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Differentiability of functions in the Zygmund class

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Abstract

Let f be a function in the Zygmund class in the euclidean space. It is proved that the Hausdorff dimension of the set of points where f has bounded divided differences, is bigger or equal to one. Furthermore, if f is in the Small Zygmund class, then the Hausdorff dimension of the set of points where f is differentiable, is bigger or equal to one. The sharpness of these results is also discussed.

1. *Introduction*

Given two parameters $b > 1$ and $0 < \alpha \leqslant 1$, in 1876, Weierstrass considered the functions given by the series

$$\sum_{n=0}^{\infty} b^{-n\alpha} \cos(2\pi b^n x), \quad x \in \mathbb{R},$$

and proved that they are continuous and nowhere differentiable if the parameters satisfy $b^{1-\alpha} \geqslant 1 + 3\pi/2$. After a series of contributions by Dini, Bromwich, Hadamard and others, in 1916, Hardy gave a proof under the optimal assumptions $b > 1$ and $0 < \alpha \leqslant 1$; see [7]. The critical case $\alpha = 1$ is the hardest one and it corresponds to the function

$$f_b(x) = \sum_{n=0}^{\infty} b^{-n} \cos(2\pi b^n x), \quad x \in \mathbb{R}, \tag{1.1}$$

which is in the Zygmund class. The Zygmund class $\Lambda_*(\mathbb{R}^d)$ is the space of bounded continuous functions $f\colon \mathbb{R}^d \to \mathbb{R}$ for which

$$\|f\|_* = \sup\left\{\frac{|f(x+h) + f(x-h) - 2f(x)|}{\|h\|} : x, h \in \mathbb{R}^d\right\} < \infty.$$

The small Zygmund class $\lambda_*(\mathbb{R}^d)$ is the subspace formed by those functions $f \in \Lambda_*(\mathbb{R}^d)$ which satisfy

$$\lim_{\|h\|\to 0} \sup_{x\in\mathbb{R}^d} \frac{|f(x+h) + f(x-h) - 2f(x)|}{\|h\|} = 0.$$

These spaces were introduced by Zygmund in the 1940s when he observed that the conjugate function of a Lipschitz function in the unit circle does not need to be Lipschitz but it is in the Zygmund class [21]. For $0 < \alpha \leqslant 1$, let $\Lambda_\alpha(\mathbb{R}^d)$ be the Hölder class of bounded functions $f\colon \mathbb{R}^d \to \mathbb{R}$ for which there exists a constant $C = C(f)$ such that $|f(x+h) - f(x)| \leqslant C\|h\|^\alpha$, for any $x, h \in \mathbb{R}^d$. It is well known that $\Lambda_1(\mathbb{R}^d) \subset \Lambda_*(\mathbb{R}^d) \subset \Lambda_\alpha(\mathbb{R}^d)$ for any $0 < \alpha < 1$ and actually the Zygmund class $\Lambda_*(\mathbb{R}^d)$ is the natural substitute of $\Lambda_1(\mathbb{R}^d)$ in many different contexts. For instance, the Hilbert transform of a compactly supported function in $\mathbb{R}$ may

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not be in $\mathbb{R}$, while standard Calderón–Zygmund operators map compactly supported functions in $\Lambda_*(\mathbb{R}^d)$ (respectively, in $\Lambda_\alpha(\mathbb{R}^d)$ for some fixed $0 < \alpha < 1$) into $\Lambda_*(\mathbb{R}^d)$ (respectively, into $\Lambda_\alpha(\mathbb{R}^d)$); see [6]. The Zygmund class can also be described in terms of harmonic extensions, Bessel potentials or best polynomial approximation and again it is the natural substitute of the Lipschitz class $\Lambda_1(\mathbb{R}^d)$ in these contexts; see [18, Chapter 5; 21, 22].

A classical result of Rademacher says that any function in $\Lambda_1(\mathbb{R}^d)$ is differentiable at almost every point. However, functions in the Zygmund class as the Hardy–Weierstrass function f_b given in (1.1), may not be differentiable at any point. More generally, let g be an almost periodic function of class $\mathcal{C}^2$ in the real line. Then for any $b > 1$, the function

$$f(x) = \sum_{n=0}^{\infty} b^{-n} g(b^n x), \quad x \in \mathbb{R}$$

is in the Zygmund class $\Lambda_*(\mathbb{R})$ and under mild assumptions on the function g, Heurteaux has proved that f is nowhere differentiable [8]. It is worth mentioning that if we allow having an infinite derivative, then every Zygmund class function defined in the real line has derivative on a set of points of Hausdorff dimension 1; see [14, p. 237].

In the 1960s, Stein and Zygmund proved a series of nice results relating differentiability properties of functions with the size of certain natural square functions. Let f be a measurable function defined in an open set $\Omega \subset \mathbb{R}^d$. Then the set of points of Ω where f is differentiable coincides, except at most for a set of Lebesgue measure zero, with the set of points $x \in \Omega$ for which there exists $\delta = \delta(x) > 0$ such that the following two conditions hold:

$$\sup_{\|h\|<\delta} \frac{|f(x+h) + f(x-h) - 2f(x)|}{\|h\|} < \infty,$$

$$\int_{\|h\|<\delta} \frac{|f(x+h) + f(x-h) - 2f(x)|^2}{\|h\|^{d+2}}\, dm(h) < \infty.$$

Here dm denotes Lebesgue measure in $\mathbb{R}^d$. Hence, differentiability can be described, modulo sets of Lebesgue measure zero, by the size of a quadratic expression involving second differences; see [18, p. 262]. However, these nice results do not apply in the situation we will consider in this paper where we study differentiability properties of functions in sets which may have Lebesgue measure zero.

Let f_b be the Weierstrass function given by (1.1). As mentioned before $f_b \in \Lambda_*(\mathbb{R})$ and one can see that

$$\limsup_{h\to 0} \frac{|f_b(x+h) - f_b(x)|}{|h|} = \infty,$$

at almost every point $x \in \mathbb{R}$. Actually, a Law of the Iterated Logarithm governs the growth of the divided differences of f_b (see [2]). Similarly, there exist functions in the small Zygmund class which are differentiable at almost no point. However, it was already observed by Zygmund [21] that any function in $\lambda_*(\mathbb{R})$ is differentiable at a dense set of points of the real line. Similarly, a function in the Zygmund class $\Lambda_*(\mathbb{R})$ has bounded divided differences at a dense set of points. In the 1980s, Makarov proved that Zygmund functions on the real line have bounded divided differences at sets of Hausdorff dimension one. See [12] and also [1, 16, 17] for related results.

THEOREM A (Makarov). (a) *Let $f \in \Lambda_*(\mathbb{R})$. Then the set*

$$\left\{ x \in \mathbb{R} : \limsup_{h\to 0} \frac{|f(x+h) - f(x)|}{|h|} < \infty \right\}$$

has Hausdorff dimension 1.

(b) *Let* $f \in \lambda_*(\mathbb{R})$. *Then the function* f *is differentiable at a set of points of Hausdorff dimension* 1.

The main purpose of this paper is to study the situation in higher dimensions. Given a function $f \in \Lambda_*(\mathbb{R}^d)$ and a unit vector $e \in \mathbb{R}^d$, let $E(f,e)$ be the set of points where the divided differences of f in the direction of e are bounded, that is,

$$E(f,e) = \left\{ x \in \mathbb{R}^d : \limsup_{\mathbb{R} \ni t \to 0} \frac{|f(x+te) - f(x)|}{|t|} < \infty \right\}.$$

There exist functions $f \in \Lambda_*(\mathbb{R}^d)$ such that, for any unit vector $e \in \mathbb{R}^d$, the set $E(f,e)$ has Lebesgue measure zero. However, the one-dimensional result of Makarov gives that for any function $f \in \Lambda_*(\mathbb{R}^d)$ and any fixed unit vector $e \in \mathbb{R}^d$, the set $E(f,e)$ has Hausdorff dimension d; see also [**10**, **13**]. Similarly, for a function $f \in \lambda_*(\mathbb{R}^d)$ and a unit vector $e \in \mathbb{R}^d$, the set

$$\left\{ x \in \mathbb{R}^d : \lim_{\mathbb{R} \ni t \to 0} \frac{f(x+te) - f(x)}{t} \text{ exists} \right\}$$

may have Lebesgue measure zero but it has Hausdorff dimension d. For a fixed direction e, the divided differences in this direction, $(f(x+te) - f(x))/t,\ x \in \mathbb{R}^d$, satisfy a certain mean value property with respect to Lebesgue measure in $\mathbb{R}^d$. This is the main point in the proof of Makarov's result as well as in the arguments leading to the fact that $\dim E(f,e) = d$. In this paper, we want to study the size of the set $E(f)$ of points where the divided differences in any direction are simultaneously bounded, that is,

$$E(f) = \left\{ x \in \mathbb{R}^d : \limsup_{\|h\| \to 0} \frac{|f(x+h) - f(x)|}{\|h\|} < \infty \right\}.$$

Let $\{e_i : i = 1, \dots, d\}$ be the canonical basis of $\mathbb{R}^d$. If $f \in \Lambda_*(\mathbb{R}^d)$, then it turns out that

$$E(f) = \bigcap_{i=1}^{d} E(f, e_i).$$

So, the main difficulty in the higher dimensional situation is to obtain a simultaneous control of the divided differences in different directions e_i, $i = 1, \dots, d$. The first main result of this paper is the following.

THEOREM 1. (a) *Let* f *be a function in* $\Lambda_*(\mathbb{R}^d)$. *Then the set* $E(f)$ *has Hausdorff dimension bigger or equal to* 1.

(b) *Let* f *be a function in* $\lambda_*(\mathbb{R}^d)$. *Then* f *is differentiable at a set of points of Hausdorff dimension bigger or equal to* 1.

The result is local in the sense that given $f \in \Lambda_*(\mathbb{R}^d)$ and a cube $Q \subset \mathbb{R}^d$ the set $E(f) \cap Q$ has Hausdorff dimension bigger or equal 1. Similarly, given $f \in \lambda_*(\mathbb{R}^d)$ and a cube $Q \subset \mathbb{R}^d$, the function f is differentiable at a set of points in the cube Q which has Hausdorff dimension bigger or equal than 1.

The proof of this result consists of constructing a Cantor-type set on which the function f has bounded divided differences. The construction of the Cantor-type set uses a stopping time argument based on a certain one-dimensional mean value property that the divided differences of f satisfy. Roughly speaking, the divided differences distribute their values in a certain uniform way when measured with respect to length. This is the main new idea in the proof and it allows us to obtain a simultaneous control of the behavior of the divided differences in the coordinate directions. Moreover, the result is sharp in the following sense.

THEOREM 2. *There exists a function f in the small Zygmund class $\lambda_*(\mathbb{R}^d)$ such that the set $E(f)$ has Hausdorff dimension* 1.

The one-dimensional case may suggest that a natural candidate for the function f in Theorem 2 is a lacunary series. However, this is not the case. Actually, it turns out that natural lacunary series f in $\Lambda_*(\mathbb{R}^d)$ satisfy $\dim E(f) = d$ (see [5]). Instead, the function f will be constructed as $f = \sum g_k$, where $\{g_k\}$ will be a sequence of smooth functions defined recursively with $\sum \|g_k\|_\infty < \infty$. The main idea is to construct them in such a way that $\nabla g_{k+1}(x)$ is *almost orthogonal* to $\nabla \sum_{j=1}^{k} g_j(x)$ and $\sum \|\nabla g_k(x)\|^2 = \infty$ for *most* points $x \in \mathbb{R}^d$. Since one cannot hope to achieve both requirements at all points $x \in \mathbb{R}^d$, an exceptional set A appears. It turns out that the function f is in the Small Zygmund class and it is not differentiable at any point in $\mathbb{R}^d \setminus A$. The construction provides the convenient *one-dimensional* estimates of the size of the set A.

It is a pleasure to thank David Preiss for an illuminating idea concerning Theorem 2 and to the anonymous referees for pointing out that the idea of gradient moving orthogonally is behind Reifenberg's Topological Disk Theorem [15] and has been used to construct somewhat similar functions in the so-called *gradient problem* posed by C. E. Weil. A function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies the *Denjoy–Clarkson property* if for any open interval $I \subset \mathbb{R}$, then $f^{-1}(I)$ is either empty or has n-dimensional positive measure. The gradient problem asks whether differentiable functions in $\mathbb{R}^n$ satisfy the Denjoy–Clarkson property. A classical theorem says that the answer is positive if $n = 1$ but the problem has been open for a long time when $n > 1$. Buczolich [3] proved that $f^{-1}(I)$ is either empty or has Hausdorff dimension at least 1 for any open interval $I \subset \mathbb{R}$ and any differentiable $f : \mathbb{R}^n \to \mathbb{R}$; see also [20]. Buczolich [4] finally solved the gradient problem in the negative giving a counterexample in $\mathbb{R}^2$. While the parallelism between our Theorem 1 and [3] is clear, part of the techniques that we use in the proof of Theorem 2 are behind the proof of Reifenberg's Topological Disk Theorem [15] and also in the construction of [4].

The paper is organized as follows. Next section contains a result on Hausdorff dimension of certain Cantor sets which will be used in the proof of Theorem 1. The third section is devoted to the proof of Theorem 1. Theorem 2 is proved in Section 4 while last section is devoted to martingale versions of our results.

The letter C will denote a constant, whose value may change from line to line, only depending on the dimension d. Similarly, $C(N)$ denotes a constant depending on the parameter N and the dimension.

2. *Hausdorff dimension of Cantor-type sets*

For $n = 1, 2, \ldots$, let $\mathcal{D}_n$ be the family of pairwise disjoint dyadic cubes of the form

$$\left[\frac{k_1}{2^n}, \frac{k_1+1}{2^n}\right) \times \cdots \times \left[\frac{k_d}{2^n}, \frac{k_d+1}{2^n}\right),$$

where $k_1, \ldots, k_d$ are integers. For $x \in \mathbb{R}^d$, let $Q_n(x)$ be the unique dyadic cube of $\mathcal{D}_n$ which contains the point x.

Given a number $0 \leqslant \alpha \leqslant d$, the Hausdorff α-content of a set $E \subset \mathbb{R}^d$ is defined as

$$M_\alpha(E) = \inf \left\{ \sum_j \ell(Q_j)^\alpha \right\},$$

where the infimum is taken over all coverings of E by cubes $\{Q_j\}$ and $\ell(Q_j)$ denotes the sidelength of Q_j. The Hausdorff dimension of E is defined as $\dim E = \inf\{\alpha \geqslant 0 : M_\alpha(E) = 0\}$.

Next result will be used to compute the Hausdorff dimension of certain sets appearing in Theorem 1. We will only use the case $s = 1$. In dimension $d = 1$, the result can be found in [9] or [11].

LEMMA 1. *Fix $0 < s \leqslant d$. For $n = 1, 2, \dots$, let $\mathcal{A}(n)$ be a collection of closed dyadic cubes in $\mathbb{R}^d$ with pairwise disjoint interiors. Assume that the families are nested, that is,*

$$\bigcup_{Q\in\mathcal{A}(n+1)} Q \subseteq \bigcup_{Q\in\mathcal{A}(n)} Q.$$

Assume also that there exist two constants $0 < \eta_0 < K_0 < 1$, satisfying the following conditions.

(a) *For any $Q \in \mathcal{A}(n+1)$ with $Q \subset \tilde{Q}$ for a certain $\tilde{Q} \in \mathcal{A}(n)$, one has $\ell(Q) \leqslant \eta_0 \ell(\tilde{Q})$.*
(b) *For any $\tilde{Q} \in \mathcal{A}(n)$, one has*

$$\sum \ell(Q)^s \geqslant K_0 \ell(\tilde{Q})^s,$$

where the sum is taken over all $Q \in \mathcal{A}(n+1)$ contained in $\tilde{Q}$.

(c) *Furthermore, assume that there exists a constant $\tilde{K} > 0$ such that for any cube $Q \subset \mathbb{R}^d$ and any $n = 1, 2, \dots$ one has*

$$\sum \ell(\tilde{Q})^s \leqslant \tilde{K} \ell(Q)^s,$$

where the sum is taken over all cubes $\tilde{Q}$ in $\mathcal{A}(n)$ contained in Q.
Then

$$\dim \left(\bigcap_{n=1}^{\infty} \bigcup_{Q\in\mathcal{A}(n)} Q \right) \geqslant s - \frac{\log K_0}{\log \eta_0}.$$

It is worth mentioning that although the constant $\tilde{K}$ does not appear in the conclusions, in the case $0 < s < d$ one cannot achieve any non-trivial lower bound for the dimension assuming only (a) and (b).

Proof of Lemma 1. The proof is standard and proceeds by constructing a probability measure $\mu \geqslant 0$ supported in $E = \bigcap_n \bigcup_{Q\in\mathcal{A}(n)} Q$ for which there exists a constant $C > 0$ such that

$$\mu(Q) \leqslant C \ell(Q)^{\alpha}, \tag{2.1}$$

for any dyadic cube Q, where $\alpha = s - \log K_0 / \log \eta_0$. Then it follows that $M_\alpha(E) > 0$ and completes the proof.

The measure μ will be obtained as a weak limit of certain measures μ_n which will be defined recursively as follows. Let $m_{|E}$ denote the Lebesgue measure in $\mathbb{R}^d$ restricted to the set $E \subset \mathbb{R}^d$. Without loss of generality, we can assume that $\mathcal{A}(1)$ consists of the unit cube $Q^{(1)}$. We choose $\mu_1 = m_{|Q^{(1)}}$. Fix a positive integer n and assume by induction that μ_n has been defined. Then consider the family $\mathcal{A}(n+1)$ and define

$$\mu_{n+1} = \sum_{Q\in\mathcal{A}(n+1)} \frac{\mu_{n+1}(Q)}{m(Q)} m_{|Q},$$

where the masses $\mu_{n+1}(Q)$ are defined as follows. Given $Q \in \mathcal{A}(n+1)$, let $\tilde{Q}$ be the dyadic cube in the previous family $\mathcal{A}(n)$ which contains Q. Also, let $\mathcal{A}(\tilde{Q})$ be the family of dyadic cubes in $\mathcal{A}(n+1)$ contained in $\tilde{Q}$. Then $\mu_{n+1}(Q)$ is defined by the relation

$$\frac{\mu_{n+1}(Q)}{\ell(Q)^s} = \frac{\mu_n(\tilde{Q})}{\sum_{Q^* \in \mathcal{A}(\tilde{Q})} \ell(Q^*)^s}.$$

Observe that

$$\sum_{Q \in \mathcal{A}(\tilde{Q})} \mu_{n+1}(Q) = \mu_n(\tilde{Q}). \tag{2.2}$$

Hence $\mu_{n+1}(\mathbb{R}^d) = \mu_n(\mathbb{R}^d)$ and iterating, we deduce that μ_n are probability measures. Let μ be a weak limit of μ_n. Then μ is a probability measure supported in $\bigcap_n \bigcup_{Q \in \mathcal{A}(n)} Q$. The rest of the proof is devoted to show the growth condition (2.1). Observe that (2.2) tells that $\mu(Q) = \mu_n(Q)$ for any cube $Q \in \mathcal{A}(n)$. Let $Q \in \mathcal{A}(n+1)$. As before, we denote by $\tilde{Q}$ the cube in the family $\mathcal{A}(n)$ with $Q \subset \tilde{Q}$ and by $\mathcal{A}(\tilde{Q})$ the family of dyadic cubes in $\mathcal{A}(n+1)$ contained in $\tilde{Q}$. Then (2.2) and property (b) yield

$$\frac{\mu(Q)}{\ell(Q)^s} = \frac{\mu_{n+1}(Q)}{\ell(Q)^s} = \frac{\mu_n(\tilde{Q})}{\sum_{Q^* \in \mathcal{A}(\tilde{Q})} \ell(Q^*)^s} \leqslant \frac{1}{K_0} \frac{\mu_n(\tilde{Q})}{\ell(\tilde{Q})^s}.$$

Iterating this inequality, we obtain

$$\frac{\mu(Q)}{\ell(Q)^s} \leqslant K_0^{-n-1}, \quad Q \in \mathcal{A}(n+1). \tag{2.3}$$

We now prove the estimate (2.1). Let R be a dyadic cube in $\mathbb{R}^d$, $\ell(R) \leqslant 1$. Choose a positive integer n such that $\eta_0^{n+1} \leqslant \ell(R) \leqslant \eta_0^n$. Then

$$\mu(R) \leqslant \sum \mu(Q),$$

where the sum is taken over cubes Q in the family $\mathcal{A}(n+1)$ with $Q \cap R \neq \emptyset$. Since the cubes in $\mathcal{A}(n+1)$ have sidelength smaller than η_0^{n+1}, if $Q \cap R \neq \emptyset$, then we deduce that $Q \subset 3R$, where $3R$ is the cube concentric to R whose sidelength is $3\ell(R)$. Hence, estimate (2.3) tells

$$\mu(R) \leqslant K_0^{-n-1} \sum_{\substack{Q \in \mathcal{A}(n+1) \\ Q \subset 3R}} \ell(Q)^s,$$

and property (c) gives that

$$\mu(R) \leqslant K_0^{-n-1} \tilde{K} 3^s \ell(R)^s.$$

Now

$$K_0^{n+1} = \eta_0^{(n+1)\log K_0/\log \eta_0} \geqslant \eta_0^{\log K_0/\log \eta_0} \ell(R)^{\log K_0/\log \eta_0},$$

and we deduce that

$$\mu(R) \leqslant 3^s \tilde{K} \eta_0^{-\log K_0/\log \eta_0} \ell(R)^{s - \log K_0/\log \eta_0},$$

which gives (2.1) and completes the proof. □

3. *Theorem* 1

Let $\{e_1, e_2, \ldots, e_d\}$ be the canonical basis of $\mathbb{R}^d$. Let Q be a cube in $\mathbb{R}^d$ with edges parallel to the coordinate axis. We will say that Q has the origin at $x \in \mathbb{R}^d$ if its vertices are the points $x + \ell(Q)(\varepsilon_1 e_1 + \varepsilon_2 e_2 + \cdots + \varepsilon_d e_d)$, where $\varepsilon_j \in \{0, 1\}$, $j = 1, \ldots, d$ and $\ell(Q)$ is the sidelength of Q. Given a cube Q in $\mathbb{R}^d$ with origin x and sidelength $\ell(Q)$ and given a function $f\colon \mathbb{R}^d \to \mathbb{R}$, let $V(Q)$ be its discrete gradient at the cube Q given by

$$V(Q) = \sum_{j=1}^{d} \frac{f(x + \ell(Q)e_j) - f(x)}{\ell(Q)} e_j.$$

If f is a function in the Zygmund class, then the vector $V(Q)$ behaves as a sort of gradient as the following lemma shows.

Lemma 2. *Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be a function in the Zygmund class $\Lambda_*(\mathbb{R}^d)$ and let Q be a cube in $\mathbb{R}^d$. Assume that a and b are two points in $\mathbb{R}^d$ for which there is a constant $C > 0$ such that* $\operatorname{dist}(a, Q) \leqslant C\ell(Q)$ *and* $\|b - a\| \leqslant C\ell(Q)$. *Then there exists a constant $K = K(C, d)$ only depending on C and the dimension d such that*

$$|f(b) - f(a) - \langle V(Q), b - a\rangle| \leqslant K\|f\|_* \ell(Q).$$

Proof. Without loss of generality, we may assume that Q is the unit cube in $\mathbb{R}^d$ and $2\|b\| + \|a\| \leqslant 1$. Since the function f is in the Zygmnund class, we have

$$|f(a) + f(b - a) - 2f(b/2)| \leqslant \|f\|_*$$

and

$$|f(b) + f(0) - 2f(b/2)| \leqslant \|f\|_*.$$

Hence,

$$|f(b) - f(a) - (f(b - a) - f(0))| \leqslant 2\|f\|_*. \tag{3.1}$$

So we can also assume that the point a is the origin.

Let us first discuss the particular case when b lies in a coordinate axis, that is, $b = b_k e_k$ for a certain $k = 1, 2, \dots, d$, with $b_k \in \mathbb{R}$, $|b_k| \leqslant \frac{1}{2}$. Since the function f lies in the Zygmund class, the divided differences of f in a fixed direction can grow at most as a fixed multiple of the logarithm of the increment. Hence, there exists a universal constant K_1 such that

$$\left| \frac{f(b) - f(0)}{b_k} - (f(e_k) - f(0)) \right| \leqslant K_1 \|f\|_* \log \frac{1}{\|b\|}.$$

Since $|b_k| = \|b\| \leqslant \frac{1}{2}$, we deduce that

$$|f(b) - f(0) - b_k(f(e_k) - f(0))| \leqslant K_1\|f\|_*.$$

We now discuss the general case. Write $b = \sum_{k=1}^{d} b_k e_k$, where $|b_k| \leqslant \frac{1}{2}$, $k = 1, 2, \dots, d$. Then

$$f(b) - f(0) = f(b_1 e_1) - f(0) + \sum_{k=2}^{d} f\left(\sum_{j=1}^{k} b_j e_j\right) - f\left(\sum_{j=1}^{k-1} b_j e_j\right).$$

Applying (3.1), for any $k = 2, \dots, d$ we get

$$\left| f\left(\sum_{j=1}^{k} b_j e_j\right) - f\left(\sum_{j=1}^{k-1} b_j e_j\right) - (f(b_k e_k) - f(0)) \right| \leqslant 2\|f\|_*.$$

Now each term is in the situation of the particular case discussed in the previous paragraph and we obtain

$$|f(b_1e_1) - f(0) - b_1(f(e_1) - f(0))| \leqslant K_1\|f\|_*,$$
$$\left| f\left(\sum_{j=1}^{k} b_j e_j\right) - f\left(\sum_{j=1}^{k-1} b_j e_j\right) - b_k(f(e_k) - f(0))\right| \leqslant (K_1+2)\|f\|_*,$$

for any $k = 2, \dots, d$. Hence,

$$\left| f(b) - f(0) - \sum_{k=1}^{d} b_k(f(e_k) - f(0))\right| \leqslant (K_1+2)\, d\|f\|_*,$$

which completes the proof. □

As a consequence of the lemma, in the proof of Theorem 1, instead of studying the divided differences of a function in the Zygmund class, we can restrict attention to the behavior of the discrete gradient.

Corollary 1. (a) *Let f be a function in the Zygmund class. Let $x \in \mathbb{R}^d$. Then $x \in E(f)$ if and only if $\limsup_{n\to\infty} \|V(Q_n(x))\| < \infty$, where $V(Q)$ denotes the discrete gradient of f at the cube Q.*

(b) *Let f be a function in the small Zygmund class. Then f is differentiable at a point $x \in \mathbb{R}^d$ if and only if $V(Q_n(x))$ has limit when $n \to \infty$.*

The proof of Theorem 1 consists of constructing a Cantor-type set on which the function f has bounded divided differences. The construction of the Cantor-type set uses a stopping time argument based on the following auxiliary result which contains the main idea of the construction. The key point is that, when the divided differences are large, the discrete gradient distributes its values in a certain uniform way when measured with respect to length.

If f is in the Zygmund class $\Lambda_*(\mathbb{R}^d)$ and $V(Q)$ is its discrete gradient at the cube Q, then consider the function

$$w(\delta) = \sup\{|V(Q') - V(Q)|\}, \tag{3.2}$$

where the supremum is taken over all pairs of cubes $Q' \subset Q \subset \mathbb{R}^d$ with $2\ell(Q') = \ell(Q) \leqslant \delta$. It follows from Lemma 2 that there exists a constant C only depending on the dimension such that $w(\delta) \leqslant C\|f\|_*$, for any $\delta > 0$.

Proposition 1. *Let f be a function in the Zygmund class $\Lambda_*(\mathbb{R}^d)$ satisfying*

$$H^1\left(\left\{x \in \mathbb{R}^d : \limsup_{\|h\|\to 0} \frac{|f(x+h) - f(x)|}{\|h\|} < \infty\right\}\right) = 0. \tag{3.3}$$

Consider the corresponding function $w(\delta)$ defined in (3.2) and assume that $w(\delta) \leqslant 1$ for any $\delta > 0$. Fix $0 < \varepsilon < 1$. Then there is a positive constant $C_0 = C_0(\varepsilon, d)$ such that for any $\delta > 0$ there is $M_0 = M_0(\delta) > 0$ with the following property. If $M > M_0$ and $Q \subset \mathbb{R}^d$ is a dyadic cube

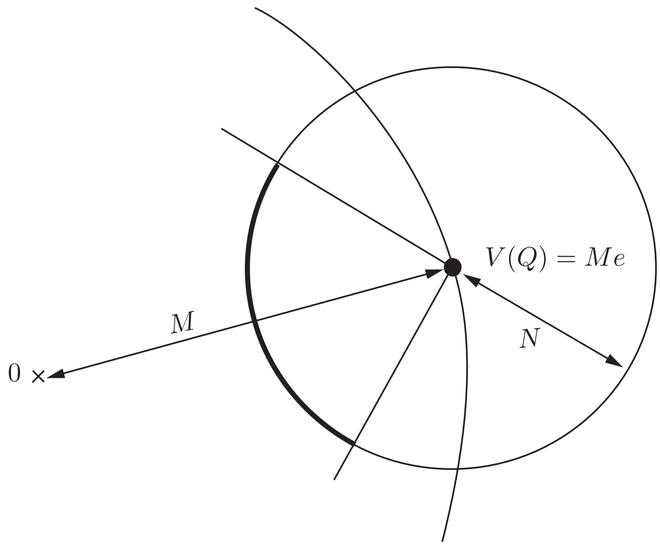

FIGURE 1. *The bold region corresponds to the possible locations of* $V(Q_j^*)$ *when* $u = -e$.

with $l(Q) < \delta$ *and* $V(Q) = Me$ *where* $e \in \mathbb{R}^d$, $\|e\| = 1$, *then for each unit vector* $u \in \mathbb{R}^d$, *there is a finite family* $\mathcal{Q}^* = \{Q_j^*\}$ *of dyadic subcubes of* Q *such that the following conditions are satisfied.*

(a) *For any* j, *one has* $\ell(Q_j^*) \leqslant 2^{-\varepsilon M}\ell(Q)$ *and* $\varepsilon M \leqslant \|V(Q_j^*) - Me\| \leqslant \varepsilon M + w(\ell(Q))$.

(b) *If* $x \in Q_j^*$ *and* $-\log_2 \ell(Q) \leqslant n < -\log_2 \ell(Q_j^*)$ *for some* j *and* n, *then* $\|V(Q_n(x)) - Me\| \leqslant \varepsilon M$.

(c) *One has* $\sum_j \ell(Q_j^*) \geqslant C_0^{-1}\ell(Q)$, *where the sum is taken over all cubes* $Q_j^* \in \mathcal{Q}^*$. *Moreover, for any cube* R *contained in* Q, *one has* $\sum_j \ell(Q_j^*) \leqslant C_0\ell(R)$, *where the sum is taken over all cubes* $Q_j^* \in \mathcal{Q}^*$ *contained in* R.

(d) *For any* j, *one has* $\langle V(Q_j^*) - Me, u\rangle \geqslant \frac{2}{3}\varepsilon^2 M$.

Let us first explain the main idea of the proof. Let Q be a dyadic cube with $V(Q) = Me$, consider the family $\{Q_j\}$ of maximal dyadic subcubes Q_j of Q satisfying

$$\|V(Q) - V(Q_j)\| \geqslant \varepsilon M.$$

It easily follows that the whole family $\{Q_j\}$ satisfies conditions (a) and (b). In particular, $V(Q_j)$ lays, up to a bounded term, in the sphere centered at $V(Q)$ of radius εM. Condition (d) tells that we are only interested in the subfamily $\mathcal{Q}^* = \{Q_j^*\}$ of $\{Q_j\}$ of cubes Q_j^* such that the corresponding discrete gradient $V(Q_j^*)$ lies in a certain cone with vertex at $V(Q)$; see Figure 1. Condition (c) tells that when measured with respect to length, there is a fixed proportion of cubes $\{Q_j^*\}$ satisfying (d). This is the main point in the result. We will construct a polygonal path joining two parallel faces of a duplicate of Q on which the discrete gradient satisfies, up to bounded terms, a certain mean value property. The family $\mathcal{Q}^*$ will be chosen as a subfamily of the cubes $\{Q_j\}$ which have a non-empty intersection with this polygonal and the mean value property will lead to estimate (c).

Proof of Proposition 1. Without loss of generality, we may assume that $\|f\|_* = 1$. Let $N = \varepsilon M$. The proof is presented in several steps.

3.1. *Reduction to the case $u = -e$*

Assume that we have proved the proposition in the case $u = -e$ and let us show it for another unit vector u. Consider the function $g(x) = f(x) - M\langle u + e, x\rangle$. It is clear that $g \in \Lambda_*(\mathbb{R}^d)$ and $\|g\|_* = \|f\|_*$. Moreover, the discrete gradient V_g of the function g verifies that for any dyadic cube R,

$$V_g(R) = V(R) - M(u + e).$$

Then $V_g(Q) = -Mu$ and, according to our assumption, we can choose the family $\{Q_j^*\}$ of cubes given by the proposition, for which $N \leqslant \|V_g(Q_j^*) - V_g(Q)\| \leqslant N + w(\ell(Q))$ and $\langle V_g(Q_j^*) - V_g(Q), -V_g(Q)/\|V_g(Q)\|\rangle \geqslant \frac{2}{3}\varepsilon N$. Hence, $N \leqslant \|V(Q_j^*) - V(Q)\| \leqslant N + w(\ell(Q))$ and $\langle V(Q_j^*) - Me, u\rangle \geqslant \frac{2}{3}\varepsilon N$, for any j.

3.2. *Covering the cube Q*

Let $\{Q_j\}$ be the family of maximal dyadic cubes contained in Q such that

$$\|V(Q) - V(Q_j)\| \geqslant N.$$

The family of cubes $\mathcal{Q}^*$ in the statement will be a subfamily of $\{Q_j\}$. The maximality of the cubes $\{Q_j\}$ imply that for any j we have

(i) $N \leqslant \|V(Q_j) - Me\| \leqslant N + w(\ell(Q))$;
(ii) $\|V(Q_n(x)) - Me\| < N$ if $x \in Q_j$ for some j and the integer n satisfies $-\log_2 \ell(Q) \leqslant n < -\log_2 \ell(Q_j)$.

Moreover, the condition $w(\delta) \leqslant 1$ gives that $\ell(Q_j) \leqslant 2^{-N}\ell(Q)$ for any j. So the whole family $\{Q_j\}$ satisfy the first two properties in the statement and the rest of the proof consists of finding a subfamily verifying the last two properties. Since condition (3.3) holds, Corollary 1 tells that

$$H^1\left(\left\{x \in \mathbb{R}^d \,:\, \limsup_{n\to\infty} \|V(Q_n(x))\| < \infty\right\}\right) = 0,$$

and hence $H^1(Q \setminus \bigcup Q_j) = 0$. Fix a small constant $\eta > 0$ with $\eta \leqslant 1/4M\sqrt{d}$ and find a covering of $Q \setminus \bigcup Q_j$ by dyadic cubes $\{R_j\}$ such that

$$\sum_j \ell(R_j) < \eta\ell(Q). \tag{3.4}$$

We can assume that no R_ℓ is contained in $\bigcup Q_j$. If $\frac{1}{2}Q$ denotes the closed cube, centered at the center of Q, with sidelength $\ell(Q)/2$, then it is clear that $\frac{1}{2}Q \subset \bigcup \tilde{Q}_j \cup \tilde{R}_j$, where $\{\tilde{Q}_j\}$ and $\{\tilde{R}_j\}$ are, respectively, the open cubes centered at the centers of Q_j and R_j with double sidelength. Compactness of $\frac{1}{2}Q$ allows us to obtain a finite covering that we will denote by $C_1, C_2, \ldots, C_n$, that is, $\frac{1}{2}Q \subset C_1 \cup \cdots C_n$. Now, we label each C_j with a unit vector v_j in the following way:

(i) if $C_j = \tilde{Q}_k$ for some k, then choose $v_j = V(Q_k)/\|V(Q_k)\|$;
(ii) if $C_j = \tilde{R}_k$ for some k, then choose $v_j = e$.

Using the cubes $\{C_i : i = 1, \ldots, n\}$ and their labels, we will construct a polygonal path Γ, starting from a point in $\frac{1}{2}Q$ in such a way that $\operatorname{diam}(\Gamma)$ is comparable to $\ell(Q)$. The polygonal will follow the directions of the vectors $\{v_j\}$ and will have vertices at certain points $a_1, a_2, \ldots, a_n$ such that all of them, except the last one, lie in $\frac{1}{2}Q$.

3.3. *Choosing the first vertices a_j*

Let a_1 be a fixed point in the cube $\frac{1}{4}Q$. Next vertices are defined recursively. Assuming that $a_j \in \frac{1}{2}Q$ has been defined, let us describe the choice of a_{j+1}. Since $a_j \in \frac{1}{2}Q$, there is a cube C_k, $k = k(j)$, containing this point, and recall that attached to this cube there is a unit vector v_k. Let us define

$$a_{j+1} = a_j + \operatorname{diam}(C_k)v_k.$$

Since C_k is an open cube, it is clear that $a_{j+1} \notin C_k$. Since $\varepsilon < 1$, there exists a constant $\alpha_0 < \pi/2$ such that the angle α_k between v_k and e satisfies $|\alpha_k| < \alpha_0$ for any k. Let $\pi(x)$ be the scalar product defined by $\pi(x) = \langle x, e\rangle$, $x \in \mathbb{R}^d$. Observe that $\pi(a_{j+1} - a_j) > (\cos\alpha_0)\operatorname{diam} C_k$. Since there are finitely many cubes C_k, iterating this process, in a finite number of steps we find a vertex not contained in $\frac{1}{2}Q$. We stop the process whenever we obtain a vertex a_n verifying $a_n \notin \frac{1}{2}Q$. Observe that there exists a constant C_0 such that $\ell(Q)/4 < \|a_n - a_1\| < C_0\ell(Q)$.

Let $a_1, a_2, \dots, a_n$ be the points obtained by the previous algorithm and let us denote by J' the subset of $\{1, 2, \dots, n\}$ formed by the indices j for which the selected cube C_k was a cube in the family $\{\tilde{R}_m\}$. Put $J = \{1, 2, \dots, n\} \setminus J^{'}$. Write

$$\begin{aligned} f(a_n) - f(a_1) &= \sum_{j=1}^{n-1}(f(a_{j+1}) - f(a_j)) \\ &= \sum_{j\in J'}(f(a_{j+1}) - f(a_j)) + \sum_{j\in J}(f(a_{j+1}) - f(a_j)). \end{aligned}$$

Since $\|a_n - a_1\| < C_0\ell(Q)$, we can apply Lemma 2 to obtain

$$|f(a_n) - f(a_1) - \langle V(Q), a_n - a_1\rangle| \leqslant C\ell(Q), \tag{3.5}$$

where C is an absolute constant only depending on the dimension. Since $V(Q) = Me$, we have $\langle V(Q), a_n - a_1\rangle = M(\pi(a_n) - \pi(a_1))$ and (3.5) tells

$$\left|\sum_{j\in J'}(f(a_{j+1}) - f(a_j)) + \sum_{j\in J}(f(a_{j+1}) - f(a_j)) - M(\pi(a_n) - \pi(a_1))\right| \leqslant C\ell(Q). \tag{3.6}$$

The family $\mathcal{Q}^*$ will be a subfamily of $\{Q_j : j \in J\}$ while the first sum in (3.6) will be an error term.

3.4. *Estimating the error term*

If $j \in J'$, then the cube C_k, $k = k(j)$, containing a_j belongs to the family $\{\tilde{R}_m\}$. So, write $C_k = \tilde{R}_\ell$. Since R_ℓ is not contained in $\bigcup Q_j$, we have that $\|V(R_\ell) - Me\| < N$. Hence, by Lemma 2, there exists a fixed constant C, such that $|f(a_{j+1}) - f(a_j)| \leqslant (M + N + C)\|a_{j+1} - a_j\| \leqslant 2M\|a_{j+1} - a_j\|$ if M is sufficiently large. Since by (3.4) the amount of length collected by these squares is small, we make the trivial estimate on them:

$$\sum_{j\in J'}|f(a_{j+1}) - f(a_j)| \leqslant 4\sqrt{d}M\sum_{j\in J'}\ell(R_j) \leqslant 4\eta\sqrt{d}M\ell(Q).$$

Since $\eta > 0$ was chosen such that $\eta < 1/4M\sqrt{d}$, estimate (3.6) gives that

$$\left|\sum_{j\in J}(f(a_{j+1}) - f(a_j)) - M(\pi(a_n) - \pi(a_1))\right| \leqslant \tilde{C}\ell(Q), \tag{3.7}$$

where $\tilde{C}$ is a constant only depending on the dimension.

3.5. *Choosing the cubes of the family* $\mathcal{Q}^*$

Assume $j \in J$. This means that $a_j \in C_k \in \{\tilde{Q}_k\}$. To simplify the notation, reorder the family $\{\tilde{Q}_k\}$ such that $a_j \in \tilde{Q}_j$ for $j \in J$. Consequently, we have

$$a_{j+1} = a_j + \operatorname{diam}(\tilde{Q}_j) v_j.$$

For $j = 1, 2, \ldots, n$, consider the segment between a_j and a_{j+1}, and let Γ be the polygonal with vertices $\{a_1, \ldots, a_n\}$. Since $\|a_{j+1} - a_j\|$ is comparable to $\ell(Q_j)$, we can apply Lemma 2 again to get

$$|f(a_{j+1}) - f(a_j) - \langle V(Q_j), a_{j+1} - a_j \rangle| \leqslant C\ell(Q_j). \tag{3.8}$$

Since the angle α_j between the vectors v_j and e satisfies $|\alpha_j| < \alpha_0 < \pi/2$, we have that $\|a_{j+1} - a_j\| \leqslant C_1 \pi(a_{j+1} - a_j)$ for any j, where $C_1 = 1/\cos\alpha_0$. Hence,

$$\sum \ell(Q_j) \leqslant C_1 \|a_n - a_1\| \leqslant C_2 \ell(Q). \tag{3.9}$$

The choice of a_{j+1} gives that for any $j \in J$, we have

$$\langle V(Q_j), a_{j+1} - a_j \rangle = \|V(Q_j)\| \|a_{j+1} - a_j\| = \frac{\|V(Q_j)\|^2}{\langle V(Q_j), e\rangle}(\pi(a_{j+1}) - \pi(a_j)).$$

So estimates (3.7)–(3.9) give the following one-dimensional mean value property:

$$\sum_{j\in J} \frac{\|V(Q_j)\|^2}{\langle V(Q_j), e\rangle}(\pi(a_{j+1}) - \pi(a_j)) = M(\pi(a_n) - \pi(a_1)) + 0(\ell(Q)). \tag{3.10}$$

Here $0(\ell(Q))$ denotes a quantity which is a bounded by a fixed proportion, independent of Q, of $\ell(Q)$. Now, a simple calculation gives

$$\begin{aligned} \frac{\|V(Q_j)\|^2}{\langle V(Q_j), e\rangle} &= \frac{\|V(Q_j) - Me\|^2 + 2M\langle V(Q_j) - Me, e\rangle + M^2}{\langle V(Q_j) - Me, e\rangle + M} \\ &= M\left(2 - \frac{1 - (1/M^2)\|V(Q_j) - Me\|^2}{1 + (1/M)\langle V(Q_j) - Me, e\rangle}\right). \end{aligned} \tag{3.11}$$

Decompose the set J into *good* and *bad* indices, that is, $J = B \cup G$ where

$$\begin{aligned} G &= \{j \in J \,:\, \langle V(Q_j) - Me, e\rangle \leqslant -\tfrac{2}{3}\varepsilon N\}, \\ B &= J \setminus G. \end{aligned}$$

The family of cubes $\mathcal{Q}^*$ we are looking for is precisely $\{Q_j : j \in G\}$. It is clear that property (d), with $u = -e$, holds. Let R be a cube in $\mathbb{R}^d$ and assume that $Q_j^* \in \mathcal{Q}^*$ is contained in R. Then there exists a vertex a_j of the polygonal Γ with $a_j \in 2Q_j^* \subset 2R$. Since $\ell(Q_j^*) \leqslant \|a_{j+1} - a_j\| \leqslant (\cos^{-1}\alpha_0)\pi(a_{j+1} - a_j)$, we deduce that there exists a constant $C_3 = C_3(\alpha_0, d)$ such that $\sum \ell(Q_j^*) < C_3 \ell(R)$ where the sum is taken over all cubes Q_j^* contained in R. This is the second part of statement (c).

3.6. *Estimates on* G *and* B

The stopping time process described in Paragraph 2 in what follows gives that $\|V(Q_j) - Me\|^2 \geqslant N^2$. Hence, $\langle V(Q_j) - Me, e\rangle \geqslant -N$ and if $j \in B$, then we also have $\langle V(Q_j) - Me, e\rangle \geqslant -2\varepsilon N/3$. Consequently, using (3.11) we obtain the following estimates:

(i) if $j \in G$, then one has $\|V(Q_j)\|^2/\langle V(Q_j), e\rangle \geqslant M(1 - \varepsilon)$;
(ii) if $j \in B$, then one has $\|V(Q_j)\|^2/\langle V(Q_j), e\rangle \geqslant M(2 - (1 - \varepsilon^2)/(1 - 2\varepsilon^2/3)) = M(1 + C(\varepsilon))$;

where $C(\varepsilon) = \varepsilon^2/(3 - 2\varepsilon^2) > 0$. Hence, equation (3.10) gives

$$\pi(a_n) - \pi(a_1) + \frac{O(\ell(Q))}{M} \geqslant (1-\varepsilon)\sum_{j\in G}(\pi(a_{j+1}) - \pi(a_j)) + (1 + C(\varepsilon))\sum_{j\in B}(\pi(a_{j+1}) - \pi(a_j)). \tag{3.12}$$

Since $\sum \ell(R_j) < \eta\ell(Q) < \ell(Q)/M$, we have that

$$\sum_{j\in J}\pi(a_{j+1}) - \pi(a_j) = \pi(a_n) - \pi(a_1) + \frac{O(\ell(Q))}{M},$$

and we deduce that

$$\pi(a_n) - \pi(a_1) + \frac{O(\ell(Q))}{M} \geqslant (\pi(a_n) - \pi(a_1))(1 + C(\varepsilon)) - (\varepsilon + C(\varepsilon))\sum_{j\in G}(\pi(a_{j+1}) - \pi(a_j)). \tag{3.13}$$

Therefore, there exists a constant $C_1(\varepsilon) > 0$ such that

$$\sum_{j\in G}(\pi(a_{j+1}) - \pi(a_j)) \geqslant C_1(\varepsilon)(\pi(a_n) - \pi(a_1)).$$

Since $\pi(a_{j+1}) - \pi(a_j)$ and $\pi(a_n) - \pi(a_1)$ are comparable, respectively, to $\ell(Q_j)$ and $\ell(Q)$, we obtain (c). □

Remark 1. Taking $u = -e$ in part (d), we obtain that for any $j = 1, 2, \ldots$, one has

$$\begin{aligned}\|V(Q_j^*)\|^2 &= \|V(Q_j^*) - Me\|^2 + M^2 + 2M\langle V(Q_j^*) - Me, e\rangle\\ &\leqslant (\varepsilon M + w(\ell(Q)))^2 + M^2 - 4\varepsilon^2M^2/3\\ &= M^2(1 - \varepsilon^2/3) + 2w(\ell(Q))\varepsilon M + w(\ell(Q))^2.\end{aligned}$$

Taking the parameters such that $M\varepsilon > 20$ and assuming $w(\ell(Q)) \leqslant 1$, we easily deduce that $\|V(Q_j^*)\| < MC_2(\varepsilon)$, where $C_2(\varepsilon) = (1 - \varepsilon^2/6)^{1/2} < 1$. So starting from a cube Q such that the discrete gradient satisfies $\|V(Q)\| = M$, Proposition 1 provides a family $\{Q_j^*\}$ of dyadic subcubes of Q satisfying $\|V(Q_j^*)\| < MC_2(\varepsilon) < M$, as well as properties (a)–(c). This fact will be crucial in the proof of Theorem 1.

Remark 2. If $0 < \varepsilon < 1$ is taken close to 1, then our arguments show that one can replace in part (d) the constant $2\varepsilon/3$ by a constant close to 1. This means that $V(Q_j^*)$ lies in a cone with vertex at the point Me and small aperture. This fact will be used in the proof of certain refinements of Theorem 1.

We can now proceed to prove Theorem 1.

Proof of Theorem 1. Let us first discuss part (a). Let f be a function in the Zygmund class and let $V(Q)$ denote its discrete gradient at a cube Q. We can assume that the function $w(\delta)$ defined in Proposition 1 satisfies $w(\delta) \leqslant 1$ for any $\delta > 0$. Corollary 1 tells that we can restrict our attention to the behavior of the discrete gradient on dyadic cubes. We can assume that condition (3.3) is satisfied and that $V(Q^1) = 0 \in \mathbb{R}^d$ where Q^1 is the unit cube in $\mathbb{R}^d$. We will construct a Cantor-type set on which the function f has bounded divided differences. The generations $\mathcal{A}(n)$ of the Cantor-type set will be defined recursively. Fix a number $0 < \varepsilon < 1$, say $\varepsilon = \frac{1}{2}$. Fix a large number $M > M_0(1)$, where $M_0(\delta)$ is the constant appearing in Proposition 1.

The first generation $\mathcal{A}(1)$ consists of the unit cube Q^1. The second generation is constructed as follows. Consider the maximal dyadic cubes $\{R_j\}$ contained in Q^1 such that $\|V(R_j)\| \geqslant M$. The maximality and the estimate $w(\delta) \leqslant 1$ give that $\|V(R_j)\| \leqslant M+1$ for any $j = 1, 2, \ldots$. Let e_1 be the first vector of the canonical basis of $\mathbb{R}^d$ and let L be the segment given by $L = \{te_1 : 0 \leqslant t \leqslant 1\}$ and consider the subfamily $\mathcal{R}$ of $\{R_j\}$ formed by those R_j which have a non-empty intersection with L. By (3.3) every point of L, except at most for a set of zero length, is contained in a cube of the family $\mathcal{R}$. Let Π be the projection over the first coordinate, that is, $\Pi(x) = \langle x, e_1 \rangle e_1$, $x \in \mathbb{R}^d$. Since $\{\Pi(R_j) : R_j \in \mathcal{R}\}$ are pairwise disjoint, for any cube $Q \subset \mathbb{R}^d$ we have

$$\sum \ell(R_j) \leqslant \ell(Q),$$

where the sum is taken over all cubes $R_j \in \mathcal{R}$ contained in Q. Assume that $N = \varepsilon M = M/2 > 20$. Apply Proposition 1 in each cube $R \in \mathcal{R}$ to obtain a family of dyadic cubes $\mathcal{Q}(R) = \{Q_j\}$ contained in R satisfying conditions (a)–(d). In particular by Remark 1, we have $\|V(Q_j)\| < M$ for any $Q_j \in \mathcal{Q}(R)$. The second generation $\mathcal{A}(2)$ of the Cantor-type set is defined as the union of the families of cubes $\{\mathcal{Q}(R) : R \in \mathcal{R}\}$. Observe that if $x \in Q$ for some $Q \in \mathcal{A}(2)$ and $n \leqslant -\log_2 \ell(Q)$, then property (b) of Proposition 1 gives that $\|V(Q_n(x))\| \leqslant M + 1 + N \leqslant 2M + 1$. Property (a) of Proposition 1 tells that $\ell(Q) < 2^{-N}$ for any $Q \in \mathcal{A}(2)$, while Property (c) gives that

$$\sum \ell(Q) \geqslant C_0^{-1},$$

where the sum is taken over all cubes $Q \in \mathcal{A}(2)$. Moreover, if R is a cube in $\mathbb{R}^d$, then we have $\sum \ell(Q) \leqslant C_0 \ell(R)$, where the sum is taken over all cubes $Q \in \mathcal{A}(2)$ contained in R. The construction continues inductively. Assume that we have defined the generation $\mathcal{A}(n)$ of cubes satisfying $\|V(Q)\| \leqslant M$ for any $Q \in \mathcal{A}(n)$. In each $Q \in \mathcal{A}(n)$, we act as in the first step, that is, we consider the family $\{R_j\}$ of maximal dyadic cubes $R_j = R_j(Q)$ contained in Q such that $\|V(R_j)\| \geqslant M$. Let $a = a(Q)$ be the origin of Q. Consider the segment $L = L(Q) = \{a + te_1 \colon 0 \leqslant t \leqslant \ell(Q)\}$ and the subfamily $\mathcal{R}(Q)$ of $\{R_j\}$ formed by those R_j which have a non-empty intersection with L. Finally, in each $R \in \mathcal{R}(Q)$ apply Proposition 1 to obtain a family of dyadic cubes $\mathcal{Q}(R) = \{Q_j\}$ contained in R satisfying conditions (a)–(d). The generation $\mathcal{A}(n+1)$ is given by the cubes $\{Q_j \colon Q_j \in \mathcal{R}(Q), Q \in \mathcal{A}(n)\}$. Properties (a) and (c) of Proposition 1 give conditions (a)–(c) in Lemma 1 with the parameters $s = 1$, $\eta_0 = 2^{-N}$, $K_0 = C_0^{-1}$ and $\tilde{K} = C_0$. Hence, the Hausdorff dimension of the set

$$E = E(M) = \bigcap_{n=1}^{\infty} \bigcup_{Q \in \mathcal{A}(n)} Q$$

is bigger than $1 - \log_2(C_0)/N$. Since for any point $x \in E$, we have $\|V(Q_n(x))\| \leqslant 2M + 1$, for any $n = 1, 2, \ldots$, we deduce that at any point in $E = E(M)$ the function f has bounded divided differences. Since $N = M/2$ can be taken large the proof of (a) is completed.

We now discuss part (b) of Theorem 1. Since the proof is similar to the previous one we will only sketch the argument. Let f be a function in the small Zygmund class. Corollary 1 tells that it is enough to show that the Hausdorff dimension of the set of points where the discrete gradient converges is bigger or equal to 1. We will construct a Cantor-type set on which the discrete gradient converges. The generations $\mathcal{A}(n)$ of the Cantor-type set will be defined recursively. The main idea is to observe that in Proposition 1 if the function f is in the small Zygmund class, at small scales one can use small parameters M. More precisely, the proof of Proposition 1 gives that one can take $M_0 = M_0(\delta)$ with $M_0(\delta) \to 0$ as $\delta \to 0$. So, choose an increasing sequence of integers $N(n) \to \infty$ as $n \to \infty$ such that

$$\sum_n M_0(2^{-N(n)}) < \infty. \tag{3.14}$$

The first generation $\mathcal{A}(1)$ consists of the unit cube Q^1. Assume by induction that we have defined the generation $\mathcal{A}(n)$ satisfying $\ell(Q) \leqslant 2^{-N(n)}$ for any $Q \in \mathcal{A}(n)$. In each $Q \in \mathcal{A}(n)$, we use a slight variation of the construction of the generations in part (a) with the purpose that the cubes of generation $n+1$ contained in Q have sidelength smaller than $2^{-N(n+1)}$ and still have discrete gradient close to $V(Q)$. Fix $Q \in \mathcal{A}(n)$ and argue as in part (a) with the ball centered at the origin of radius M replaced by the ball centered at $V(Q)$ of radius $2M_0(2^{-N(n)})$, to obtain an intermediate family of cubes $\{R_j\}$ such that $\|V(R_j) - V(Q)\| \leqslant 2M_0(2^{-N(n)})$ for any $j = 1, 2, \ldots$. In each intermediate cube R_j, we can repeat this process finitely many times to obtain a family of cubes $\mathcal{R}(Q)$ such that $\ell(R) \leqslant 2^{-N(n+1)}$ and $\|V(R) - V(Q)\| \leqslant 2M_0(2^{-N(n)})$ for any $R \in \mathcal{R}(Q)$. The family $\mathcal{A}(n+1)$ will be formed by the dyadic cubes in $\bigcup \mathcal{R}(Q)$, where the union is taken over all cubes $Q \in \mathcal{A}(n)$. Consider the set $E = \bigcap_n \bigcup_{Q \in \mathcal{A}(n)} Q$. As before Lemma 1 tells that the Hausdorff dimension of E is bigger or equal to one. Also condition (3.14) tells that for any $x \in E$ the sequence $V(Q_n(x))$ converges as $n \to \infty$. □

4. *Theorem* 2

The purpose of this section is to construct a function f in the small Zygmund class $\lambda_*(\mathbb{R}^d)$ for which the set $E(f)$ has Hausdorff dimension 1. Let us first explain the main idea of the construction. Fix a sequence of positive numbers $0 \leqslant \varepsilon_k \leqslant 1$, with $\lim \varepsilon_k = 0$ such that $\sum \varepsilon_k^2 = \infty$. The function f will be constructed as $f = \sum g_k$, where $\{g_k\}$ will be a sequence of smooth functions defined recursively with $\sum ||g_k||_\infty < \infty$. Certain size estimates of g_k and its gradient will give that f is in the Small Zygmund Class. Assume that the functions $\{g_j : j = 0, \ldots, k\}$ have been defined. The idea is to construct the function g_{k+1} in such a way that $\nabla g_{k+1}(x)$ is *almost orthogonal* to $\nabla \sum_{j=1}^{k} g_j(x)$ and $|\nabla g_{k+1}(x)|$ is comparable to ε_{k+1} for *most* points $x \in \mathbb{R}^d$. Since one cannot hope to achieve both requirements at all points $x \in \mathbb{R}^d$, an exceptional set appears and *one-dimensional* estimates of the size of this set will be needed. The construction is easier in dimension $d = 2$ because the boundary of a square has dimension 1 while when $d > 2$ some extra requirements will be needed to obtain the convenient one-dimensional estimates.

Pick $g_0 \equiv 0$. Fix an integer $k > 0$ and assume by induction that $f_k = \sum_{j=0}^{k} g_j$ has been defined. Assume also by induction that ∇f_k is uniformly continuous in $\mathbb{R}^d$ and that

$$M_k = \sup_{\mathbb{R}^d} \|\nabla f_k\| < \infty.$$

The construction of the function g_{k+1} is presented in several steps. The reader only interested in dimension $d = 2$ could skip Paragraphs 3, 4 and 5 in what follows.

(1) Choose a positive number $\eta_k > 0$ with $\eta_k < 2^{-k}/M_k$ and $\eta_k \leqslant \varepsilon_k^3$ and find $N_k > 0$ such that

$$\|\nabla f_k(z) - \nabla f_k(w)\| < \eta_k, \tag{4.1}$$

if $\|z - w\| \leqslant 2^{-N_k}$.

(2) Consider the collection $\mathcal{Q} = \{Q_j : j = 1, 2, \ldots\}$ of pairwise disjoint open-closed dyadic cubes of generation N_k. So $\mathbb{R}^d = \bigcup Q_j$, $\ell(Q_j) = 2^{-N_k}$. Let a_j be the center of Q_j, $j = 1, 2, \ldots$. All cubes in the construction will be in this family or in a family obtained by translating $\{Q_j : j = 1, 2, \ldots\}$ by a fixed vector. When $d = 2$ let $e_1(a_j)$ be a unit vector orthogonal to $\nabla f_k(a_j)$.

(3) This paragraph only applies when the dimension $d > 2$. We will choose a set of unit vectors $\{e_\ell(a_j) : \ell = 1, \ldots, d-1\}$ associated to each cube Q_j, $j = 1, 2, \ldots$ which are *essentially* an orthonormal basis of the hyperplane orthogonal to $\nabla f_k(a_j)$. More precisely, we claim that there exists a constant $C > 0$ only depending on the dimension, such that for each cube $Q_j \in \mathcal{Q}$,

$j = 1, 2, \dots$ one can choose unit vectors $\{e_\ell(a_j) \colon \ell = 1, \dots, d-1\}$ satisfying the following three properties.

(a) $|\langle \nabla f_k(a_j), e_\ell(a_j)\rangle| \leqslant C\eta_k^{1/2}$ for $\ell = 1, \dots, d-1$.

(b) For any collection $\{Q_{j(\ell)} \colon \ell = 1, \dots, d-1\}$ of non-necessarily distinct $d-1$ cubes in the family $\mathcal{Q}$ with $\bigcap_{\ell=1}^{d-1} \bar{Q}_{j(\ell)} \neq \emptyset$, the corresponding vectors $\{e_\ell(a_{j(\ell)}) : \ell = 1, 2, \dots, d-1\}$ form a C-Riesz set, that is, for any set $\{\alpha_\ell\}$ of real numbers, one has

$$C^{-1} \sum_{l=1}^{d-1} |\alpha_\ell|^2 \leqslant \left\| \sum_{\ell=1}^{d-1} \alpha_\ell e_\ell(a_{j(\ell)}) \right\|^2 \leqslant C \sum_{\ell=1}^{d-1} |\alpha_\ell|^2.$$

(c) The angle between any coordinate axis and any $e_\ell(a_j)$, $\ell = 1, \dots, d-1$, $j = 1, 2, \dots$, is between, say, $\pi/6$ and $\pi/3$.

The first condition tells that the unit vectors $\{e_\ell(a_j) \colon \ell = 1, \dots, d-1\}$ are, up to a small error, orthogonal to the gradient $\nabla f_k(a_j)$. Condition (b) tells that the unit vectors act, up to constants, as an orthonormal basis. Property (c) tells that any hyperplane with normal unit vector $e_\ell(a_j)$ is far from being parallel to any face of any dyadic cube. These facts will be used later. Let us explain the choice of the vectors $\{e_\ell(a_j) \colon \ell = 1, \dots, d-1\}$.

Let $\mathcal{A}$ be the set of indices j such that $\|\nabla f_k(a_j)\| \geqslant \eta_k^{1/2}$. If $\mathcal{A}$ is empty, then one can take $\{e_\ell(a_j) \colon \ell = 1, \dots, d-1\}$ to be any fixed orthonormal system satisfying (c). Then (a) and (b) with $C = 1$ are satisfied. So, assume that $\mathcal{A}$ is non-empty. We first construct the unit vectors corresponding to indices in $\mathcal{A}$. Observe that by (4.1), one has

$$\left\| \frac{\nabla f_k(a_j)}{\|\nabla f_k(a_j)\|} - \frac{\nabla f_k(a_{j'})}{\|\nabla f_k(a_{j'})\|} \right\| < 2\eta_k^{1/2} \quad \text{if } j, j' \in \mathcal{A} \text{ and } \bar{Q}_j \cap \bar{Q}_{j'} \neq \emptyset.$$

Hence, the orthogonal hyperplanes $M_j = \{x \in \mathbb{R}^d \colon \langle x, \nabla f_k(a_j)\rangle = 0\}$ deviate smoothly for those j in $\mathcal{A}$ corresponding to contiguous cubes. Now, for every $j \in \mathcal{A}$ one can choose an orthonormal basis $\{e_\ell(a_j) \colon \ell = 1, \dots, d-1\}$ of M_j which for any $\ell = 1, \dots, d-1$ satisfies

$$\|e_\ell(a_j) - e_\ell(a_{j'})\| < C\eta_k^{1/2} \quad \text{if } j, j' \in \mathcal{A} \text{ and } \bar{Q}_j \cap \bar{Q}_{j'} \neq \emptyset. \tag{4.2}$$

Moreover, when $d > 2$, the dimension of M_j is $d-1 \geqslant 2$ and one can take $\{e_\ell(a_j) \colon \ell = 1, \dots, d-1\}$ verifying also (c). Observe that since $\{e_\ell(a_j) \colon \ell = 1, \dots, d-1\}$ is an orthonormal basis of M_j, the continuity property (4.2) gives (b) for indices j in the set $\mathcal{A}$. We now need to construct $\{e_\ell(a_j) \colon \ell = 1, \dots, d-1\}$ when $j \notin \mathcal{A}$. The construction proceeds recursively and the choice of the vectors $\{e_\ell(a_j) \colon \ell = 1, \dots, d-1\}$ will depend on the vectors $\{e_\ell(a_k) \colon \ell = 1, \dots, d-1\}$ corresponding to cubes Q_k with $\bar{Q}_j \cap \bar{Q}_k \neq \emptyset$, for which $\{e_\ell(a_k)\}$ will have been previously defined. We start with the family $\mathcal{A}_1$ of indices $j \notin \mathcal{A}$ so that $\bar{Q}_j \cap \bar{Q}_{j'} \neq \emptyset$ for some index $j' \in \mathcal{A}$. Fix $j \in \mathcal{A}_1$. Observe that since $j \notin \mathcal{A}$, property (a) with $C = 1$ will be satisfied as soon as the vectors $\{e_\ell(a_j) \colon \ell = 1, \dots, d-1\}$ are chosen of unit norm. We first choose the vector $e_1(a_j)$. For every set of non-necessarily distinct indices $j(2), \dots, j(d-1) \in \mathcal{A}$ such that $\bar{Q}_j \cap \bigcap_{\ell=2}^{d-1} \bar{Q}_{j(\ell)} \neq \emptyset$, let $\tilde{M} = \tilde{M}(j, j(2), \dots, j(d-1))$ be the subspace in $\mathbb{R}^d$ generated by the vectors $\{e_\ell(a_{j(\ell)}) \colon \ell = 2, \dots, d-1\}$. Since $j \in \mathcal{A}_1$, we can always choose $j(2) = \cdots = j(d-1) \in \mathcal{A}$ with $\bar{Q}_j \cap \bar{Q}_{j(2)} \neq \emptyset$. Note also that since $j(2), \dots, j(d-1) \in \mathcal{A}$, the vectors $\{e_\ell(a_{j(\ell)}) \colon \ell = 2, \dots, d-1\}$ have already been defined. Observe that for a fixed j, the number of possible choices of indices and hence of subspaces of this form, is bounded by a constant only depending on the dimension. Hence, one can choose the unit vector $e_1(a_j)$ to be at a fixed positive angle, only depending on the dimension, to any such subspace $\tilde{M}$ and also verifying (c). This guarantees (b) for the vectors $\{e_1(a_j), e_\ell(a_{j(\ell)}) \colon \ell = 2, \dots, d-1\}$. We fix $e_1(a_j)$ with these properties. Once $e_1(a_j)$ has been chosen, we choose $e_2(a_j)$ similarly. Now the set of indices

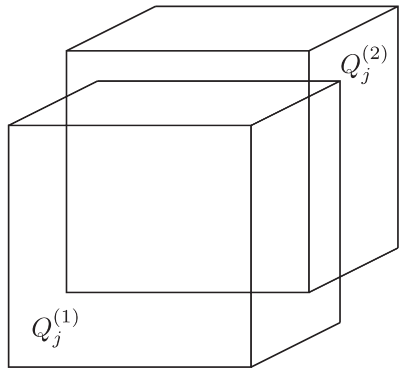

FIGURE 2. *The cubes* $Q_j^{(1)}$ *and* $Q_j^{(2)}$ *in* $\mathbb{R}^3$.

will be of the form $j(1), j(3), \dots, j(d-1)$ where $j(3), \dots, j(d-1) \in \mathcal{A}$ and either $j(1) \in \mathcal{A}$ or $j(1) = j$, such that

$$\bar{Q}_j \cap \bigcap_{\substack{\ell=1 \\ \ell \neq 2}}^{d-1} \bar{Q}_{j(\ell)} \neq \emptyset,$$

and we choose $e_2(a_j)$ to be a unit vector at a fixed positive angle to any subspace in $\mathbb{R}^d$ generated by $\{e_\ell(a_{j(\ell)})\colon \ell = 1, 3, \dots, d-1\}$ and also verifying (c). The vectors $e_3(a_j), \dots, e_{d-1}(a_j)$ are defined similarly. So, the unit vectors $\{e_\ell(a_j)\colon \ell = 1, \dots, d-1\}$ are constructed for any index $j \in \mathcal{A}_1$. Next, using the same procedure, one constructs the unit vectors $\{e_\ell(a_j)\colon \ell = 1, \dots, d-1\}$ for the set $\mathcal{A}_2$ of indices $j \notin \mathcal{A} \cup \mathcal{A}_1$ such that $\bar{Q}_j \cap \bar{Q}_{j'} \neq \emptyset$ for some $j' \in \mathcal{A}_1$. The construction continues inductively.

(4) This paragraph only applies when the dimension $d > 2$. In that case we will need $d-1$ different collections of translated dyadic cubes in order to obtain the convenient one-dimensional estimates. The idea is that the exceptional set will be contained in an intersection of boundaries of cubes which are in different translated collections. This fact will be used later when computing the dimension of the exceptional set.

Denote $Q_j^{(1)} = Q_j$, $j = 1, 2, \dots,$ and for $\ell = 2, \dots, d-1$, let $\{Q_j^{(\ell)}\colon j = 1, 2, \dots\}$ be the collection of pairwise disjoint half-open cubes of sidelength $\ell(Q_j^{(\ell)}) = \ell(Q_j) = 2^{-N_k}$ and center $a_j^{(\ell)}$ given by

$$a_j^{(\ell)} = a_j + \frac{(\ell-1)}{10(d-1)} 2^{-N_k} \mathbf{1}, \quad j = 1, 2, \dots,$$

where $\mathbf{1} = d^{-1/2}(1, \dots, 1) \in \mathbb{R}^d$. In other words, consider $Q_j^{(\ell)} = \lambda(\ell) + Q_j$ where $\lambda(\ell) = (\ell - 1)2^{-N_k}\mathbf{1}/10(d-1)$, $\ell = 2, \dots, d-1$. Observe that for any $\ell = 1, \dots, d-1$, the cubes $\{Q_j^{(\ell)}\colon j = 1, 2, \dots\}$ are pairwise disjoint, $\mathbb{R}^d = \bigcup_j Q_j^{(\ell)}$ and moreover cubes in different families $\{Q_j^{(\ell)}\colon j = 1, 2, \dots\}$ corresponding to different indices ℓ, intersect nicely; see Figure 2. More precisely, for any set of distinct indices $\mathcal{F} \subset \{1, \dots, d-1\}$ of cardinality n, each set of the form

$$\bigcap_{\ell \in \mathcal{F}} \partial Q_{j(\ell)}^{(\ell)}$$

is contained in at most $C(d)$ distinct $(d-n)$-dimensional planes of $\mathbb{R}^d$ parallel to a coordinate hyperplane. Here ∂Q means the boundary of the cube Q. To see this, observe that if $\ell_1 \neq \ell_2$, then the choice of the centers of $\{Q_j^{(\ell)}\}$ guarantees that two parallel faces of $Q_{j(\ell_1)}^{(\ell_1)}$ and $Q_{j(\ell_2)}^{(\ell_2)}$, with $\ell_1 \neq \ell_2$ never meet. So the points in $\bigcap_{\ell\in\mathcal{F}} \partial Q^{(\ell)}_{j(\ell)}$ must lie into at most $C(d)$ distinct $(d-\#\mathcal{F})$-dimensional planes of $\mathbb{R}^d$.

(5) This paragraph applies only when the dimension $d > 2$. Let $\pi_\ell(a_j)$ denote a hyperplane in $\mathbb{R}^d$ with normal unit vector $e_\ell(a_j)$. The content of this paragraph is the existence of a fixed constant $C(d)$ with the following property. For any collection of $d-1$ distinct cubes $\{Q_{j(\ell)}\colon \ell = 1,\dots,d-1\}$ of the family $\mathcal{Q}$ such that $\bigcap_{\ell=1}^{d-1} \bar{Q}_{j(\ell)} \neq \emptyset$ and for any collection $\mathcal{F} \subset \{1,\dots,d-1\}$ of distinct indices, the set

$$\bigcap_{\ell\in\mathcal{F}} \partial Q^{(\ell)}_{j(\ell)} \cap \bigcap_{\ell\in\{1,\dots,d-1\}\setminus\mathcal{F}} \pi_\ell(a_{j(\ell)})$$

is contained in at most $C(d)$ lines in $\mathbb{R}^d$.

This follows from the fact that the set under consideration is contained in a union of at most $C(d)$ sets each of them being an intersection of $d-1$ hyperplanes such that the angle between two of them is bigger than a fixed constant only depending on the dimension (and hence this set is contained in a line). To see this observe that by Paragraph 4, the points in $\bigcap_{\ell\in\mathcal{F}} \partial Q^{(\ell)}_{j(\ell)}$ must lie into at most $C(d)$ distinct $(d-\#\mathcal{F})$-dimensional planes parallel to a coordinate hyperplane of $\mathbb{R}^d$ while by property (c) in Paragraph 3, the angle between the hyperplanes $\{\pi_\ell(a_{j(\ell)})\colon \ell \in \{1,\dots,d-1\}\setminus\mathcal{F}\}$ and any coordinate hyperplane is bigger than $\pi/6$ and by (b) of Paragraph 3 the angle between two different $\pi_\ell(a_{j(\ell)})$ is also bigger than a fixed constant.

(6) In this paragraph, standard bump functions adapted to the collection $\{Q^{(\ell)}_j : j = 1,2,\dots\}$ are constructed. Pick a small positive number $\alpha_k > 0$, $\alpha_k < 2^{-N_k-2}$ and a function $w_k\colon \mathbb{R}^d \longrightarrow \mathbb{R}$, $0 \leqslant w_k \leqslant 1$ with

$$\begin{aligned} &w_k \equiv 0 \quad \text{on } \mathbb{R}^d \setminus [-2^{-N_k-1}, 2^{-N_k-1}]^d,\\ &w_k \equiv 1 \quad \text{on } [-2^{-N_k-1}+\alpha_k, 2^{-N_k-1}-\alpha_k]^d,\\ &\sup_{x\in\mathbb{R}^d} \alpha_k \|\nabla w_k(x)\| = C(d) < \infty. \end{aligned}$$

Recall that the center of the cube $Q^{(\ell)}_j$ was denoted by $a^{(\ell)}_j$ and consider the function $w^{(\ell)}_j(x) = w_k(x - a^{(\ell)}_j)$, $x \in \mathbb{R}^d$. Hence, $w^{(\ell)}_j \equiv 0$ on $\mathbb{R}^d \setminus Q^{(\ell)}_j$, $w^{(\ell)}_j \equiv 1$ on $(1-\alpha_k 2^{N_k+1})Q^{(\ell)}_j$ and $\alpha_k\|\nabla w^{(\ell)}_j(x)\| \leqslant C(d)$ for any $x \in \mathbb{R}^d$.

(7) In this paragraph, a one-dimensional function with *small* size and *large* derivative is constructed. Pick a small positive number $\sigma_k > 0$ with $\sigma_k < \varepsilon_k\alpha_k\eta_k/(1+M_k)$. Pick an integer $n_k > 2\sigma_k^{-1}$ and a small positive number $\beta_k < C(d)2^{-N_k}/n_k$. Consider a smooth one-dimensional periodic function $\phi_k\colon \mathbb{R} \to \mathbb{R}$, satisfying $\phi_k(x+1) = \phi_k(x)$, for any $x \in \mathbb{R}$, such that

$$\begin{aligned} &\sup_{x\in\mathbb{R}} |\phi_k(x)| \leqslant \sigma_k,\\ &\sup_{x\in\mathbb{R}} |\phi'_k(x)| \leqslant \varepsilon_k, \end{aligned}$$

and such that $|\phi'_k(x)| = \varepsilon_k$ for *most* points $x \in \mathbb{R}$. More concretely, we require that the set $\{x \in [-1,1]\colon |\phi'_k(x)| \neq \varepsilon_k\}$ can be covered by $4n_k$ intervals $\{J_i\}$ of length β_k. This can be done by smoothing the function $\psi_k(t) = \varepsilon_k \inf\{|t - i/n_k|\colon i \in \mathbb{Z}\}$. Observe that $\|\psi_k\|_\infty \leqslant \varepsilon_k/n_k < \varepsilon_k\sigma_k < \sigma_k$ and $|\psi'_k(x)| = \varepsilon_k$ if $x \neq i/n_k$ for any $i \in \mathbb{Z}$; see Figure 3. Observe that the lengths of the intervals $\{J_i\}$ where the regularization is performed can be taken as small as desired.

(8) As explained before, the main idea is to construct a function g_{k+1} whose gradient is *essentially* orthogonal to ∇f_k, where $f_k = \sum_{j=0}^k g_j$. To achieve this, roughly speaking, in each cube $Q_j = Q^{(1)}_j$ the function g_{k+1} will look like $\sum_{\ell=1}^{d-1} \phi_k(\langle x - a_j, e_\ell(a_j)\rangle)$, because the gradient of this function is a linear combination of the vectors $\{e_\ell(a_j)\colon \ell = 1,\dots,d-1\}$ which by (a)

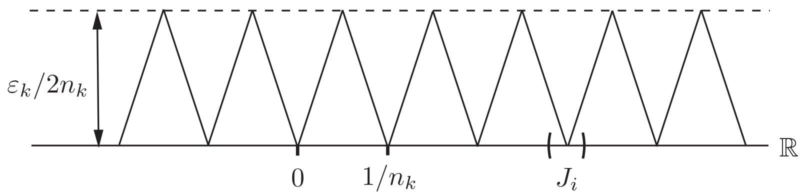

FIGURE 3. *The graph of the function ψ_k.*

of Paragraph 3 are almost orthogonal to $\nabla f_k(a_j)$. The functions $\{w_j^{(\ell)}\}$ of Paragraph 6 will be used to paste together the pieces corresponding to different cubes. More concretely, consider the function

$$g_{k+1}(x) = \sum_{\ell=1}^{d-1} \sum_j w_j^{(\ell)}(x) \phi_k(\langle x - a_j, e_\ell(a_j)\rangle), \quad x \in \mathbb{R}^d,$$

and $f_{k+1} = f_k + g_{k+1}$. Observe that for every $\ell = 1, \dots, d-1$ there is at most one single non-vanishing term in the inner sum. Also observe that for every $\ell = 1, \dots, d-1$ we use the center a_j of $Q_j = Q_j^{(1)}$ and the vectors $\{e_\ell(a_j) : \ell = 1, \dots, d\}$ corresponding to $\nabla f_k(a_j)$. Since $\|w_j^{(\ell)}\|_\infty \leqslant 1$, $\|\nabla w_j^{(\ell)}\|_\infty \leqslant C(d)/\alpha_k$, $\|\phi_k\|_\infty \leqslant \sigma_k$ and $\|\phi_k'\|_\infty \leqslant \varepsilon_k$, we deduce that

$$\sup_{x\in\mathbb{R}^d} |g_{k+1}(x)| \leqslant (d-1)\sigma_k,$$
$$\sup_{x\in\mathbb{R}^d} |\nabla g_{k+1}(x)| \leqslant C\left(\frac{\sigma_k}{\alpha_k} + \varepsilon_k\right) \leqslant 2C\varepsilon_k,$$

because σ_k was chosen such that $\sigma_k < \varepsilon_k \alpha_k$. Here C is a constant only depending on the dimension.

All quantifiers needed to define g_{k+1} have already appeared. They are small numbers which have to be chosen in the right order, such that at each step the new one is smaller than a certain function of the previous ones. Let us summarize the order. Given $\varepsilon_k > 0$ with $\sum \varepsilon_k^2 = \infty$, the quantifiers η_k, 2^{-N_k} are chosen to satisfy (4.1). Given these three quantities, the small number α_k is chosen in Paragraph 6 when the bump functions $w_j^{(\ell)}$ are defined. Finally, the last three ones σ_k, $1/n_k$ and β_k are chosen in Paragraph 7 when the functions ϕ_k are introduced. We continue recursively and we can assume that

$$\sum_{j\geqslant k} \sigma_j \leqslant 2\sigma_k.$$

(9) In this paragraph, we show that $\nabla g_{k+1}(x)$ is almost orthogonal to $\nabla f_k(x)$. This fact will be crucial in the construction. Fix $x \in \mathbb{R}^d$. For $\ell = 1, \dots, d-1$, let $j(\ell)$ be the index such that $x \in Q_{j(\ell)}^{(\ell)}$. Since $\|\nabla w_j^{(\ell)}\|_\infty \leqslant C/\alpha_k$ and $\|\phi_k\|_\infty \leqslant \sigma_k$, we have

$$\nabla g_{k+1}(x) = \sum_{\ell=1}^{d-1} w_{j(\ell)}^{(\ell)}(x) \phi_k'(\langle x - a_{j(\ell)}, e_\ell(a_{j(\ell)})\rangle)\, e_\ell(a_{j(\ell)}) + O(\sigma_k/\alpha_k). \tag{4.3}$$

Here the notation $O(A_k)$ means a vector (or a quantity) whose norm (or modulus) is bounded by a fixed proportion, independent of k and x, of A_k. Since $\bigcap_{\ell=1}^{d-1} Q_{j(\ell)}^{(\ell)} \neq \emptyset$, we have $\bigcap_{\ell=1}^{d-1} \bar{Q}_{j(\ell)} \neq \emptyset$ and condition (a) of Paragraph 3, tells that

$$|\langle \nabla f_k(a_{j(\ell)}), e_\ell(a_{j(\ell)})\rangle| \leqslant C\eta_k^{1/2}.$$

Since by (4.1), one has $\|\nabla f_k(x) - \nabla f_k(a_{j(\ell)})\| \leqslant C\eta_k$, we deduce that $|\langle \nabla f_k(x), e_\ell(a_{j(\ell)})\rangle| \leqslant 2C\eta_k^{1/2}$. Since $\|\phi_k'\|_\infty \leqslant \varepsilon_k$ from (4.3), we deduce that

$$\langle \nabla f_k(x), \nabla g_{k+1}(x)\rangle = O(\eta_k^{1/2}\varepsilon_k) + O\left(\sup_{\mathbb{R}^d} |\nabla f_k| \sigma_k/\alpha_k\right).$$

Since $\sigma_k < \varepsilon_k \eta_k \alpha_k / \sup_{\mathbb{R}^d} |\nabla f_k|$, we deduce that there exists a constant C only depending on the dimension such that for any $x \in \mathbb{R}^d$ one has

$$|\langle \nabla f_k(x), \nabla g_{k+1}(x)\rangle| \leqslant C\varepsilon_k \eta_k^{1/2}. \tag{4.4}$$

(10) We have observed that $\|\nabla g_{k+1}\|_\infty \leqslant C\varepsilon_k$. Next we will show that $\|\nabla g_{k+1}(x)\|$ is comparable to ε_k for all points $x \in \mathbb{R}^d$ except possibly for a set of points A_k for which one has a certain one-dimensional estimate. Here the one dimensional information given by the choice of the vectors $\{e_\ell(a_j)\}$ and the cubes $\{Q_j^{(\ell)}\}$ explained in Paragraph 5 will be used.

Fix $x \in \mathbb{R}^d$. As before, for $\ell = 1, \dots, d-1$, let $j(\ell)$ be the index such that $x \in Q_{j(\ell)}^{(\ell)}$. The formula (4.3) tells that

$$\begin{aligned}\|\nabla g_{k+1}(x)\| &\geqslant \left\| \sum_{\ell=1}^{d-1} w_{j(\ell)}^{(\ell)}(x)\phi_k'(\langle x - a_{j(\ell)}, e a_{j(\ell)}\rangle) e_\ell(a_{j(\ell)})\right\| - O(\sigma_k/\alpha_k) \\ &\geqslant C \sum_{\ell=1}^{d-1} w_{j(\ell)}^{(\ell)}(x) |\phi_k'(\langle x - a_{j(\ell)}, e_\ell(a_{j(\ell)})\rangle)| - O(\sigma_k/\alpha_k) \end{aligned} \tag{4.5}$$

because, since $x \in \bigcap_{\ell=1}^{d-1} Q_{j(\ell)}^{(\ell)}$, we have $\bigcap_{\ell=1}^{d-1} \bar{Q}_{j(\ell)}^{(\ell)} \neq \emptyset$ and consequently, by condition (b) in Paragraph 3, the vectors $\{e_\ell(a_{j(\ell)})\colon \ell = 1, \dots, d-1\}$ form a Riesz set. Now recall that $w_{j(\ell)}^{(\ell)} \equiv 1$ on $(1 - 2^{N_k+1}\alpha_k) Q_{j(\ell)}^{(\ell)}$ and $|\phi_k'(\langle x - a_{j(\ell)}, e_\ell(a_{j(\ell)})\rangle)| = \varepsilon_k$ if $\langle x - a_{j(\ell)}, e_\ell(a_{j(\ell)})\rangle$ is not contained in any of the $4n_k$ intervals in $[-1, 1]$ of length β_k which appeared in the construction of the function ϕ_k in Paragraph 7. Let $J \subset [-1, 1]$ be one of such intervals and let $c(J)$ be its center. Let $\pi_\ell(J) = \pi_\ell(a_{j(\ell)})(J)$ be the hyperplane of $\mathbb{R}^d$ given by

$$\pi_\ell(J) = \{x \in \mathbb{R}^d \colon \langle x, e_\ell(a_{j(\ell)})\rangle = \langle a_{j(\ell)}, e_\ell(a_{j(\ell)})\rangle + c(J)\},$$

and let $\tilde{\pi}_\ell(J) = \tilde{\pi}_\ell(a_{j(\ell)})(J)$ be the neighborhood of $\pi_\ell(J)$ given by

$$\tilde{\pi}_\ell(J) = \{x \in \mathbb{R}^d \colon |\langle x, e_\ell(a_{j(\ell)})\rangle - \langle a_{j(\ell)}, e_\ell(a_{j(\ell)})\rangle - c(J)| < \beta_k\}.$$

Observe that if $x \notin \bigcup_J \tilde{\pi}_\ell(J)$, then $|\phi_k'(\langle x - a_{j(\ell)}, e_\ell(a_{j(\ell)})\rangle)| = \varepsilon_k$. Now given a collection of cubes $\{Q_{j(\ell)}^{(\ell)} \colon \ell = 1, 2, \dots, d-1\}$ such that $\bigcap_{\ell=1}^{d-1} Q_{j(\ell)}^{(\ell)} \neq \emptyset$, let $A(\{Q_{j(\ell)}^{(\ell)} : \ell = 1, \dots, d-1\})$ be the union over all possible collections of distinct indices $\mathcal{F} \subset \{1, \dots, d-1\}$, of sets of the form

$$A(\mathcal{F}) = \bigcap_{\ell \in \mathcal{F}} (Q_{j(\ell)}^{(\ell)} \setminus (1 - 2^{N_k+1}\alpha_k) Q_{j(\ell)}^{(\ell)}) \cap \bigcap_{\ell \in \{1,\dots,d-1\} \setminus \mathcal{F}} \bigcup\nolimits_J \tilde{\pi}_\ell(a_{j(\ell)})(J).$$

Here the union $\bigcup_J$ is taken over the $4n_k$ intervals $J \subset [-1, 1]$ appearing in the construction of the function ϕ_k. Since $\bigcap_{\ell=1}^{d-1} Q_{j(\ell)}^{(\ell)} \neq \emptyset$ one has $\bigcap_{\ell=1}^{d-1} \bar{Q}_{j(\ell)} \neq \emptyset$ and by Paragraph 5 each set

$$\bigcap_{\ell \in \mathcal{F}} \partial Q_{j(\ell)}^{(\ell)} \cap \bigcap_{\ell \in \{1,\dots,d-1\} \setminus \mathcal{F}} \pi_\ell(a_{j(\ell)})$$

is contained in at most $C(d)$ lines in $\mathbb{R}^d$. Since given $\varepsilon_k, \eta_k, N_k > 0$, the quantifiers α_k and β_k can be taken arbitrarily small, each set $A(\mathcal{F})$ is contained in a *small* neighborhood of $C(d)$ lines. Let $A_k = \bigcup A(\{Q_{j(\ell)}^{(\ell)} : \ell = 1, \dots, d-1\})$ where the union is taken over all possible collections $\{Q_{j(\ell)}^{(\ell)} : \ell = 1, \dots, d-1\}$ of cubes for which $\bigcap_{\ell=1}^{d-1} Q_{j(\ell)}^{(\ell)} \neq \emptyset$. Since the set $A(\{Q_{j(\ell)}^{(\ell)} : \ell = 1, \dots, d-1\})$ is a finite union of $A(\mathcal{F})$ and in each bounded set there are at most $C(d)$ collections of cubes $\{Q_{j(\ell)}^{(\ell)}\}$, there exists $\delta_k > 0$, depending on the previous quantifiers ε_k, η_k,

N_k, α_k, β_k with $\delta_k \to 0$, as $k \to \infty$ and a collection of dyadic cubes $\{R_j^{(k)}\}_j$ with sidelength smaller than δ_k such that $A_k \subset \bigcup_j R_j^{(k)}$ and

$$\sum_{j\colon R_j^{(k)} \cap \{\|x\|<N\} \neq \emptyset} \ell(R_j^{(k)}) \leqslant C(N,d), \tag{4.6}$$

for any $N > 0$. Assume now that $x \notin A_k$. As before, for any $\ell = 1, \ldots, d-1$, let $j(\ell)$ be the index such that $x \in Q_{j(\ell)}^{(\ell)}$. Let $\mathcal{F}$ be the set of indices ℓ in $\{1, \ldots, d-1\}$ such that $x \in (1 - 2^{N_k+1}\alpha_k)Q_{j(\ell)}^{(\ell)}$ and hence $w_{j(\ell)}^{(\ell)}(x) = 1$. Hence, $x \in \bigcap_{\ell \notin \mathcal{F}} Q_{j(\ell)}^{(\ell)} \setminus (1 - 2^{N_k+1}\alpha_k)Q_{j(\ell)}^{(\ell)}$ and since $x \notin A_k$, then $x \notin \bigcap_{\ell \in \mathcal{F}} \bigcup_J \tilde{\pi}_\ell(a_{j(\ell)})(J)$. Hence, there exists $\ell \in \mathcal{F}$ with $x \notin \bigcup_J \tilde{\pi}_\ell(a_{j(\ell)})(J)$ and we deduce that $|\phi_k'(\langle x - a_{j(\ell)}, e_\ell(a_{j(\ell)})\rangle)| = \varepsilon_k$. Therefore, using (4.5) we get

$$\begin{aligned}\|\nabla g_{k+1}(x)\| &\geqslant C \sum_{\ell=1}^{d-1} w_{j(\ell)}^{(\ell)}(x) |\phi_k'(\langle x - a_{j(\ell)}, e_\ell(a_{j(\ell)})\rangle)| - O(\sigma_k/\alpha_k) \\ &\geqslant C\varepsilon_k - O(\sigma_k/\alpha_k).\end{aligned}$$

Since $\sigma_k/\varepsilon_k\alpha_k$ was taken arbitrarily small, we deduce that

$$\|\nabla g_{k+1}(x)\| \geqslant C\varepsilon_k \quad \text{for } x \in \mathbb{R}^d \setminus A_k. \tag{4.7}$$

(11) We now prove that there exists a fix constant $C = C(d) > 0$ such that if k is sufficiently large, then one has

$$\|\nabla(f_k + g_{k+1})(x)\|^2 \geqslant \|\nabla f_k(x)\|^2 + C\varepsilon_k^2, \tag{4.8}$$

for any $x \in \mathbb{R}^d \setminus A_k$. To prove (4.8), write

$$\|\nabla(f_k + g_{k+1})(x)\|^2 = \|\nabla f_k(x)\|^2 + \|\nabla g_{k+1}(x)\|^2 + 2\langle \nabla f_k(x), \nabla g_{k+1}(x)\rangle.$$

By (4.4), we have $\langle \nabla f_k(x), \nabla g_{k+1}(x)\rangle = O(\varepsilon_k \eta_k^{1/2})$ while (4.7) tells that $\|\nabla g_{k+1}(x)\| \geqslant C\varepsilon_k$ if $x \in \mathbb{R}^d \setminus A_k$. Therefore, for any $x \in \mathbb{R}^d \setminus A_k$ one has

$$\|\nabla(f_k + g_{k+1})(x)\|^2 \geqslant \|\nabla f_k(x)\|^2 + C\varepsilon_k^2 - O(\varepsilon_k \eta_k^{1/2}).$$

Since η_k was chosen in Paragraph 1 such that $\eta_k < \varepsilon_k^3$ we deduce (4.8).

(12) In this paragraph, we define the function f in $\lambda_*(\mathbb{R}^d)$ whose divided differences are unbounded at any point except for a set of Hausdorff dimension 1. Since $\|g_{k+1}\|_\infty \leqslant (d-1)\sigma_k$ and $\sum \sigma_k < \infty$, we can consider

$$f = \sum_{k=0}^{\infty} g_k.$$

The function f is continuous in $\mathbb{R}^d$. In the next paragraphs, it will be shown that $f \in \lambda_*(\mathbb{R}^d)$, f has unbounded divided differences at any point of the set $\mathbb{R}^d \setminus A$, where

$$A = \bigcap_{j=1}^{\infty} \bigcup_{k \geqslant j} A_k,$$

and that A has σ-finite length.

(13) We first show that $f \in \lambda_*(\mathbb{R}^d)$. We will show that the functions $f_k = \sum_{j=0}^k g_j$ are uniformly in the small Zygmund class. Fix an integer $k > 0$. Let $h \in \mathbb{R}^d$ and let $\{N_j\}$ be the quantifiers appearing in Paragraph 1. We use the notation $\Delta_2 f(x,h) = f(x+h) + f(x-h) - 2f(x)$, for $x, h \in \mathbb{R}^d$. We distinguish two possible situations.

(a) Assume $\|h\| < 2^{-N_k}$. Since (4.1) tells that $\|\nabla f_k(z) - \nabla f_k(w)\| < \eta_k$ if $\|z - w\| < 2^{-N_k}$, then we have $|\Delta_2 f_k(x, h)| \leqslant \eta_k \|h\|$. On the other hand, since $\|\nabla g_{k+1}\|_\infty \leqslant C\varepsilon_k$, we deduce that

$$\begin{aligned}|\Delta_2 g_{k+1}(x, h)| &\leqslant |g_{k+1}(x + h) - g_{k+1}(x)| + |g_{k+1}(x) - g_{k+1}(x - h)|\\ &\leqslant 2C\varepsilon_k \|h\|.\end{aligned}$$

So, we obtain that

$$|\Delta_2 f_{k+1}(x, h)| \leqslant (\eta_k + 2C\varepsilon_k)\|h\|. \tag{4.9}$$

(b) Assume $2^{-N_{j+1}} < \|h\| \leqslant 2^{-N_j}$ for some $j < k$. We have

$$|\Delta_2 g_{k+1}(x, h)| \leqslant 4\|g_{k+1}\|_\infty \leqslant 4(d - 1)\sigma_k \leqslant \eta_k 2^{-N_k} < \eta_k \|h\|.$$

Therefore, we have

$$|\Delta_2 f_{k+1}(x, h)| \leqslant |\Delta_2 f_k(x, h)| + \eta_k \|h\|.$$

Iterating this estimate, we obtain that if $2^{-N_{j+1}} < \|h\| \leqslant 2^{-N_j}$, then one has

$$|\Delta_2 f_{k+1}(x, h)| \leqslant |\Delta_2 f_{j+1}(x, h)| + \left(\sum_{i=j+1}^{k} \eta_i\right) \|h\|.$$

Now, case (a) applies to $\Delta_2 f_{j+1}(x, h)$ and by (4.9) we obtain $|\Delta_2 f_{j+1}(x, h)| \leqslant (\eta_j + 2C\varepsilon_j)\|h\|$. Hence, if $2^{-N_{j+1}} \leqslant \|h\| \leqslant 2^{-N_j}$, then we get

$$|\Delta_2 f_{k+1}(x, h)| \leqslant \left(\sum_{i=j}^{k} \eta_i + 2C\varepsilon_j\right) \|h\|. \tag{4.10}$$

Now, since $\sum \eta_i < \infty$ and $\varepsilon_j \to 0$, we deduce that f is in $\lambda_*(\mathbb{R}^d)$.

(14) Let $A = \bigcap_j \bigcup_{k \geqslant j} A_k$. In this paragraph, we show that f has unbounded divided differences at the points of $\mathbb{R}^d \setminus A$. Let $z, x \in \mathbb{R}^d$. One has

$$f(z) - f(x) = f_k(z) - f_k(x) + \sum_{j>k} (g_j(z) - g_j(x)).$$

Since $\|g_j\|_\infty \leqslant (d - 1)\sigma_{j-1}$, we deduce that

$$|f(z) - f(x) - (f_k(z) - f_k(x))| \leqslant 2(d - 1) \left(\sum_{j>k} \sigma_j\right).$$

On the other hand, applying (4.1) one gets

$$|f_k(z) - f_k(x) - \langle \nabla f_k(x), z - x\rangle| \leqslant 2\eta_k \|z - x\|,$$

if $\|z - x\| \leqslant 2^{-N_k}$. Hence,

$$|f(z) - f(x) - \langle \nabla f_k(x), z - x\rangle| \leqslant 2\eta_k \|z - x\| + 2(d - 1) \sum_{j>k} \sigma_j,$$

if $\|z - x\| \leqslant 2^{-N_k}$. Since $\sum_{j \geqslant k} \sigma_j \leqslant 2\sigma_k$, we have

$$\left| \frac{f(z) - f(x)}{\|z - x\|} - \left\langle \nabla f_k(x), \frac{(z - x)}{\|z - x\|} \right\rangle \right| \leqslant 2\eta_k + 2(d - 1)\sigma_k/\|z - x\|, \tag{4.11}$$

if $\|z - x\| \leqslant 2^{-N_k}$. We can now prove that

$$\left\{ x \in \mathbb{R}^d \colon \limsup_{z \to x} \frac{|f(z) - f(x)|}{\|z - x\|} < \infty \right\} \subseteq A. \tag{4.12}$$

Indeed, fix $x \notin A$, that is, $x \in \bigcap_{k \geqslant j}(\mathbb{R}^d \setminus A_k)$ for a certain index j. Applying (4.8) in Paragraph 11, for any $k \geqslant j$ with j sufficiently large, we have

$$\|\nabla f_{k+1}(x)\|^2 \geqslant \|\nabla f_k(x)\|^2 + C\varepsilon_k^2,$$

and iterating

$$\|\nabla f_{k+1}(x)\|^2 \geqslant C \sum_{i=j}^{k} \varepsilon_i^2.$$

Since $\sum \varepsilon_k^2 = \infty$, we deduce that

$$\lim_{k \to \infty} \|\nabla f_k(x)\| = \infty.$$

Now choose z in (4.11) such that $\|z - x\| = \sigma_k$ and such that $z - x$ is an scalar positive multiple of $\nabla f_k(x)$. We deduce that

$$\left| \frac{f(z) - f(x)}{\|z - x\|} - \|\nabla f_k(x)\| \right| \leqslant 2\eta_k + 2(d-1).$$

Consequently,

$$\limsup_{z \to x} \frac{|f(z) - f(x)|}{\|z - x\|} = \infty,$$

which proves (4.12).

(15) Finally, we only have to show that A has σ-finite length. Recall that (4.6) tells that there exist $\delta_k \to 0$ and a collection of cubes $\{R_j^{(k)} : j = 1, 2, \ldots\}$, $\ell(R_j^{(k)}) \leqslant \delta_k$ such that

$$A_k \subset \bigcup_j R_j^{(k)}$$

$$\sum_{j\,:\; R_j^{(k)} \cap \{\|x\| < N\} \neq \emptyset} \ell(R_j^{(k)}) \leqslant C(N, d),$$

for any N. Then $A = \bigcap_j \bigcup_{k \geqslant j} A_k$ has σ-finite length. This completes the proof of Theorem 2.

It is worth mentioning that the set $E(f)$ has σ-finite length (and hence Hausdorff dimension 1). It seems likely that one could combine the construction above with one-dimensional constructions to produce a function $f \in \lambda_*(\mathbb{R}^d)$ such that $E(f)$ has zero length. However, there seems to be some technical difficulties in following this plan and we have not done it.

5. *Conservative martingales*

Let Q_0 be the unit cube of $\mathbb{R}^d$. A sequence of functions $\{S_n : n = 1, 2, \ldots\}$, $S_n \colon Q_0 \to \mathbb{R}$, is a dyadic martingale if for any $n = 1, 2, \ldots$, then the function S_n is constant on each dyadic cube of generation n and

$$\frac{1}{|Q|} \int_Q S_{n+1}\, dm = S_{n\,|Q},$$

for any dyadic cube Q of generation n. Here dm is Lebesgue measure in $\mathbb{R}^d$. This corresponds to the standard notion of martingale when the probability space is given by Lebesgue measure in the unit cube and the filtration is the one generated by the dyadic decomposition. A dyadic martingale $\{S_n\}$ is in the Bloch space if there exists a constant $C = C(\{S_n\}) > 0$ such that

$$|S_{n\,|Q} - S_{n\,|Q'}| \leqslant C,$$

for any pair of dyadic cubes Q, Q' with $\ell(Q) = \ell(Q') = 2^{-n}$ and $\bar{Q} \cap \bar{Q}' \neq \emptyset$. The infimum of the constants $C > 0$ satisfying the estimate above is called the Bloch (semi)norm of $\{S_n\}$ and will be denoted by $\|S_n\|_*$. The Little Bloch space is the subspace of those dyadic martingales $\{S_n\}$ in the Bloch space for which

$$\lim_{n\to\infty} \sup |S_{n|Q} - S_{n|Q'}| \to 0,$$

where the supremum is taken over all pairs of dyadic cubes Q, Q' with $\ell(Q) = \ell(Q') = 2^{-n}$ and $\bar{Q} \cap \bar{Q}' \neq \emptyset$. It is well known that a Bloch dyadic martingale $\{S_n\}$ may converge at no point, that is, it may happen that

$$\lim_{n\to\infty} S_n(x)$$

does not exist for any $x \in Q_0$. However, a Bloch dyadic martingale is bounded at a set of maximal Hausdorff dimension, that is, $\dim\{x \in Q_0 \colon \limsup_{n\to\infty} |S_n(x)| < \infty\} = d$. It is worth mentioning that the set above may have volume zero. Similarly, the set of points where a dyadic martingale in the Little Bloch space converges may have volume zero but it has always maximal Hausdorff dimension; see [**11**]. So the situation is analogous to the one described in the introduction for the divided differences of a function in the Zygmund class.

Let $\{e_i \colon i = 1, \ldots, d\}$ be the canonical basis of $\mathbb{R}^d$. A sequence of mappings $\{\mathbf{S}_n\}$, $\mathbf{S}_n \colon Q_0 \to \mathbb{R}^d$, $n = 1, 2, \ldots$, is called a dyadic vector-valued martingale if for any $i = 1, 2, \ldots, d$, then the corresponding component $\{\langle \mathbf{S}_n, e_i\rangle\}_n$ is a dyadic martingale. A dyadic vector-valued martingale $\{\mathbf{S}_n\}$ satisfies the Bloch condition if so does each of its components $\{\langle \mathbf{S}_n, e_i\rangle\}_n$, $i = 1, \ldots, d$. Similarly, $\{\mathbf{S}_n\}$ is in the Little Bloch space if for any $i = 1, \ldots, d$, then the scalar martingale $\{\langle \mathbf{S}_n, e_i\rangle\}$ is in the Little Bloch space. In contrast with the scalar case, when $d \geqslant 2$ there exist Bloch vectorial dyadic martingales $\{\mathbf{S}_n\}$ in $Q_0 \subset \mathbb{R}^d$ for which

$$\limsup_{n\to\infty} \|\mathbf{S}_n(x)\| = \infty,$$

for any $x \in Q_0$; see [**11**, p. 34; **19**]. However, the proof of Theorem 1 suggests that there is a natural class of Bloch dyadic vectorial martingales which are bounded at a set of points which has Hausdorff dimension bigger or equal to one.

Let $d \geqslant 2$. A Bloch dyadic vector-valued martingale $\{\mathbf{S}_n\}$, $\mathbf{S}_n \colon Q_0 \to \mathbb{R}^d$, is called conservative if there exists a constant $C = C(\{\mathbf{S}_n\}) > 0$ such that for any dyadic subcube Q of Q_0 with $\ell(Q) = 2^{-n}$ and any polygonal $\gamma \subset \bar{Q}$ which intersects two different parallel faces of Q one has

$$\left| \int_\gamma \mathbf{S}_{n+k}\, d\gamma - \int_\gamma \mathbf{S}_n\, d\gamma \right| \leqslant C\ell(Q), \tag{5.1}$$

for any $k = 1, 2, \ldots$. Here $\int_\gamma \mathbf{S}_m d\gamma$ denotes the line integral of $\mathbf{S}_m$ through γ, that is, if the curve γ is parameterized by $\gamma \colon [0,1] \to \mathbb{R}^d$, then the line integral is

$$\int_\gamma \mathbf{S}_m\, d\gamma = \int_0^1 \langle \mathbf{S}_m(\gamma(t)), \gamma'(t)\rangle\, dt.$$

Property (5.1) should be understood as a one-dimensional mean value property. For instance, taking $Q = Q_0$, $n = 0$ and γ a unit segment in the direction of e_i, one deduces that for any $i = 1, \ldots, d$, one has

$$\sup_k \left| \int_0^1 \langle \mathbf{S}_k(te_i), e_i\rangle\, dt - \langle \mathbf{S}_0 | Q_0, e_i\rangle \right| \leqslant C.$$

So, the mean of $\langle \mathbf{S}_k, e_i\rangle$ over a unit segment in the direction e_i is, up to bounded terms, the value of $\langle \mathbf{S}_0, e_i\rangle$ on the unit cube. Given a function in the Zygmund class there is a natural conservative Bloch dyadic vectorial martingale which governs the behavior of the divided differences of the function. Actually, let $f \colon \mathbb{R}^d \to \mathbb{R}$ be a function in the Zygmund class. For a dyadic cube $Q \subset Q_0 \subset \mathbb{R}^d$ of generation n, the value $\mathbf{S}_n \mid Q$ of the vectorial martingale is defined as the vector in $\mathbb{R}^d$ whose components are given by

$$\langle \mathbf{S}_n \mid Q, e_i \rangle = \frac{1}{\ell(Q)^d} \left(\int_{Q^+(i)} f\, dA - \int_{Q^-(i)} f\, dA \right), \quad i = 1, \ldots, d, \tag{5.2}$$

where $Q^+(i)$ and $Q^-(i)$ are the two opposite faces of Q which are orthogonal to e_i such that the ith coordinate of the points of $Q^+(i)$ is bigger than the ith coordinate of the points of $Q^-(i)$. Here dA is the $(d-1)$-dimensional Lebesgue measure. Let us now check that the martingale $\{\mathbf{S}_n\}$ is conservative. Let Q be a dyadic cube with $\ell(Q) = 2^{-n}$. Let γ be a polygonal which intersects two different parallel faces of Q at the points, say, A and B. Given an integer $k > 0$, choose points $A_0 = A, A_1, \ldots, A_N = B$ in $\gamma \cap Q$ with $2^{-n-k-1} \leqslant \|A_{i+1} - A_i\| < 2^{-n-k}$ for any $i = 0, \ldots, N-1$ and let Q_i be the dyadic subcube of Q which contains A_i with $\ell(Q_i) = 2^{-n-k}$. Lemma 2 tells that there exists an absolute constant $C > 0$ such that

$$|f(B) - f(A) - \langle \mathbf{S}_n | Q, B - A \rangle| \leqslant C \|f\|_* \|B - A\|.$$

Similarly, for any $j = 0, \ldots, N-1$, we have

$$|f(A_{j+1}) - f(A_j) - \langle \mathbf{S}_{n+k} | Q_j, A_{j+1} - A_j \rangle| \leqslant C \|f\|_* \|A_{j+1} - A_j\|.$$

Then condition (5.1) follows easily. The proof of Theorem 1 applies in this context and one can obtain the following result.

THEOREM 3. (a) *Let $\{\mathbf{S}_n\}$ be a conservative Bloch dyadic vectorial martingale in $Q_0 \subset \mathbb{R}^d$. Then the set $\{x \in Q_0\colon \limsup_{n\to\infty} \|\mathbf{S}_n(x)\| < \infty\}$ has Hausdorff dimension bigger or equal to one.*

(b) *Let $\{\mathbf{S}_n\}$ be a conservative dyadic vectorial martingale in $Q_0 \subset \mathbb{R}^d$. Assume that $\{\mathbf{S}_n\}$ is in the Little Bloch space. Then the set*

$$\left\{ x \in Q_0\colon \lim_{n\to\infty} \mathbf{S}_n(x) \text{ exists} \right\}$$

has Hausdorff dimension bigger or equal to 1.

It is worth mentioning that Theorem 3 applied to the martingale defined in (5.2) implies Theorem 1.

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