. Assume $\sum(1 - |z_n|)\delta_{z_n}$ is a Carleson measure (for the Hardy space H^2). Is it then true that there exists $\lambda \in \partial\mathbb{D}$ such that*

$$\sum_{z\,:\,I_\lambda(z)=0} (1 - |z|)\delta_z$$

is a Carleson measure?

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