The Journal of Fourier Analysis and Applications

Volume 8, Issue 6, 2002

Doubling Properties of A_∞

María José González and Artur Nicolau

Communicated by Guido Weiss

ABSTRACT. *We present a new characterization of A_∞ weights in terms of Carleson measures that involve the doubling properties of the weight. We prove that a doubling weight ω belongs to A_∞ if and only if*

$$\left|1-\frac{\omega(I_z^+)}{\omega(I_z^-)}\right|^2\frac{dx\,dy}{y}$$

is a Carleson measure in $\mathbb{R}_2^+$, where $z=x+iy$, and I_z^+, I_z^- denote the right and left half of the interval $I_z=(x-y,x+y)$. A similar result holds in $\mathbb{R}^n$, $n>1$.

1. Introduction

The purpose of this note is to prove a new characterization of A_∞ weights. We shall present a criterion in terms of Carleson measures involving the doubling properties of the weight. We will consider weights on $\mathbb{R}$, and show at the end of this article how the argument can be modified to obtain the equivalent result in $\mathbb{R}^n$, $n>1$. A locally integrable positive function ω is in the class A_∞ if for any α, $0<\alpha<1$, there exists β, $0<\beta<1$, such that for all intervals I and all subsets $E\subset I$,

$$|E|\leqslant\alpha\,|I|\Rightarrow\omega(E)\leqslant\beta\omega(I)$$

where $\omega(E)=\displaystyle\int_E\omega\,dx$. It is then an immediate consequence that any $\omega\in A_\infty$ is a doubling weight, that is, it is a locally integrable function for which there exists a constant $c>0$ such that for any interval $I\subset\mathbb{R}$

$$\omega\left(\tilde{I}\right)\leqslant c\omega(I)$$

Math Subject Classifications. primary 42B25, 26B35.

Keywords and Phrases. A_∞ weights, Carleson measures, doubling condition.

Acknowledgements and Notes. Supported in part by the DGICYT grant PB98-0872 and CIRIT grant 2000 SGR00059.

ISSN 1069-5869

where $\tilde{I}$ denotes the interval with the same center as I, and double its length. An equivalent definition of A_∞ weights, which in fact it is the one we will use, is the following: A weight ω is in the class A_∞ if and only if

$$\left|\log \fint_I \omega - \fint_I \log \omega\right| \leqslant c \tag{1.1}$$

for some constant $c > 0$, and all intervals I, where $\fint_I \omega = \frac{1}{|I|}\int_I \omega$. See [4] for this and several other equivalent definitions as well as for the connection of this theory with singular integrals and BMO.

Our result is related to the work R. Fefferman, C. Kenig, and J. Pipher in [2], in particular they show that a doubling weight ω is in the class A_∞ exactly when

$$\frac{\left|\nabla_x(\omega * \Phi_y)\right|^2}{\left|\omega * \Phi_y\right|^2} y\, dx\, dy$$

is a Carleson measure on $\mathbb{R}^2_+$, where Φ is any non-negative function in the Schwarz class $\mathcal{S}(\mathbb{R})$, with $\int_{\mathbb{R}} \Phi\, dx = 1$, and $\Phi_y(x) = y^{-1}\Phi(x/y)$. Recall that a measure μ on $\mathbb{R}^2_+$ is called a Carleson measure if there exists $c > 0$, such that for any square Q of the form $Q = I \times [0, I]$, $\mu(Q) \leqslant c\,|I|$. The smoothness condition $\Phi \in \mathcal{S}(\mathbb{R})$ is essential. Actually, if $P(t) = \pi^{-1}(1+t^2)^{-1}$ is the Poisson kernel, the condition

$$\frac{\left|\nabla_x(\omega * P_y)\right|^2}{\left|\omega * P_y\right|^2} y\, dx\, dy$$

describes a class of weights, called invariant-A_∞, which is smaller than the class of A_∞ weights. See [5].

In [2] the dyadic situation is also considered: A weight ω is a dyadic doubling weight if $\omega(\tilde{I}) \leqslant c\omega(I)$, for all dyadic intervals I, where $\tilde{I}$ is the smallest dyadic interval properly containing I, and $\omega \in A^d_\infty$ if (1.1) holds for any dyadic interval I. In this context they provide a characterization of A^d_∞ in terms of the doubling properties of ω over dyadic intervals. More precisely, a dyadic doubling weight $\omega \in A^d_\infty$ if and only if for each dyadic interval I,

$$\sum_{J \subseteq I} \left(1 - \frac{2\omega(J)}{\omega\left(\tilde{J}\right)}\right)^2 |J| \leqslant c\,|I| \tag{1.2}$$

where the sum is taken over all dyadic intervals J contained in I. Related results can be found in [1, 5]. The characterization of A_∞ we prove can be understood as a non-dyadic version of (1.2). Before stating the theorem we need to introduce some notation: Given a point $z = (x, y) \in \mathbb{R}^2_+$, denote by I_z the interval on $\mathbb{R}$ centered at x and length y, and by I_z^+, I_z^- the right and left half of I_z, respectively.

Theorem 1.

A doubling weight ω belongs to A_∞ if and only if

$$\left|1 - \frac{\omega\left(I_z^+\right)}{\omega\left(I_z^-\right)}\right|^2 \frac{dx\, dy}{y}$$

is a Carleson measure in $\mathbb{R}^2_+$.

Results in a similar spirit can be found in [3], where relations between the doubling properties of a measure and its regularity properties such as being singular are studied. The rest of this article will be devoted to the proof of the theorem and to discuss its extension to $\mathbb{R}^n,\ n > 1$.

2. Proof of Theorem 1

Proof. Given $x \in \mathbb{R}$, denote by $\Gamma(x)$ the cone with vertex at x,

$$\Gamma(x) = \left\{ z = (t, y) \in \mathbb{R}^2_+,\ |x - t| < y,\ 0 < y < 1 \right\} .$$

Let us consider the area function

$$A(x) = \left(\int_{\Gamma(x)} \log \frac{2^{-1}\omega(I_z)}{\omega\left(I_z^+\right)^{1/2} \omega\left(I_z^-\right)^{1/2}} \frac{dm(z)}{y^2} \right)^{1/2}$$

and its truncated version $A_l^2(x)$ where the integral is defined over the truncated cone

$$\Gamma_l(x) = \Gamma(x) \cup \{0 \leqslant y \leqslant l\} .$$

Note that $\dfrac{2^{-1}\omega(I_z)}{\omega(I_z^+)^{1/2}\omega(I_z^-)^{1/2}}$ is exactly the ratio between the arithmetic mean and the geometric mean of the densities of ω over the intervals I_z^+ and I_z^-, that is $\omega(I_z^+)/\left|I_z^+\right|$ and $\omega(I_z^-)/\left|I_z^-\right|$. Since ω is a doubling weight the quantity

$$\eta(z) = \log \frac{2^{-1}\omega(I_z)}{\omega\left(I_z^+\right)^{1/2} \omega\left(I_z^-\right)^{1/2}}$$

is uniformly bounded by a constant only depending on the doubling constant of ω, that will be denoted by ρ. Note also, that the doubling condition of ω implies that $\eta(z)$ is comparable to $d^2(z) = \left| 1 - \dfrac{\omega(I_z^+)}{\omega(I_z^-)} \right|^2$ with comparison constant only depending on ρ. So $\eta(z)$ measures the error done by ω when doubling at the interval I_z.

To avoid notation, throughout the rest of this article c will always denote a quantity, which may change from line to line, and which is bounded by a constant only depending on ρ.

The theorem will be an easy consequence of the following estimate

$$\log \fint_I \omega - \fint_I \log \omega = \fint_I A^2_{|I|}(x)\, dx + c . \tag{2.1}$$

Let us first prove the theorem assuming (2.1). Denote by $Q = I \times [0, |I|]$ the Carleson square associated to the interval $I = [a, b]$. By Fubini and the fact that $\eta(z)$, $z = (t, y) \in \mathbb{R}^2_+$, is uniformly bounded, we obtain

$$\int_I A^2_{|I|}(x)\, dx = \int_I \left(\int_{\Gamma_{|I|}(x)} \frac{\eta(z)}{y^2}\, dm(z) \right) dx = \int_Q \frac{\eta(z)}{y}\, dm(z) + c\,|I| .$$

Hence

$$\fint_I A^2_{|I|}(x)\,dx = \frac{1}{|I|}\int_Q \frac{\eta(z)}{y}\,dm(z) + c\,. \tag{2.2}$$

So, assuming (2.1), since $\eta(z)$ is comparable to $d^2(z)$, the characterization of A_∞ given in Theorem 1 holds if and only if $d^2(z)y^{-1}\,dm(z)$ is a Carleson measure as we wanted to prove.

Next, observe that as a consequence of (2.2), (2.1) is equivalent to

$$\log \fint_I \omega - \fint_I \log\omega = \frac{1}{|I|}\int_Q \frac{\eta(z)}{y}\,dm(z) + c\,. \tag{2.3}$$

So, it only remains to show that (2.3) holds. Set $z = (x, y) \in \mathbb{R}^2_+$, and define

$$\Phi(y) = \int_{x\in I} \log \frac{2^{-1}\omega(I_z)}{\omega\left(I_z^+\right)^{1/2}\omega\left(I_z^-\right)^{1/2}}\,dx$$

then

$$\int_Q \frac{\eta(z)}{y}\,dm(z) = \int_0^{|I|}\frac{\Phi(y)}{y}\,dy = \int_{|I|/2}^{|I|}\sum_{k=0}^{\infty}\Phi\left(\frac{y}{2^k}\right)\frac{dy}{y}\,. \tag{2.4}$$

We will use some cancellation properties to estimate the sum $\sum \Phi(y/2^k)$. In fact, $\Phi(y) + \Phi(y/2)$ can be rewritten as

$$\begin{aligned}\Phi(y)+\Phi(y/2) = &\int_I \log\frac{\omega(x-y/2, x+y/2)}{\omega(x-y,x)^{1/2}\omega(x,x+y)^{1/2}}\,dx\\ &+\int_I \log\frac{\omega(x-y,x+y)/2^2}{\omega(x-y/2,x)^{1/2}\omega(x,x+y/2)^{1/2}}\,dx\,.\end{aligned}$$

To estimate the first integral, observe that

$$\begin{aligned}&\int_a^b \log\omega(x-y/2,x+y/2)\,dx - \frac12\int_a^b\log\omega(x-y,x)\,dx\\ &\quad-\frac12\int_a^b\log\omega(x,x+y)\,dx = \int_a^b\log\omega(x-y/2,x+y/2)\,dx\\ &\quad-\frac12\int_{a+y/2}^{b+y/2}\log\omega(x-y/2,x+y/2)\,dx - \frac12\int_{a-y/2}^{b-y/2}\log\omega(x,x+y/2)\,dx\\ &=\frac12\left(\int_a^{a+y/2}\log\omega(x-y/2,x+y/2)\,dx - \int_{a-y/2}^{a}\log\omega(x-y/2,x+y/2)\,dx\right)\\ &\quad+\frac12\left(\int_{b-y/2}^{b}\log\omega(x-y/2,x+y/2)\,dx - \int_b^{b+y/2}\log\omega(x-y/2,x+y/2)\,dx\right)\,,\end{aligned}$$

which can be bounded by cy, where c depends on the doubling constant of w. Therefore,

$$\Phi(y)+\Phi(y/2) = cy + \int_a^b \log\frac{\omega(x-y,x+y)/2^2}{\omega(x-y/2,x)^{1/2}\omega(x,x+y/2)^{1/2}}\,dx\,.$$

Repeating this argument k_0 times, we obtain

$$\sum_{k=1}^{k_0} \Phi\left(y/2^k\right) = cy + \int_a^b \log \frac{\omega(x-y, x+y)/2^{k_0+1}}{\omega\left(x - y/2^{k_0}, x\right)^{1/2} \omega\left(x, x + y/2^{k_0}\right)^{1/2}}\, dx\ .$$

Using the doubling condition of ω again, and setting $\delta = |I|/2_0^k$ we get

$$\begin{aligned}\int_{|I|/2}^{|I|} \sum_{k=1}^{k_0} \Phi\left(y/2^k\right) \frac{dy}{y} &= \sum_{k=1}^{k_0} \Phi\left(|I|/2^k\right) + c\,|I| \\ &= c\,|I| + \int_a^b \log \frac{\omega(x-|I|, x+|I|)/2\,|I|}{(\omega(x-\delta, x)/\delta)^{1/2}(\omega(x, x+\delta)/\delta)^{1/2}}\, dx \\ &= c_1\,|I| + |I| \log \frac{\omega(I)}{|I|} - \int_a^b \log \frac{\omega(x-\delta, x+\delta)}{2\delta}\, dx\ .\end{aligned}$$

So, letting $\delta \to 0$, by (2.4)

$$\int_Q \frac{\eta(z)}{y}\, dm(z) = c_1\,|I| + |I| \log \frac{\omega(I)}{|I|} - \int_I \log \omega(x)\, dx\ ,$$

which proves (2.3), and therefore Theorem 1. $\square$

Remark. The theorem can be extended to $\mathbb{R}^n,\ n > 1$, The area function to consider in this case would be

$$A^2(x) = \int_{\Gamma(x)} \log \frac{a.m(Q(w, y))}{g.m(Q(w, y))}\, dw \frac{dy}{y^{n+1}},\ \ x \in \mathbb{R}^n\ ,$$

where $Q(w, y)$ denotes the cube in $\mathbb{R}^n$ centered at $w \in \mathbb{R}^n$ of sidelength y, and $a.m$ ($g.m$) denote the arithmetic (geometric) mean of the density of ω over the 2^n disjoint subcubes, Q_k, of Q of sidelength $y/2$. Then, it can be shown ([3], Lemma 2.2) that

$$d^2(w, y) \simeq \log \frac{a.m(Q(w, y))}{g.m(Q(w, y))}$$

where

$$d^2(w, y) = \max_k \left|1 - \frac{2^n \omega(Q_k)}{\omega(Q)}\right|^2\ .$$

So, $d(w, y)$ measures the error done by ω when doubling at the cube Q centered at w and sidelength y. Following similar arguments as in the one dimensional case, it can be proved that a doubling weight ω belongs to $A_\infty(\mathbb{R}^n)$ if and only if $d^2(w, y)\, dx\, dy/y$ is a Carleson measure in $\mathbb{R}_+^{n+1}$. We omit the details.

References

[1] Buckley, S.M. (1993). Summation conditions on weights, *Michigan Math. J.,* **40**, 153–170.

[2] Fefferman, R., Kenig, C., and Pipher, J. (1991). The theory of weights and the Dirichlet problem for elliptic equations, *Ann. of Math.,* **134**, 65–124.

[3] Gonzales, M.J. and Nicolau, A. Multiplicative square functions, preprint.

[4] Stein, E.M. (1993). *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscilating Integrals*, Princeton University Press.

[5] Treil, S., Volberg, A. and Zheng, D. (1997). Hilbert transform, Toeplitz operators and Hankel operators and invariant A_∞ weights, *Rev. Mat. Iberoamericana*, **13**, 319–360.

Received May 07, 2001

Departamento de Matemáticas, Universidad de Cádiz, Apartado 40, 11510 Puerto Real (Cádiz), Spain
e-mail:majose.gonzalez@uca.es

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Spain
e-mail: artur@mat.uab.es