Convergence of linear combinations of iterates of an inner function

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Abstract

Let f be an inner function with $f(0) = 0$ which is not a rotation and let f^n be its n-th iterate. Let $\{a_n\}$ be a sequence of complex numbers. We prove that the series $\sum a_n f^n(\xi)$ converges at almost every point ξ of the unit circle if and only if $\sum |a_n|^2 < \infty$. The main step in the proof is to show that under this assumption, the function $F = \sum a_n f^n$ has bounded mean oscillation. We also prove that F is bounded on the unit disc if and only if $\sum |a_n| < \infty$. Finally we describe the sequences of coefficients $\{a_n\}$ such that F belongs to other classical function spaces, as the disc algebra and the Dirichlet class.

Keywords. Inner function, bounded mean oscillation, Hardy spaces, Dirichlet class.

1 Introduction

Inner functions are analytic mappings from the unit disc $\mathbb{D}$ of the complex plane into itself whose radial limits are of modulus one at almost every point of the unit circle $\partial\mathbb{D}$. Inner functions are a central notion in Complex and Functional Analysis. See for instance [Ga]. Any inner function f induces a map from $\partial\mathbb{D}$ into itself defined at almost every point $\xi \in \partial\mathbb{D}$ by $f(\xi) = \lim_{r\to 1} f(r\xi)$. Let f^n denote the n-th iterate of f, which is again an inner function. The main purpose of this paper is to study the convergence of linear combinations of iterates of an inner function and to show that in a certain sense, the iterates $f^n \colon \partial\mathbb{D} \to \partial\mathbb{D}$ behave as independent random variables. This phenomena has been extensively studied in the context of random series of functions ([Ka]) but it also occurs in other settings where independence is not present, as in the theory of lacunary series (see Section 6 of Chapter V and Section 5 of Chapter XVI of [Zy]).

We start discussing pointwise convergence. Let $\{X_n\}$ be a sequence of independent random variables with mean 0 and finite variances $V(X_n)$. The classical Khintchine–Kolmogorov Convergence Theorem asserts that $\sum X_n$ converges almost surely if and only if $\sum V(X_n) < \infty$. In our context we have the following analogous result.

Theorem 1.1. *Let f be an inner function with $f(0) = 0$ which is not a rotation and let $\{a_n\}$ be a sequence of complex numbers. Then the following conditions are equivalent:*

(a) The series $\sum_{n=1}^{\infty} a_n f^n(\xi)$ converges at almost every point $\xi \in \partial\mathbb{D}$.

(b) The set $\left\{\xi \in \partial\mathbb{D} : \sup_N \left|\sum_{n=1}^{N} a_n f^n(\xi)\right| < \infty\right\}$ has positive Lebesgue measure.

(c) The complex numbers $\{a_n\}$ satisfy $\sum_{n=1}^{\infty} |a_n|^2 < \infty$.

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Let m denote normalized Lebesgue measure in $\partial\mathbb{D}$. For $0 < p < \infty$ let $\mathbb{H}^p(\mathbb{D})$ be the classical Hardy space of analytic functions F in $\mathbb{D}$ such that

$$\|F\|_p^p = \sup_{r<1} \int_{\partial\mathbb{D}} |F(r\xi)|^p \, dm(\xi) < \infty.$$

Let $\mathbb{H}^\infty(\mathbb{D})$ be the algebra of bounded analytic functions F in $\mathbb{D}$ and $\|F\|_\infty = \sup\{|F(z)| : z \in \mathbb{D}\}$. Let BMOA be the space of analytic functions F in $\mathbb{D}$ such that

$$\|F\|^2_{\mathrm{BMOA}(\mathbb{D})} = \sup_{z\in\mathbb{D}} \int_{\partial\mathbb{D}} |F(\xi) - F(z)|^2 P(z,\xi) \, dm(\xi) < \infty.$$

Here $P(z,\xi) = (1-|z|^2)|\xi - z|^{-2}$ is the Poisson kernel. The subspace VMOA is formed by the functions $F \in$ BMOA such that the integral above tends to 0 as $|z|$ tends to 1. Any function $F \in \mathbb{H}^p(\mathbb{D})$, $0 < p \le \infty$, has non-tangential limit, denoted by $F(\xi)$, at almost every point $\xi \in \partial\mathbb{D}$. Moreover if $p \ge 1$, then

$$F(z) = \int_{\partial\mathbb{D}} F(\xi) P(z,\xi) \, dm(\xi), \quad z \in \mathbb{D}.$$

We have $\mathbb{H}^\infty(\mathbb{D}) \subseteq \mathrm{BMOA}(\mathbb{D}) \subseteq \mathbb{H}^p(\mathbb{D})$ for any $0 < p < \infty$. Actually $\mathrm{BMOA}(\mathbb{D})$ is the natural substitute of $\mathbb{H}^\infty(\mathbb{D})$ in several results of the theory of Hardy spaces. See [Ga] for all these well known results.

The proof of Theorem 1.1 is based in the following BMOA type estimate.

Theorem 1.2. *Let f be an inner function with $f(0) = 0$ which is not a rotation. Then there exists a constant $C = C(f) > 0$ such that for any positive integer N, any set $\{a_n : n = 1, \dots, N\}$ of complex numbers and any point $z \in \mathbb{D}$, we have*

$$\int_{\partial\mathbb{D}} \left| \sum_{n=1}^N a_n (f^n(\xi) - f^n(z)) \right|^2 P(z,\xi) \, dm(\xi) \le C \sum_{n=1}^N |a_n|^2 (1 - |f^n(z)|^2).$$

Let f be an analytic selfmapping of $\mathbb{D}$ with $f(0) = 0$ which is not a rotation. The classical Denjoy–Wolff Theorem asserts that f^n tends to 0 uniformly on compact sets of $\mathbb{D}$. See Chapter V of [Sh]. The proof of Theorem 1.2 uses the interplay between those dynamical properties of the inner function f as a self mapping of $\partial\mathbb{D}$ and those as a self mapping of $\mathbb{D}$. In particular we use a result due to Pommerenke ([Po]) which provides an exponential decay in the Denjoy–Wolff Theorem.

It was proved in [NS2] that $\sum a_n f^n$ converges in $\mathbb{H}^2(\mathbb{D})$ if and only if $\sum |a_n|^2 < \infty$. As a consequence of Theorem 1.2 we will show that under this last condition, we actually have $\sum a_n f^n \in \mathrm{BMOA}(\mathbb{D})$.

Corollary 1.3. *Let f be an inner function with $f(0) = 0$ which is not a rotation and let $\{a_n\}$ be a sequence of complex numbers. Assume $\sum_{n=1}^{\infty} |a_n|^2 < \infty$.*

(a) Then $F = \sum_{n=1}^{\infty} a_n f^n \in \mathrm{BMOA}(\mathbb{D})$ and for almost every $\xi \in \partial\mathbb{D}$ we have

$$\lim_{r\to 1} F(r\xi) = \sum_{n=1}^{\infty} a_n f^n(\xi). \tag{1.1}$$

Moreover there exists a constant $C = C(f) > 0$ only depending on f, such that

$$C^{-1} \sum_{n=1}^{\infty} |a_n|^2 \le \|F\|^2_{\mathrm{BMOA}(\mathbb{D})} \le C \sum_{n=1}^{\infty} |a_n|^2.$$

(b) If f is a finite Blaschke product, then $F = \sum_{n=1}^{\infty} a_n f^n \in \mathrm{VMOA}$.

The following Khintchine type estimate follows easily from the previous result.

Corollary 1.4. *Let f be an inner function with $f(0) = 0$ which is not a rotation and let $\{a_n\}$ be a sequence of complex numbers. Assume $\sum_{n=1}^{\infty} |a_n|^2 < \infty$. Then $\sum_{n=1}^{\infty} a_n f^n \in \mathbb{H}^p(\mathbb{D})$ for any $0 < p < \infty$. Moreover for any $0 < p < \infty$, there exists a constant $C(p, f) > 0$ such that for any sequence of complex numbers $\{a_n\}$ we have*

$$C(p,f)^{-1} \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2} \leq \left\| \sum_{n=1}^{\infty} a_n f^n \right\|_p \leq C(p,f) \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2} .$$

By Theorem 1.1, if $\sum |a_n|^2 < \infty$, the series $\sum a_n f^n(\xi)$ converges at almost every $\xi \in \partial\mathbb{D}$, while if $\sum |a_n|^2 = \infty$, the series $\sum a_n f^n(\xi)$ converges at almost no point $\xi \in \partial\mathbb{D}$. If the coefficients satisfy the stronger condition $\sum |a_n| < \infty$, it is clear that $\sum a_n f^n \in \mathbb{H}^\infty(\mathbb{D})$. Our next result says that this condition is also necessary.

Theorem 1.5. *Let f be an inner function with $f(0) = 0$ which is not a rotation. Let $\{a_n\}$ be a sequence of complex numbers with $\sum_{n=1}^{\infty} |a_n|^2 < \infty$ and $F = \sum_{n=1}^{\infty} a_n f^n$. Assume there exists an arc $I \subset \partial\mathbb{D}$ such that* $\sup\{|F(\xi)| : \xi \in I\} < \infty$. *Then $\sum_{n=1}^{\infty} |a_n| < \infty$.*

The proof of Theorem 1.5 is the most technical part of the paper. The main idea is to construct by induction a sequence of arcs $I_k \subset I$ and a sequence M_k of positive integers tending to infinity such that

$$\frac{1}{m(I_k)} \int_{I_k} \operatorname{Re} F \, dm \geq A \sum_{j=1}^{M_k} |a_j| - B, \quad k = 1, 2, \ldots, \tag{1.2}$$

where A and B are positive constants independent of k. We start showing that if $|f(z)|$ is sufficiently small, the two terms in the conclusion of Theorem 1.2 are comparable. This provides a way of finding arcs $I_k \subset I$ and of splitting $F = \sum_j F_j$ into blocks F_k such that

$$\frac{1}{m(I_k)} \int_{I_k} \sum_{j=1}^{k} \operatorname{Re} F_j \, dm \geq C \sum_{j=1}^{k} \| \operatorname{Re} F_j \|_2, \quad k = 1, 2, \ldots$$

Since the blocks F_j have a uniformly bounded number of terms, $\| \operatorname{Re} F_j \|_2$ can be estimated from below by the sum of the modulus of the corresponding coefficients and (1.2) follows easily.

Let $\overline{\mathbb{H}^\infty(\mathbb{D})}$ denote the closure of $\mathbb{H}^\infty(\mathbb{D})$ in $\mathrm{BMOA}(\mathbb{D})$. This space was studied in [GJ]. See also [NS1] and [SS]. Theorems 1.1, 1.5 and Corollary 1.3 lead to the following result.

Corollary 1.6. *Let f be an inner function with $f(0) = 0$ which is not a rotation. Let $\{a_n\}$ be a sequence of complex numbers with $\sum_{n=1}^{\infty} |a_n|^2 < \infty$ but $\sum_{n=1}^{\infty} |a_n| = \infty$. Then $F = \sum_{n=1}^{\infty} a_n f^n \in \overline{\mathbb{H}^\infty(\mathbb{D})}$. Moreover*

$$F(\xi) = \lim_{r \to 1} F(r\xi) = \sum_{n=1}^{\infty} a_n f^n(\xi)$$

at almost every $\xi \in \partial\mathbb{D}$ but $\sup\{|F(\xi)| : \xi \in I\} = \infty$ *for any arc $I \subset \partial\mathbb{D}$.*

Theorems 1.1 and 1.5 and Corollaries 1.3 and 1.4 can be understood as analogues to classical results in the theory of lacunary series (see Section 6 of Chapters V and VI of [Zy]). However it should be noted that no lacunarity assumption is needed in our results.

The paper is organized as follows. Next section is devoted to the proof of Theorems 1.1, 1.2 and Corollaries 1.3 and 1.4. In Section 3 we start collecting several auxiliary results which are used in the proof of Theorem 1.5. In Section 4 we describe in terms of the coefficients, those linear combinations of iterates which belong to certain classical function spaces, such as the disc algebra, the Dirichlet space or the Bloch space. Finally last section is devoted to state some related open questions we have not explored.

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2 BMO estimates and pointwise convergence

Let $\rho(z,w)$ denote the pseudohyperbolic distance between the points $z,w \in \mathbb{D}$ defined as $\rho(z,w) = |z-w||1-\overline{w}z|^{-1}$. Schwarz's Lemma asserts that any analytic selfmapping f of $\mathbb{D}$ contracts hyperbolic distances, that is, $\rho(f(z),f(w)) \leq \rho(z,w)$ for any $z,w \in \mathbb{D}$. Equivalently $D_h(f)(z) \leq 1$, $z \in \mathbb{D}$, where

$$D_h(f)(z) = \frac{(1-|z|^2)|f'(z)|}{1-|f(z)|^2}, \quad z \in \mathbb{D}, \tag{2.1}$$

is the hyperbolic derivative of f at the point z. Our first auxiliary result is a quantitative version of the Denjoy–Wolff Theorem which is essentially due to Pommerenke ([Po]). Its short proof is included for the sake of completeness.

Lemma 2.1. *Let $f \in H^\infty(\mathbb{D})$, $\|f\|_\infty \leq 1$ with $f(0) = 0$ which is not a rotation.*

(a) Then, for any $0 < r < 1$, there exists $0 < c = c(r,f) < 1$ such that $1-|z| \leq c(1-|f(z)|)$ if $|z| \geq r$.

(b) Assume $f'(0) \neq 0$. Then there exists $0 < r_0 = r_0(f) < 1$ such that

$$|f^n(z)| \leq r_0^{-1}|f'(0)|^n|z|, \quad n \geq 1, \quad \textit{if} \quad |z| \leq r_0.$$

(c) Assume $f'(0) = 0$. Then $|f^n(z)| \leq |z|^{2^n}$, $z \in \mathbb{D}$.

Proof. Since $f(0) = 0$, Schwarz's Lemma gives

$$\rho\left(\frac{f(z)}{z}, f'(0)\right) \leq |z|, \quad z \in \mathbb{D}.$$

Denote $\psi(z) = z(z+|f'(0)|)(1+|f'(0)|z)^{-1}$, $z \in \mathbb{D}$, to obtain

$$|f(z)| \leq \psi(|z|), \quad z \in \mathbb{D}. \tag{2.2}$$

Then

$$1-|f(z)| \geq 1-\psi(|z|) = (1-|z|)\frac{1+|z|}{1+|f'(0)||z|}.$$

Given $0 < r < 1$, taking $c = (1+r)^{-1}(1+|f'(0)|r)$ the estimate in (a) follows.

We now prove (b). For $n = 1, 2, \dots$, consider the function $g_n(z) = \psi^n(z)|f'(0)|^{-n}$. It is known that $\{g_n\}$ converges uniformly on compact sets of $\mathbb{D}$ to an analytic function g on $\mathbb{D}$, known as the Königs function of ψ, which satisfies $g(\psi(z)) = |f'(0)|g(z)$, $z \in \mathbb{D}$. See [Sh, pp. 89–93]. Moreover for $0 \leq x \leq 1$ we have

$$g_{n+1}(x) = \frac{\psi^{n+1}(x)}{|f'(0)|^{n+1}} = g_n(x)\frac{1+|f'(0)|^{n-1}g_n(x)}{1+|f'(0)|^{n+1}g_n(x)} \geq g_n(x).$$

Hence $g_n(x) \le g(x)$, $n = 1, 2, \dots, 0 < x < 1$. Note that there exists a constant $\delta = \delta(|f'(0)|) > 0$ such that ψ is univalent in $\{z \in \mathbb{D} : |z| < \delta\}$. Hence g_n is also univalent in $\{z \in \mathbb{D} : |z| < \delta\}$. By Koebe Distortion Theorem, there exists $0 < r_0 = r_0(f) < 1$ such that $|g(w)| < 1$ if $|w| < r_0$. By Schwarz's Lemma we deduce $|g(w)| \le r_0^{-1}|w|$ if $|w| < r_0$. Applying (2.2), for any $|z| < r_0$ we have

$$|f^n(z)| \le \psi^n(|z|) \le |f'(0)|^n g(|z|) \le r_0^{-1}|f'(0)|^n|z|, \quad n = 1, 2, \dots$$

This proves (b). Assume now $f'(0) = 0$. Note that (2.2) gives $|f(z)| \le |z|^2$, $z \in \mathbb{D}$. Iterating we obtain (c). □

For future reference we state the following easy consequence of Lemma 2.1.

Corollary 2.2. *Let $f \in \mathbb{H}^\infty(\mathbb{D})$, $\|f\|_\infty \le 1$ with $f(0) = 0$ which is not a rotation. Then there exist constants $0 < r_0 = r_0(f) < 1$ and $0 < c_0 = c_0(f) < 1$ such that*

$$|f^n(z)| \le r_0^{-1}c_0^n|z| \quad \text{if} \quad |z| \le r_0.$$

Proof. If $f'(0) \ne 0$ let $r_0 = r_0(f)$ be the constant given by part (b) of Lemma 2.1 and $c_0 = |f'(0)|$. If $f'(0) = 0$, pick any $0 < r_0 < 1$. Part (c) of Lemma 2.1 gives $|f^n(z)| \le r_0^{2^n-1}|z|$, $|z| \le r_0$. Since $2^n - 1 \ge n$, $n \ge 1$, we can take $c_0 = r_0$. □

For future reference we also state the following auxiliary result whose proof can be found in [NS2, Theorem 9].

Lemma 2.3. *Let f be an inner function with $f(0) = 0$ which is not a rotation. Let $\{a_n\}$ be a sequence of complex numbers. Then*

$$\frac{1 - |f'(0)|}{1 + |f'(0)|}\sum_{n=1}^{N}|a_n|^2 \le \left\|\sum_{n=1}^{N} a_n f^n\right\|_2^2 \le \frac{1 + |f'(0)|}{1 - |f'(0)|}\sum_{n=1}^{N}|a_n|^2, \quad N = 1, 2, \dots$$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Without loss of generality we can assume $f^n(z) \ne 0$ for any $n \ge 1$. Denote $F = \sum_{n=1}^{N} a_n f^n$. Then

$$\int_{\partial\mathbb{D}} |F(\xi) - F(z)|^2 P(z, \xi)\, dm(\xi) = \sum_{n=1}^{N} |a_n|^2 c_{n,n} + 2\,\mathrm{Re}\sum_{n=1}^{N-1}\sum_{k>n}^{N} \overline{a}_n a_k c_{k,n},$$

where

$$c_{k,n} = \int_{\partial\mathbb{D}} \overline{(f^n(\xi) - f^n(z))}(f^k(\xi) - f^k(z))P(z, \xi)\, dm(\xi), \quad k, n = 1, \dots, N.$$

For $k \ge n$, the function $\overline{f^n(\xi)}f^k(\xi) = f^k(\xi)/f^n(\xi)$, $\xi \in \partial\mathbb{D}$, has an analytic extension to $\mathbb{D}$. Then

$$c_{k,n} = \frac{f^k(z)}{f^n(z)} - f^k(z)\overline{f^n(z)} = \frac{f^k(z)(1 - |f^n(z)|^2)}{f^n(z)}, \quad k \ge n.$$

Hence

$$\int_{\partial\mathbb{D}} |F(\xi) - F(z)|^2 P(z, \xi) = A + 2\,\mathrm{Re}\, B, \tag{2.3}$$

where

$$A = \sum_{n=1}^{N} |a_n|^2 (1 - |f^n(z)|^2),$$

$$B = \sum_{n=1}^{N-1}\sum_{k>n}^{N} \overline{a}_n a_k \frac{f^k(z)(1 - |f^n(z)|^2)}{f^n(z)}.$$

The idea is that the cross term B can be estimated by the diagonal term A. Let $0 < r_0 = r_0(f) < 1$ and $0 < c_0 = c_0(f) < 1$ be the constants appearing in Corollary 2.2. The Denjoy-Wolff Theorem says that the iterates f^n converge to 0 uniformly on compacts of $\mathbb{D}$. So we can consider the smallest positive integer $\ell = \ell(z)$ such that $|f^\ell(z)| \le r_0$. Corollary 2.2 gives that $|f^k(z)| \le r_0^{-1} c_0^{k-n} |f^n(z)|$ if $k \ge n > \ell$. Assume $\ell < N$. We have

$$\sum_{n\ge\ell}^{N-1} |a_n| \frac{1-|f^n(z)|^2}{|f^n(z)|} \sum_{k>n}^{N} |a_k||f^k(z)| \le r_0^{-1} \sum_{n\ge\ell}^{N-1} |a_n| \sum_{k>n}^{N} |a_k| c_0^{k-n}.$$

Writting $j = k - n$ and applying Cauchy–Schwarz's inequality, last double sum can be bounded by

$$\sum_{j=1}^{N-\ell} c_0^j \sum_{n\ge\ell}^{N-j} |a_n||a_{n+j}| \le (1-c_0)^{-1} \sum_{n\ge\ell}^{N} |a_n|^2.$$

Since $|f^n(z)| \le |f^\ell(z)| \le r_0$ for $n \ge \ell$, last expression above is bounded by

$$(1-r_0^2)^{-1}(1-c_0)^{-1} \sum_{n\ge\ell}^{N} |a_n|^2 (1-|f^n(z)|^2).$$

Hence we only need to estimate $B_1 + B_2$ where

$$B_1 = \sum_{n=1}^{\ell-1} |a_n| \frac{1-|f^n(z)|^2}{|f^n(z)|} \sum_{k>n}^{\ell} |a_k||f^k(z)|,$$

$$B_2 = \sum_{n=1}^{\ell-1} |a_n| \frac{1-|f^n(z)|^2}{|f^n(z)|} \sum_{k>\ell}^{N} |a_k||f^k(z)|.$$

Since $|f^k(z)| \le |f^n(z)|$ for $k \ge n$, writting $j = k - n$ we obtain

$$B_1 \le \sum_{n=1}^{\ell-1} |a_n|(1-|f^n(z)|^2) \sum_{k>n}^{\ell} |a_k| \le \sum_{j=1}^{\ell} \sum_{n=1}^{\ell-j} |a_n||a_{n+j}|(1-|f^n(z)|^2).$$

Since $|f^n(z)| \ge r_0$ for $n < \ell$, part (a) of Lemma 2.1 provides a constant $0 < c_1 = c_1(f) < 1$ such that $1 - |f^n(z)| \le c_1^j (1 - |f^{n+j}(z)|)$ for any $n < n + j \le \ell$. Hence

$$B_1 \le 2 \sum_{j=1}^{\ell} c_1^{j/2} \sum_{n=1}^{\ell-j} |a_n|(1-|f^n(z)|)^{1/2} |a_{n+j}|(1-|f^{n+j}(z)|)^{1/2}.$$

Applying Cauchy–Schwarz's inequality, we deduce

$$B_1 \le 2(1-c_1^{1/2})^{-1} \sum_{n=1}^{\ell} |a_n|^2 (1-|f^n(z)|).$$

Finally we estimate B_2. Note that Corollary 2.2 gives $|f^k(z)| \le r_0^{-1} c_0^{k-\ell} |f^\ell(z)|$ if $k \ge \ell$. Since $|f^n(z)| \ge |f^\ell(z)|$ for $n \le \ell$, we have

$$B_2 \le r_0^{-1} \sum_{n=1}^{\ell-1} |a_n|(1-|f^n(z)|^2) \sum_{k>\ell}^{N} |a_k| c_0^{k-\ell}.$$

Cauchy–Schwarz's inequality gives

$$\sum_{k>\ell}^{N} |a_k| c_0^{k-\ell} \le (1-c_0^2)^{-1/2} \left(\sum_{k>\ell}^{N} |a_k|^2 \right)^{1/2} \tag{2.4}$$

and

$$\sum_{n=1}^{\ell-1} |a_n|(1-|f^n(z)|) \le \left(\sum_{n=1}^{\ell-1} |a_n|^2(1-|f^n(z)|)\right)^{1/2} \left(\sum_{n=1}^{\ell-1}(1-|f^n(z)|)\right)^{1/2}.$$

Since $1-|f^n(z)| \le c_1^{\ell-n}$ for $1 \le n < \ell$, last sum is bounded by $(1-c_1)^{-1}$ and we deduce

$$\sum_{n=1}^{\ell-1} |a_n|(1-|f^n(z)|) \le (1-c_1)^{-1/2} \left(\sum_{n=1}^{\ell-1} |a_n|^2(1-|f^n(z)|)\right)^{1/2}. \tag{2.5}$$

Now applying (2.4) and (2.5) we deduce

$$\begin{aligned} B_2 &\le c_2 \left(\sum_{n=1}^{\ell-1} |a_n|^2(1-|f^n(z)|^2)\right)^{1/2} \left(\sum_{k>\ell}^{N} |a_k|^2\right)^{1/2} \\ &\le \frac{c_2}{2}\left(\sum_{n=1}^{\ell-1} |a_n|^2(1-|f^n(z)|^2) + \sum_{k>\ell}^{N} |a_k|^2\right), \end{aligned}$$

where $c_2 = 2r_0^{-1}(1-c_0^2)^{-1/2}(1-c_1)^{-1/2}$. Finally observe that $|f^k(z)| \le r_0$ for $k > \ell$ to deduce

$$\sum_{k>\ell}^{N} |a_k|^2 \le (1-r_0^2)^{-1} \sum_{k>\ell}^{N} |a_k|^2(1-|f^k(z)|^2).$$

This finishes the proof if $\ell < N$. Assume now $\ell \ge N$, that is $|f^k(z)| > r_0$ for any $k < N$. Then we argue as in the estimate of B_1 replacing ℓ by N, to obtain

$$\sum_{n=1}^{N-1} |a_n| \frac{1-|f^n(z)|^2}{|f^n(z)|} \sum_{k>n}^{N} |a_k||f^k(z)| \le 2(1-c_1^{1/2})^{-1} \sum_{n=1}^{N} |a_n|^2(1-|f^n(z)|).$$

This finishes the proof. □

The following well known auxiliary result plays a fundamental role in the classical work of Paley and Zygmund, as well as in the study of pointwise convergence of random series of functions ([Ka]). Its short proof is included for the sake of completeness.

Lemma 2.4 (Paley–Zygmund inequality)**.** *Let $(X, \Omega, d\mu)$ be a probability space and $Z\colon X \to [0,\infty)$ be a positive square integrable random variable. Then for any $0 < \lambda < 1$, we have*

$$\mu\left\{x \in X : Z(x) > \lambda \int_X Z\,d\mu\right\} \ge (1-\lambda)^2 \frac{\left(\int_X Z\,d\mu\right)^2}{\int_X Z^2\,d\mu}.$$

Proof. We can assume $\int_X Z\,d\mu = 1$. Let W_λ denote the indicator function of the set $\{x \in X : Z(x) \le \lambda\}$. Cauchy–Schwarz's inequality gives

$$1 = \int_X ZW_\lambda\,d\mu + \int_X Z(1-W_\lambda)\,d\mu \le \lambda + \left(\int_X Z^2\,d\mu\right)^{1/2} \mu\{x \in X : Z(x) > \lambda\}^{1/2}.$$

□

We are now ready to prove Theorem 1.1 stated in the Introduction.

Proof of Theorem 1.1. It is obvious that (a) implies (b). We start proving that (c) implies (a). By Lemma 2.3, the series $\sum a_n f^n$ converges in $\mathbb{H}^2(\mathbb{D})$. Let $F(z) = \sum_{n=1}^{\infty} a_n f^n(z)$, $z \in \mathbb{D}$. Hence the non-tangential limit

$$F(\xi) = \lim_{z \to \not< \xi} F(z)$$

exists for almost every $\xi \in \partial\mathbb{D}$. For almost every $\xi \in \partial\mathbb{D}$ we will construct a sequence of points $\{z_N = z_N(\xi) : N = 1, 2, \dots\}$ tending non-tangentially to ξ, such that for any $\varepsilon > 0$ there exists an integer $N_0 = N_0(\varepsilon, \xi) > 0$ satisfying

$$\left| \sum_{n=1}^{N} a_n f^n(\xi) - F(z_N) \right| < \varepsilon, \quad N \geq N_0. \tag{2.6}$$

It is clear that (2.6) implies the statement in (a). By Corollary 2.2, there exist constants $0 < r_0 = r_0(f) < 1$, $0 < c_0 = c_0(f) < 1$ such that

$$|f^n(z)| \leq r_0^{-1} c_0^n |z|, \quad |z| \leq r_0, \quad n = 1, 2, \dots \tag{2.7}$$

Fix $N \geq 1$. Write $F_N = \sum_{n=1}^{N} a_n f^n$. Since $f(0) = 0$ and f^N is inner, there exists a subset $S = S_N \subset \partial\mathbb{D}$ with $m(\partial\mathbb{D} \backslash S) = 0$ such that for any $\xi \in S$ there exists $0 < r_N = r_N(\xi) < 1$ such that $|f^N(r_N \xi)| = r_0$. Since f^N tends to 0 uniformly on compact sets of $\mathbb{D}$ as $N \to \infty$, we deduce that $r_N \to 1$ as $N \to \infty$. Note also that $|f^j(r_N \xi)| \geq |f^N(r_N \xi)| = r_0$ for $1 \leq j \leq N$. For $\xi \in S$ consider the arc $I(N, \xi) = \{z \in \partial\mathbb{D} : |z - \xi| < 1 - r_N\}$. Apply Vitali's Covering Lemma (see [EG, p. 27]) to obtain a subcollection $\{I(N, \xi_k) : k = 1, 2, \dots\}$ of pairwise disjoint arcs such that $m(\partial\mathbb{D} \backslash \cup 5I(N, \xi_k)) = 0$. Here $5I$ denotes the arc in the unit circle having the same center than I and with $m(5I) = 5m(I)$. Given $\varepsilon > 0$ consider the set

$$E(N) = \bigcup_k \{\xi \in 5I(N, \xi_k) : |F_N(\xi) - F_N(\xi_k)| \geq \varepsilon\}$$

and

$$E = \bigcap_{\ell \geq 1} \bigcup_{N \geq \ell} E(N).$$

We first show that $m(E) = 0$. By the classical Borel–Cantelli Lemma, it is sufficient to show

$$\sum_{N=1}^{\infty} m(E(N)) < \infty. \tag{2.8}$$

Observe that Theorem 1.2 gives that $F \in \mathrm{BMOA}$. Since $P(r_N \xi_k, \xi)$ is comparable to $m(I(N, \xi_k))^{-1}$ for $\xi \in 5I(N, \xi_k)$, there exists a constant $c_1 = c_1(f) > 0$ such that for any $k, N \geq 1$, we have

$$\frac{1}{m(I(N, \xi_k))} \int_{5I(N, \xi_k)} |F_N(\xi) - F_N(r_N \xi_k)|^2 \, dm(\xi) \leq c_1 \sum_{n=1}^{N} |a_n|^2 (1 - |f^n(r_N \xi_k)|).$$

By part (a) of Lemma 2.1 and the choice of r_N, there exists a constant $0 < c < 1$ such that $1 - |f^n(r_N \xi_k)| \leq c^{N-n}(1 - r_0)$, $n = 1, \dots, N$. Hence

$$\frac{1}{m(I(N, \xi_k))} \int_{5I(N, \xi_k)} |F_N(\xi) - F_N(r_N \xi_k)|^2 \, dm(\xi) \leq c_1 (1 - r_0) \sum_{n=1}^{N} |a_n|^2 c^{N-n}, \quad k, N \geq 1.$$

Given $\varepsilon > 0$, we deduce that for any $k, N \geq 1$, we have

$$\frac{1}{m(I(N, \xi_k))} m(\{\xi \in 5I(N, \xi_k) : |F_N(\xi) - F_N(r_N \xi_k)| > \varepsilon\}) \leq \frac{c_1(1 - r_0)}{\varepsilon^2} \sum_{n=1}^{N} |a_n|^2 c^{N-n}.$$

Since $\{I(N,\xi_k) : k = 1, 2, \dots\}$ are pairwise disjoint, we deduce

$$m(E(N)) \leq \frac{c_1(1-r_0)}{\varepsilon^2} \sum_{n=1}^{N} |a_n|^2 c^{N-n}.$$

Then

$$\sum_{N=1}^{\infty} m(E(N)) \leq \frac{c_1(1-r_0)}{\varepsilon^2} \sum_{N=1}^{\infty} \sum_{n=1}^{N} |a_n|^2 c^{N-n} = \frac{c_1(1-r_0)}{\varepsilon^2(1-c)} \sum_{n=1}^{\infty} |a_n|^2.$$

This proves (2.8) and shows that $m(E) = 0$. Finally for almost every $\xi \in \partial\mathbb{D}\backslash E$ we will construct the sequence $\{z_N = z_N(\xi)\}$ for which (2.6) holds. Since $m(\partial\mathbb{D}\backslash \cup 5I(N,\xi_k)) = 0$, for almost every $\xi \in \partial\mathbb{D}\backslash E$ there exists $N_0 = N_0(\xi) > 0$ such that for any $N > N_0$ there exists $\xi_k \in \partial\mathbb{D}$ with $\xi \in 5I(N,\xi_k)$ and

$$|F_N(\xi) - F_N(r_N\xi_k)| < \varepsilon. \tag{2.9}$$

The choice of r_N and (2.7) gives that

$$|f^n(r_N\xi_k)| \leq c_0^{n-N}, \quad n \geq N.$$

Hence

$$\sum_{n>N} |a_n||f^n(r_N\xi_k)| \leq \sum_{n>N} |a_n| c_0^{n-N} \leq \left(1-c_0^2\right)^{-1/2} \left(\sum_{n>N} |a_n|^2\right)^{1/2}. \tag{2.10}$$

Pick $z_N = z_N(\xi) = r_N\xi_k$, where ξ_k is chosen such that $\xi \in 5I(N,\xi_k)$. Now (2.9) and (2.10) imply (2.6). Note that since $\xi \in 5I(N,\xi_k)$ and $r_N \to 1$, the points z_N converge non-tangentially to ξ as N tends to ∞.

We now prove that (b) implies (c). Given two sets $A, B \subset \partial\mathbb{D}$ we use the notation $A \overset{\text{a.e.}}{=} B$ if A and B differ at most on a set of Lebesgue measure zero. Consider the set

$$A = \left\{ \xi \in \partial\mathbb{D} : \sup_N \left| \sum_{n=1}^{N} a_n f^n(\xi) \right| < \infty \right\}.$$

Note that $f^{-k}(f^k(A)) \overset{\text{a.e.}}{=} A$ for any $k = 1, 2, \dots$ Since the mapping $f\colon \partial\mathbb{D} \to \partial\mathbb{D}$ is exact ([Ne]) we deduce that $m(A) = 0$ or $m(A) = 1$. Our assumption gives $m(A) = 1$. Write

$$F_N = \sum_{n=1}^{N} a_n f^n$$

and $s_N = \|F_N\|_2$. By Lemma 2.3 there exists a constant $c = c(f) > 1$ such that

$$c^{-1} \sum_{n=1}^{N} |a_n|^2 \leq s_N^2 \leq c \sum_{n=1}^{N} |a_n|^2, \quad N = 1, 2, \dots \tag{2.11}$$

Now the Paley–Zygmund inequality stated in Lemma 2.4 gives that for any $0 < \lambda < 1$ we have

$$m\{\xi \in \partial\mathbb{D} : |F_N(\xi)|^2 \geq \lambda s_N^2\} \geq (1-\lambda)^2 \frac{s_N^4}{\displaystyle\int_{\partial\mathbb{D}} |F_N|^4 \, dm}.$$

It was proved in [NS2, part (c) of Theorem 9] that there exists a constant $c_1 = c_1(f) > 0$ such that

$$\int_{\partial\mathbb{D}} |F_N|^4 \, dm \leq c_1 s_N^4.$$

Hence

$$m\{\xi \in \partial\mathbb{D} : |F_N(\xi)|^2 \geq \lambda s_N^2\} \geq c_1^{-1}(1-\lambda)^2, \quad 0 < \lambda < 1.$$

Since $m(A) = 1$, we deduce that s_N is bounded. Applying (2.11) we deduce $\sum_n |a_n|^2 < \infty$. □

We are now ready to prove Corollaries 1.3 and 1.4.

Proof of Corollary 1.3. By Lemma 2.3, the series $\sum a_n f^n$ converges in $\mathbb{H}^2(\mathbb{D})$. Let $F = \sum_{n\geq 1} a_n f^n$. Theorem 1.2 gives that $F \in \mathrm{BMOA}(\mathbb{D})$ and $\|F\|^2_{\mathrm{BMOA}(\mathbb{D})} \leq C \sum_{n\geq 1} |a_n|^2$. The converse estimate follows easily from Lemma 2.3 which gives

$$\|F\|^2_{\mathrm{BMOA}(\mathbb{D})} \geq \|F\|_2^2 \geq \frac{1-|f'(0)|}{1+|f'(0)|} \sum_{n=1}^{\infty} |a_n|^2.$$

The identity (1.1) follows from (2.6). This finishes the proof of (a). Assume now that f is a finite Blaschke product. Fix a number $0 < r < 1$ and let $0 < c = c(r, f) < 1$ be the constant appearing in part (a) of Lemma 2.1. For $z \in \mathbb{D}$ let $N = N(z)$ be the smallest positive integer such that $|f^N(z)| \leq r$. Note that $|f^n(z)| \leq r$ for $n \geq N$ and $1 - |f^n(z)| \leq c^{N-n}$ for $1 \leq n \leq N$. Since f is a finite Blaschke product, we have

$$\lim_{|z|\to 1} N(z) = +\infty. \tag{2.12}$$

Now

$$\sum_{n=1}^{\infty} |a_n|^2 (1 - |f^n(z)|) \leq \sum_{n=1}^{N} |a_n|^2 c^{N-n} + \sum_{n=N+1}^{\infty} |a_n|^2$$

and (2.12) gives

$$\lim_{|z|\to 1} \sum_{n=1}^{\infty} |a_n|^2 (1 - |f^n(z)|^2) = 0.$$

Theorem 1.2 finishes the proof. □

Proof of Corollary 1.4. Let $F = \sum_{n=1}^{\infty} a_n f^n$ and $s^2 = \sum_{n=1}^{\infty} |a_n|^2$. We first prove the upper estimate. Consider the distribution function $\Phi(\lambda) = m\{\xi \in \partial\mathbb{D} : |F(\xi)| > \lambda\}$, $\lambda > 0$. Corollary 1.3 and the John–Nirenberg Theorem give that there exist universal constants $A, B > 0$ such that

$$\Phi(\lambda) \leq A e^{-B\lambda/s}, \quad \lambda > 0.$$

Then

$$\|F\|_p^p = \int_0^{\infty} p\lambda^{p-1} \Phi(\lambda)\, d\lambda \leq C(p) s^p,$$

where $C(p)$ is a constant depending on A, B and p. The lower estimate follows from the following standard duality argument. We can assume $p < 2$. By Hölder's inequality

$$\int_{\partial\mathbb{D}} |F|^2\, dm \leq \left(\int_{\partial\mathbb{D}} |F|^p\, dm\right)^{1/p} \left(\int_{\partial\mathbb{D}} |F|^q\, dm\right)^{1/q},$$

where $p^{-1} + q^{-1} = 1$. By Lemma 2.3 and the upper estimate we have already proved, there exists a constant $c = c(f, p) > 0$ such that

$$\sum_{n=1}^{\infty} |a_n|^2 \leq c \left(\int_{\partial\mathbb{D}} |F|^p\, dm\right)^{1/p} \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2}.$$

This finishes the proof. □

3 Uniform estimates

Given $F \in L^1(\partial\mathbb{D})$ and $z \in \mathbb{D}$, let $F(z)$ denote the value of its harmonic extension at z, that is,

$$F(z) = \int_{\partial\mathbb{D}} F(\xi) P(z,\xi)\, dm(\xi).$$

If F is the characteristic function of a measurable set $E \subset \partial\mathbb{D}$, the corresponding function is called the harmonic measure of the set E from the point $z \in \mathbb{D}$ and will be denoted by $F(z) = w(z, E)$. A function $F \in L^1(\partial\mathbb{D})$ is in the space $\mathrm{BMO}(\partial\mathbb{D})$ if

$$\|F\|^2_{\mathrm{BMO}(\partial\mathbb{D})} = \sup_{z\in\mathbb{D}} \int_{\partial\mathbb{D}} |F(\xi) - F(z)|^2 P(z,\xi)\, dm(\xi) < \infty.$$

Equivalently, $F \in L^1(\partial\mathbb{D})$ is in $\mathrm{BMO}(\partial\mathbb{D})$ if and only if there exists a constant $C = C(F) > 0$ such that for any arc $I \subset \partial\mathbb{D}$, we have

$$\frac{1}{m(I)} \int_I |F - F_I|^2\, dm \leq C.$$

Here $F_I = m(I)^{-1} \int_I F\, dm$ denotes the mean of F on I. See Chapter VI of [Ga]. For $z \in \mathbb{D}$ let τ_z be the automorphism of $\mathbb{D}$ given by $\tau_z(w) = (w - z)(1 - \overline{z}w)^{-1}$, $w \in \mathbb{D}$. The John–Nirenberg Theorem applied to $F \circ \tau_z - F(z)$ provides two universal constants $A, B > 0$ such that

$$w(z, \{\xi \in \partial\mathbb{D} : |F(\xi) - F(z)| > \lambda\}) \leq A e^{-B\lambda/\|F\|_{\mathrm{BMO}(\partial\mathbb{D})}}, \quad \lambda > 0. \tag{3.1}$$

See [Ba]. The proof of Theorem 1.5 is based on a careful study of the oscillation of the partial sums of $\sum a_n f^n$. We start with an auxiliary result which holds for any BMO function.

Lemma 3.1. *For any $0 < c < 1$ there exists a constant $0 < \delta = \delta(c) < 1$ such that the following statement holds. Let F be a real valued function defined on $\partial\mathbb{D}$ with $\|F\|_{\mathrm{BMO}(\partial\mathbb{D})} = 1$. Let $z \in \mathbb{D}$ such that*

$$\int_{\partial\mathbb{D}} |F(\xi) - F(z)|^2 P(z,\xi)\, dm(\xi) \geq c^2. \tag{3.2}$$

Then $w(z, \{\xi \in \partial\mathbb{D} : F(\xi) - F(z) \geq \delta\}) \geq \delta$.

Proof. Given $0 \leq a < b$, let $E(a, b)$ denote the set of points $\xi \in \partial\mathbb{D}$ such that $a \leq |F(\xi) - F(z)| \leq b$. Apply the John–Nirenberg estimate (3.1) to find a constant $c_1 = c_1(c) > 0$, such that

$$\int_{\partial\mathbb{D}\setminus E(0,c_1)} |F(\xi) - F(z)|^2 P(z,\xi)\, dm(\xi) \leq \frac{c^2}{4}.$$

Since

$$\int_{E(0,c/2)} |F(\xi) - F(z)|^2 P(z,\xi)\, dm(\xi) \leq \frac{c^2}{4},$$

we deduce that

$$\int_{E(c/2,c_1)} |F(\xi) - F(z)|^2 P(z,\xi)\, dm(\xi) \geq \frac{c^2}{2}.$$

Hence

$$w\left(z, \left\{\xi \in \partial\mathbb{D} : |F(\xi) - F(z)| \geq \frac{c}{2}\right\}\right) \geq \frac{c^2}{2c_1^2}. \tag{3.3}$$

Apply the John–Nirenberg estimate (3.1) to find a constant $c_2 = c_2(c) > 0$ such that

$$\int_{\partial\mathbb{D}\setminus E(0,c_2)} |F(\xi) - F(z)| P(z,\xi)\, dm(\xi) \leq \frac{c^3}{16c_1^2}. \tag{3.4}$$

Let $0 < \delta = \delta(c) < c/2$ be a small constant to be fixed later. We will show that if $\delta > 0$ is conveniently chosen, then

$$w(z, \{\xi : F(\xi) - F(z) > \delta\}) \geq \min\left\{\frac{c^3}{16c_1^2 c_2}, \frac{c^2}{16c_1^2}\right\}. \tag{3.5}$$

We argue by contradiction. Assume

$$w(z, \{\xi : F(\xi) - F(z) > \delta\}) \leq \min\left\{\frac{c^3}{16c_1^2 c_2}, \frac{c^2}{16c_1^2}\right\}. \tag{3.6}$$

Apply (3.3) to deduce that $w(z, \{\xi : F(\xi) - F(z) < -c/2\}) \geq 7c^2/16c_1^2$. Hence

$$\int_{\{\xi : F(\xi)-F(z)\leq 0\}} (F(\xi) - F(z))P(z, \xi)\, dm(\xi) \leq \frac{-7c^3}{32c_1^2}. \tag{3.7}$$

For $0 < \gamma < \tau$ consider the set $G(\gamma, \tau) = \{\xi \in \partial\mathbb{D} : \gamma \leq F(\xi) - F(z) \leq \tau\}$. We have

$$\begin{aligned}\int_{\{\xi:F(\xi)-F(z)>0\}} (F(\xi) - F(z))P(z, \xi)\, dm(\xi) &\leq \delta + \int_{G(\delta,c_2)} (F(\xi) - F(z))P(z, \xi)\, dm(\xi)\\ &+ \int_{\{\xi:F(\xi)-F(z)>c_2\}} (F(\xi) - F(z))P(z, \xi)\, dm(\xi).\end{aligned}$$

The choice (3.4) of c_2 gives that last integral is bounded by $c^3/16c_1^2$. Moreover by (3.6), we have

$$\int_{G(\delta,c_2)} (F(\xi) - F(z))P(z, \xi)\, dm(\xi) \leq c_2 w(z, \{\xi : F(\xi) - F(z) > \delta\}) \leq \frac{c^3}{16c_1^2}.$$

We deduce

$$\int_{\{\xi:F(\xi)-F(z)>0\}} (F(\xi) - F(z))P(z, \xi)\, dm(\xi) \leq \delta + \frac{c^3}{8c_1^2}. \tag{3.8}$$

Choosing $0 < \delta < c^3/16c_1^2$ we observe that (3.7) and (3.8) contradict the identity

$$\int_{\partial\mathbb{D}} (F(\xi) - F(z))P(z, \xi)\, dm(\xi) = 0.$$

Hence choosing $0 < \delta < \min\{c/2, c^3/16c_1^2\}$, estimate (3.5) holds. This finishes the proof. □

Our next auxiliary result says that $F = \sum_{n=M}^{N} a_n f^n$ satisfies condition (3.2) in Lemma 3.1 if $|f^M(z)|$ is sufficiently small.

Lemma 3.2. *Let f be an inner function with $f(0) = 0$ which is not a rotation. Then there exists a constant $0 < \varepsilon = \varepsilon(f) < 1$ such that if $M < N$ are positive integers, $\{a_n : M \leq n \leq N\}$ are complex numbers and $z \in \mathbb{D}$ satisfies $|f^M(z)| \leq \varepsilon$, we have*

$$\int_{\partial\mathbb{D}} \left|\sum_{n=M}^{N} a_n(f^n(\xi) - f^n(z))\right|^2 P(z, \xi)\, dm(\xi) \geq \frac{1}{2}\frac{1 - |f'(0)|}{1 + |f'(0)|}\sum_{n=M}^{N} |a_n|^2.$$

Proof. By continuity we can assume that $f^n(z) \neq 0$ for any n. Write $F = \sum_{n=M}^{N} a_n f^n$. In (2.3) we already noted that

$$\begin{aligned}\int_{\partial\mathbb{D}} |F(\xi) - F(z)|^2 P(z, \xi)\, dm(\xi) &= \sum_{n=M}^{N} |a_n|^2(1 - |f^n(z)|^2)\\ &+ 2\,\mathrm{Re}\sum_{n=M}^{N-1}\sum_{k>n}^{N} \overline{a}_n a_k(1 - |f^n(z)|^2)\frac{f^k(z)}{f^n(z)}.\end{aligned} \tag{3.9}$$

The idea is to compare the expression above with

$$\int_{\partial\mathbb{D}} |F(\xi)|^2 \, dm(\xi) = \sum_{n=M}^{N} |a_n|^2 + 2\,\mathrm{Re} \sum_{n=M}^{N-1} \sum_{k>n}^{N} \overline{a}_n a_k f'(0)^{k-n}. \tag{3.10}$$

Assume $f'(0) \neq 0$. For $k > n \geq M$ consider the function $g = g_{k,n}$ defined by $g(w) = f^{k-n}(w)$, $w \in \mathbb{D}$. By part (b) of Lemma 2.1, there exists a constant $0 < r_0 = r_0(f) < 1$ such that $|g(w)| \leq r_0^{-1}|f'(0)|^{k-n}|w|$ if $|w| < r_0$. Hence there exists a constant $c(r_0) > 0$ such that

$$\left| \frac{g(w)}{w} - g'(0) \right| \leq c(r_0)|f'(0)|^{k-n}|w|, \quad |w| \leq \frac{r_0}{2}. \tag{3.11}$$

Let $0 < \varepsilon < r_0/2$ be a constant to be fixed later and assume $|f^M(z)| \leq \varepsilon$. Taking $w = f^n(z)$ in (3.11) we obtain

$$\left| \frac{f^k(z)}{f^n(z)} - f'(0)^{k-n} \right| \leq c(r_0)|f'(0)|^{k-n}|f^n(z)| \leq \varepsilon c(r_0)|f'(0)|^{k-n}, \quad k > n \geq M.$$

By part (b) of Lemma 2.1, we also have $|f^k(z)| \leq r_0^{-1}|f'(0)|^{k-M}\varepsilon$ and $|f^n(z)| \leq r_0^{-1}|f'(0)|^{n-M}\varepsilon$ if $k \geq M$ and $n \geq M$. Then

$$\left| \sum_{n=M}^{N-1} \sum_{k>n}^{N} \overline{a}_n a_k \left((1 - |f^n(z)|^2) \frac{f^k(z)}{f^n(z)} - f'(0)^{k-n} \right) \right|$$
$$\leq \sum_{n=M}^{N-1} \sum_{k>n}^{N} |a_n||a_k| (\varepsilon c(r_0)|f'(0)|^{k-n} + r_0^{-2}\varepsilon^2 |f'(0)|^{k+n-2M}).$$

As in the proof of Theorem 1.2, writting $j = k - n$ and applying Cauchy-Schwarz inequality, we find a constant $c(f) > 0$ such that

$$\sum_{n=M}^{N-1} \sum_{k>n}^{N} |a_n||a_k||f'(0)|^{k-n} \leq c(f) \sum_{n=M}^{N} |a_n|^2,$$
$$\sum_{n=M}^{N-1} \sum_{k>n}^{N} |a_n||a_k||f'(0)|^{k+n-2M} \leq c(f) \sum_{n=M}^{N} |a_n|^2.$$

We deduce that there exists a constant $c(f, r_0) > 0$ such that

$$\left| \sum_{n=M}^{N-1} \sum_{k>n}^{N} \overline{a}_n a_k \left((1 - |f^n(z)|^2) \frac{f^k(z)}{f^n(z)} - f'(0)^{k-n} \right) \right| \leq c(f, r_0)\varepsilon \sum_{n=M}^{N} |a_n|^2.$$

Applying (3.9) and (3.10), we deduce

$$\left| \int_{\partial\mathbb{D}} |F(\xi) - F(z)|^2 P(z,\xi) \, dm(\xi) - \int_{\partial\mathbb{D}} |F(\xi)|^2 \, dm(\xi) \right| \leq \varepsilon^2 \sum_{n=M}^{N} |a_n|^2 + 2c(f, r_0)\varepsilon \sum_{n=M}^{N} |a_n|^2.$$

Recall that

$$\int_{\partial\mathbb{D}} |F(\xi)|^2 \, dm(\xi) \geq \frac{1 - |f'(0)|}{1 + |f'(0)|} \sum_{n=M}^{N} |a_n|^2.$$

Choose $0 < \varepsilon < r_0/2$ small enough so that $\varepsilon^2 + 2c(f, r_0)\varepsilon < (1 - |f'(0)|)(1 + |f'(0)|)^{-1}/2$ and deduce

$$\int_{\partial\mathbb{D}} |F(\xi) - F(z)|^2 P(z,\xi) \, dm(\xi) \geq \frac{1}{2} \frac{1 - |f'(0)|}{1 + |f'(0)|} \sum_{n=M}^{N} |a_n|^2.$$

This finishes the proof if $f'(0) \neq 0$. Assume now $f'(0) = 0$. Apply part (c) of Lemma 2.1 to the inner function f^{k-n} to obtain $|f^k(z)| \leq |f^n(z)|^{2^{k-n}}$, $k > n$. Let $0 < \varepsilon < 1$ be a (small) constant to be fixed later and assume $|f^M(z)| \leq \varepsilon$. If $k > n \geq M$, we have

$$\frac{|f^k(z)|}{|f^n(z)|} \leq |f^n(z)|^{2^{k-n}-1} \leq \varepsilon^{2^{k-n}-1} \leq \varepsilon^{k-n}.$$

Writing $j = k - n$ and applying Cauchy-Schwarz's inequality, we obtain

$$\left| \sum_{n=M}^{N-1} \overline{a}_n \sum_{k>n}^{N} a_k (1 - |f^n(z)|^2) \frac{f^k(z)}{f^n(z)} \right| \leq \sum_{n=M}^{N-1} \sum_{k>n}^{N} |a_n||a_k| \varepsilon^{k-n} \leq \varepsilon(1-\varepsilon)^{-1} \sum_{n=M}^{N} |a_n|^2.$$

As before, applying (3.9) and (3.10) we deduce

$$\left| \int_{\partial\mathbb{D}} |F(\xi) - F(z)|^2 P(z,\xi)\, dm(\xi) - \int_{\partial\mathbb{D}} |F(\xi)|^2\, dm(\xi) \right| \leq (\varepsilon^2 + 2\varepsilon(1-\varepsilon)^{-1}) \sum_{n=M}^{N} |a_n|^2.$$

Since

$$\int_{\partial\mathbb{D}} |F(\xi)|^2\, dm(\xi) = \sum_{n=M}^{N} |a_n|^2$$

we only need to pick $\varepsilon > 0$ small enough so that $\varepsilon^2 + 2\varepsilon(1-\varepsilon)^{-1} < 1/2$. □

Next auxiliary result is an easy consequence of Lemmas 3.1 and 3.2.

Lemma 3.3. *Let f be an inner function with $f(0) = 0$ which is not a rotation. Then there exist two constants $0 < \varepsilon = \varepsilon(f) < 1$ and $0 < c = c(f) < 1$ such that the following statement holds. Let $M < N$ be positive integers, $z \in \mathbb{D}$ satisfying $|f^M(z)| < \varepsilon$ and $\{a_n : M \leq n \leq N\}$ a set of complex numbers. Then there exists a set $E = E(z, \{a_n\}, f) \subset \partial\mathbb{D}$ with $w(z, E) \geq c$ such that*

$$\operatorname{Re} \sum_{n=M}^{N} a_n f^n(\xi) \geq c \left(\sum_{n=M}^{N} |a_n|^2 \right)^{1/2}, \quad \xi \in E.$$

Proof. Note that for any analytic function $G \in \mathbb{H}^2(\mathbb{D})$ with $G(z) = 0$, we have

$$\int_{\partial\mathbb{D}} (\operatorname{Re} G(\xi))^2 P(z,\xi)\, dm(\xi) = \frac{1}{2} \int_{\partial\mathbb{D}} |G(\xi)|^2 P(z,\xi)\, dm(\xi). \tag{3.12}$$

Let $0 < r_0 = r_0(f) < 1$ and $0 < c_0 = c_0(f) < 1$ be the constants given in Corollary 2.2 and let $\varepsilon_0 = \varepsilon_0(f) > 0$ be the constant appearing in Lemma 3.2. Let $0 < \varepsilon < \min\{\varepsilon_0, r_0\}$ be a (small) constant to be fixed later. Lemma 3.2 and the identity (3.12) applied to the function $G = F - F(z)$, where $F = \sum_{n=M}^{N} a_n f^n$, give that

$$\int_{\partial\mathbb{D}} (\operatorname{Re} F(\xi) - \operatorname{Re} F(z))^2 P(z,\xi)\, dm(\xi) \geq c(f) \sum_{n=M}^{N} |a_n|^2,$$

where $c(f) = 4^{-1}(1 - |f'(0)|)(1 + |f'(0)|)^{-1}$. Applying Lemma 3.1 to the function

$$(\operatorname{Re} F) \|\operatorname{Re} F\|^{-1}_{\mathrm{BMO}(\partial\mathbb{D})},$$

we find a constant $\delta = \delta(f) > 0$ and a set $E = E(z, \{a_n\}, f) \subset \partial\mathbb{D}$ with $w(z, E) \geq \delta$ such that

$$\operatorname{Re} F(\xi) - \operatorname{Re} F(z) \geq \delta \|\operatorname{Re} F\|_{\mathrm{BMO}(\partial\mathbb{D})}, \quad \xi \in E.$$

By Corollary 1.3, $\|\operatorname{Re} F\|^2_{\mathrm{BMO}(\partial\mathbb{D})}$ is comparable to $\sum_{n=M}^{N} |a_n|^2$. Reducing $\delta > 0$ if necessary, we can assume

$$\operatorname{Re} F(\xi) - \operatorname{Re} F(z) \geq \delta \left(\sum_{n=M}^{N} |a_n|^2 \right)^{1/2}, \quad \xi \in E. \tag{3.13}$$

We have $|f^M(z)| < \varepsilon$. Note that by Corollary 2.2, we obtain $|f^n(z)| \leq r_0^{-1} c_0^{n-M} \varepsilon$, $n \geq M$. Cauchy-Schwarz inequality yields

$$|F(z)| \leq \varepsilon c(c_0, r_0) \left(\sum_{n=M}^{N} |a_n|^2 \right)^{1/2},$$

where $c(c_0, r_0)$ is a constant depending on c_0 and r_0. Pick $0 < \varepsilon < \min\{\varepsilon_0, r_0\}$ sufficiently small such that $\varepsilon c(c_0, r_0) \leq \delta/2$. Taking $c = \delta/2$ the proof is completed. □

The next easy auxiliary result is stated for future reference.

Lemma 3.4. *Let $F \in \mathrm{BMO}(\partial\mathbb{D})$ and $0 < c < 1$. Then there exists a constant $\varepsilon = \varepsilon(c) > 0$ such that if $I \subset \partial\mathbb{D}$ is an arc with $m(\{\xi \in I : \operatorname{Re} F(\xi) \geq c\|F\|_{\mathrm{BMO}(\partial\mathbb{D})}\}) \geq cm(I)$ and*

$$\frac{1}{m(I)} \int_I |F - F_I|^2 \, dm \leq \varepsilon \|F\|^2_{\mathrm{BMO}(\partial\mathbb{D})}, \tag{3.14}$$

then $\operatorname{Re}(F_I) \geq 2^{-1} c \|F\|_{\mathrm{BMO}(\partial\mathbb{D})}$.

Proof. Denote $E = \{\xi \in I : \operatorname{Re} F(\xi) \geq c\|F\|_{\mathrm{BMO}(\partial\mathbb{D})}\}$. We argue by contradiction. Assume $\operatorname{Re}(F_I) \leq 2^{-1} c \|F\|_{\mathrm{BMO}(\partial\mathbb{D})}$. Then the estimate (3.14) gives

$$\frac{c^2}{4} \|F\|^2_{\mathrm{BMO}(\partial\mathbb{D})} \frac{m(E)}{m(I)} \leq \varepsilon \|F\|^2_{\mathrm{BMO}(\partial\mathbb{D})}$$

and we deduce $m(E) \leq 4\varepsilon c^{-2} m(I)$. Since $m(E) \geq cm(I)$, picking $0 < \varepsilon < c^3/4$ one gets a contradiction. □

Given an arc I centered at $\xi \in \partial\mathbb{D}$ we denote by $z(I)$ the point $z(I) = (1 - m(I))\xi$. Conversely, given a point $z \in \mathbb{D} \backslash \{0\}$ let $I(z)$ be the arc in the unit circle such that $z(I(z)) = z$. Given an arc $I \subset \partial\mathbb{D}$ and a number $0 < c < 1/m(I)$, we denote by cI the arc which has the same center as I and with $m(cI) = cm(I)$. Given an arc $I \subset \partial\mathbb{D}$ we consider its dyadic decomposition $\mathcal{D}(I) = \bigcup_{n \geq 0} \mathcal{D}_n(I)$ where $\mathcal{D}_n(I)$ is the set of the 2^n pairwise disjoint subarcs of I of Lebesgue measure $2^{-n} m(I)$.

Lemma 3.5. *Let f be an inner function with $f(0) = 0$ which is not a rotation. Then there exist constants $0 < \varepsilon = \varepsilon(f) < 1$ and $0 < c = c(f) < 1$ such that the following statement holds. Let $0 < \gamma < 1$ be a constant with $1 - \gamma \leq \varepsilon$, let $M < N$ be positive integers, $\{a_n : M \leq n \leq N\}$ be complex numbers, and $z \in \mathbb{D}$ with $|f^M(z)| < \varepsilon$. Then there exist pairwise disjoint arcs $\{I_k\}$ with $c^{-1} I_k \subset c^{-1} I(z)$ for any k, such that*

(a) For any $k = 1, 2, \ldots,$ we have $\tau(1-\gamma) \leq 1 - |f^N(z(I_k))| \leq 1 - \gamma$, where $\tau > 0$ is a universal constant.

(b) $\sum m(I_k) \geq cm(I(z))$.

(c) *For any $k = 1, 2, \dots,$ we have*

$$\frac{1}{|I_k|}\int_{I_k} \operatorname{Re}\sum_{n=M}^{N} a_n f^n \, dm \geq c\left(\sum_{n=M}^{N}|a_n|^2\right)^{1/2}.$$

Proof. Denote $F = \sum_{n=M}^{N} a_n f^n$. Let $0 < \varepsilon_0 = \varepsilon_0(f) < 1$, $0 < c_0 = c_0(f) < 1$ be the constants given by Lemma 3.3. Let $0 < \varepsilon < \varepsilon_0(f)/2$ be a (small) constant to be fixed later. Assume $|f^M(z)| < \varepsilon$. Apply Lemma 3.3 to find a set $E \subset \partial\mathbb{D}$ with $w(z, E) \geq c_0$ such that

$$\operatorname{Re} F(\xi) \geq c_0\left(\sum_{n=M}^{N}|a_n|^2\right)^{1/2}, \quad \xi \in E. \tag{3.15}$$

Reducing E if necessary, we can assume that there exists a constant $c_1 = c_1(c_0) > 0$ such that $E \subset c_1^{-1}I(z)$ and $m(E) \geq c_1 m(I(z))$. Let $I_0 = c_1^{-1}I(z)$ and consider its dyadic decomposition $\mathcal{D}(I_0)$. Fix $0 < \gamma < 1$ with $1 - \gamma < \varepsilon$. Let $\mathcal{G}_0 = \{I_k : k \geq 1\}$ be the collection of maximal dyadic arcs in $\mathcal{D}(I_0)$ such that

$$|f^N(z_k)| \geq \gamma. \tag{3.16}$$

Here $z_k = z(I_k)$, $k \geq 1$. By maximality, the arcs in the collection $\mathcal{G}_0$ are pairwise disjoint. Note also that by maximality and Schwarz's Lemma, there exists a universal constant $\tau > 0$ such that

$$\tau(1-\gamma) \leq 1 - |f^N(z_k)| \leq 1 - \gamma, \quad k \geq 1. \tag{3.17}$$

This is the statement in (a). Since f^N is inner, the union of $\{I_k : k \geq 1\}$ covers almost every point of I_0. Consider the subcollection $\mathcal{G}_1 \subset \mathcal{G}_0$ of those dyadic arcs $I_k \in \mathcal{G}_0$ such that

$$m(E \cap I_k) \geq c_2 m(I_k). \tag{3.18}$$

Here $c_2 > 0$ is a small constant to be fixed later. We will show that the subcollection $\mathcal{G}_1$ satisfies the conditions (b) and (c) in the statement. Let $\mathcal{L}$ be the collection of indices k such that $I_k \in \mathcal{G}_1$. Note that

$$c_1 m(I(z)) \leq m(E) \leq \sum_{k \notin \mathcal{L}} m(E \cap I_k) + \sum_{k \in \mathcal{L}} m(I_k).$$

If $k \notin \mathcal{L}$, we have $m(E \cap I_k) \leq c_2 m(I_k)$. Hence the first sum is bounded by $c_2 c_1^{-1} m(I(z))$. Pick $c_2 > 0$ sufficiently small so that $c_1 - c_2 c_1^{-1} > c_1/2$ to deduce

$$\sum_{k \in \mathcal{L}} m(I_k) \geq \frac{c_1}{2} m(I(z)). \tag{3.19}$$

Choosing $0 < c < c_1/2$, the estimate (3.19) gives the statement in (b). Note that $|f^N(z)| \leq |f^M(z)| < \varepsilon$ and $|f^N(z_k)| \geq \gamma$, for any positive integer k. By Schwarz's Lemma $\rho(z_k, z) \geq \rho(\gamma, \varepsilon)$. Since $0 < 1 - \gamma < \varepsilon$, taking $\varepsilon > 0$ sufficiently small we deduce that $\rho(\gamma, \varepsilon)$ is close to 1 and then $m(I_k) \leq c_3(\varepsilon) m(I_0)$ for any $k \geq 1$, where $c_3(\varepsilon)$ is tends to 0 as ε tends to 0. If $c > 0$ and $\varepsilon > 0$ are taken sufficiently small we have $c^{-1}I_k \subset c^{-1}I(z)$ for any $k \geq 1$.

By Theorem 1.2, there exists a constant $c_4 = c_4(f) > 0$ such that for any $k \geq 1$ we have

$$\frac{1}{m(I_k)}\int_{I_k} |F(\xi) - F_{I_k}|^2 \, dm(\xi) \leq c_4 \sum_{n=M}^{N} |a_n|^2 (1 - |f^n(z_k)|^2).$$

By part (a) of Lemma 2.1, there exists a constant $0 < c_5 = c_5(\gamma, f) < 1$ such that $1 - |f^n(z_k)| \leq c_5^{N-n}(1 - |f^N(z_k)|)$, $n \leq N$. Since $1 - |f^N(z_k)| \leq 1 - \gamma$, we obtain

$$\frac{1}{m(I_k)} \int_{I_k} |F(\xi) - F_{I_k}|^2 \, dm(\xi) \leq 2c_4(1-\gamma) \sum_{n=M}^{N} |a_n|^2 c_5^{N-n}.$$

Recall that $m(E \cap I_k) \geq c_2 m(I_k)$ if $k \in \mathcal{L}$. Then if $1-\gamma$ is sufficiently small, by Lemma 3.4, the estimate (3.15) implies that

$$\frac{1}{m(I_k)} \int_{I_k} \operatorname{Re} F \, dm \geq \frac{c_0}{2} \left(\sum_{n=M}^{N} |a_n|^2 \right)^{1/2}.$$

Choosing $0 < c < c_0/2$, last estimate gives the statement in (c).This finishes the proof. $\square$

Our next auxiliary result is the building block of the main construction in the proof of Theorem 1.5.

Lemma 3.6. *Let f be an inner function with $f(0) = 0$ which is not a rotation. Then there exist constants $0 < \varepsilon = \varepsilon(f) < 1$ and $0 < c = c(f) < 1$ such that the following statement holds. Let $0 < \gamma < 1$ be a constant with $1 - \gamma \leq \varepsilon$, let $\{a_n\}$ be a sequence of complex numbers, let $M_1 < N_1 < M_2 < N_2$ be positive integers and*

$$F_i = \sum_{n=M_i}^{N_i} a_n f^n, \quad i = 1, 2.$$

Let $I \subset \partial\mathbb{D}$ be an arc such that $c(1-\gamma) < 1 - |f^{N_1}(z(I))| \leq 1 - \gamma$ and $|f^{M_2}(z(I))| \leq \varepsilon$. Then there exists an arc J with $c^{-1}J \subset c^{-1}I$, such that

$$c(1-\gamma) \leq 1 - |f^{N_2}(z(J))| \leq 1 - \gamma, \tag{3.20}$$

$$\frac{1}{|J|} \int_J \operatorname{Re} F_2 \, dm \geq c \left(\sum_{n=M_2}^{N_2} |a_n|^2 \right)^{1/2}, \tag{3.21}$$

$$\frac{1}{|J|} \int_J \operatorname{Re} F_1 \, dm \geq \frac{1}{|I|} \int_I \operatorname{Re} F_1 \, dm - c^{-1}(1-\gamma)^{1/2} \left(\sum_{n=M_1}^{N_1} |a_n|^2 t^{N_1 - n} \right)^{1/2}, \tag{3.22}$$

where $t = (1+\gamma)^{-1}(1 + |f'(0)|\gamma) < 1$.

Proof. Let $\varepsilon_0 > 0$ and $c_0 > 0$ be the constants appearing in Lemma 3.5. Let $0 < \varepsilon < \varepsilon_0$ be a constant to be fixed later. Let $0 < \gamma < 1$ with $1 - \gamma < \varepsilon$. Apply Lemma 3.5 to the function F_2 to find arcs J_k with $c_0^{-1} J_k \subset c_0^{-1} I$, such that $\tau(1-\gamma) \leq 1 - |f^{N_2}(z(J_k))| \leq 1 - \gamma$, $\sum m(J_k) \geq c_0 m(I)$ and

$$\frac{1}{|J_k|} \int_{J_k} \operatorname{Re} F_2 \, dm \geq c_0 \left(\sum_{n=M_2}^{N_2} |a_n|^2 \right)^{1/2}.$$

Here $\tau > 0$ is the universal constant given by part (a) of Lemma 3.5. The constant c will satisfy $0 < c < c_0$ and the arc J will be one of the arcs of the family $\{J_k\}$. Hence estimates (3.20) and (3.21) will follow from the construction. Theorem 1.2 gives that there exists a constant $c_1 = c_1(f) > 0$ such that

$$\frac{1}{c_0^{-1} m(I)} \int_{c_0^{-1} I} |\operatorname{Re} F_1 - \operatorname{Re} F_1(z(c_0^{-1} I))|^2 \, dm \leq c_1 \sum_{n=M_1}^{N_1} |a_n|^2 (1 - |f^n(z(c_0^{-1} I))|^2). \tag{3.23}$$

Note that

$$|(\operatorname{Re} F_1)_I - \operatorname{Re} F_1(z(c_0^{-1}I))| \le \frac{1}{m(I)} \int_I |\operatorname{Re} F_1 - \operatorname{Re} F_1(z(c_0^{-1}I))| dm$$
$$\le \left(\frac{1}{m(I)} \int_I |\operatorname{Re} F_1 - \operatorname{Re} F_1(z(c_0^{-1}I))|^2 dm \right)^{1/2}$$

Applying (3.23) we find a constant $c_2 = c_2(f, c_0) > 0$ such that

$$\frac{1}{m(I)} \int_{c_0^{-1}I} |\operatorname{Re} F_1 - (\operatorname{Re} F_1)_I|^2 \, dm \le c_2 \sum_{n=M_1}^{N_1} |a_n|^2 (1 - |f^n(z(c_0^{-1}I))|^2). \tag{3.24}$$

By Schwarz's Lemma, there exists a constant $c_3 = c_3(c_0) > 0$ such that $1 - |f^n(z(c_0^{-1}I))|^2 \le c_3(1 - |f^n(z(I))|^2)$ for any $n \ge 1$. Since $1 - |f^{N_1}(z(I))| \le 1 - \gamma$, part (a) of Lemma 2.1 provides a constant $0 < c_4 < 1$ such that $1 - |f^n(z(I))| \le c_4^{N_1 - n}(1 - \gamma)$, $n \le N_1$. Actually, according to Lemma 2.1 one can take $c_4 = (1+\gamma)^{-1}(1 + |f'(0)|\gamma)$. Applying (3.24) we obtain a constant $c_5 = c_5(c_0, f) > 0$ such that

$$\frac{1}{m(I)} \int_{c_0^{-1}I} |\operatorname{Re} F_1 - (\operatorname{Re} F_1)_I|^2 \, dm \le c_5(1-\gamma) \sum_{n=M_1}^{N_1} |a_n|^2 c_4^{N_1 - n}. \tag{3.25}$$

Let $K > 0$ be a large constant to be fixed later. Let $\mathcal{G}$ be the subcollection of those arcs J_k such that

$$(\operatorname{Re} F_1)_{J_k} \le (\operatorname{Re} F_1)_I - K(1-\gamma)^{1/2} \left(\sum_{n=M_1}^{N_1} |a_n|^2 c_4^{N_1-n} \right)^{1/2}.$$

Note that for any $J_k \in \mathcal{G}$ we have

$$\frac{1}{m(J_k)} \int_{J_k} |\operatorname{Re} F_1 - (\operatorname{Re} F_1)_I| dm \ge |(\operatorname{Re} F_1)_{J_k} - (\operatorname{Re} F_1)_I| \ge K(1-\gamma)^{1/2} \left(\sum_{n=M_1}^{N_1} |a_n|^2 c_4^{N_1-n} \right)^{1/2}.$$

Adding over k, we obtain

$$\sum_{J_k \in \mathcal{G}} \int_{J_k} |\operatorname{Re} F_1 - (\operatorname{Re} F_1)_I| dm \ge K(1-\gamma)^{1/2} \left(\sum_{n=M_1}^{N_1} |a_n|^2 c_4^{N_1-n} \right)^{1/2} \sum_{J_k \in \mathcal{G}} m(J_k).$$

Applying (3.25) we deduce

$$\sum_{J_k \in \mathcal{G}} m(J_k) \le \frac{c_5^{1/2}}{K} m(I).$$

Since $\sum m(J_k) \ge c_0 m(I)$, taking $K > 0$ large enough so that $c_5^{1/2} K^{-1} < c_0/2$, we deduce

$$\sum_{J_k \notin \mathcal{G}} m(J_k) \ge \frac{c_0}{2} m(I)$$

and one can take as J in the statement, any of the arcs $J_k \notin \mathcal{G}$. $\square$

We are now ready to prove Theorem1.5.

Proof of Theorem 1.5. Since $\sum a_n f^n$ converges at a set of positive measure, Theorem 1.1 gives that $\sum |a_n|^2 < \infty$. Hence the function $F = \sum a_n f^n$ is in BMOA. The plan of the proof is to find constants $A, B > 0$, a sequence of arcs $\{I_k\}$ contained in I and a sequence of positive integers $\{M_k\}$ tending to infinity such that

$$\frac{1}{|I_k|}\int_{I_k} \operatorname{Re} F\, dm \geq A \sum_{n=1}^{M_k} |a_n| - B. \tag{3.26}$$

It is clear that (3.26) finishes the proof. The construction of the arcs $\{I_k\}$ and the sequence $\{M_k\}$ is performed inductively. We start using an idea of Weiss ([We]). Let $T' < T$ be two (large) positive integers to be fixed later such that T/T' is integer. Split the sum $F = \sum a_n f^n$ into blocks of length T, that is, $F = \sum_{k\geq 0} G_k$, where

$$G_k = \sum_{n=0}^{T-1} a_{kT+n} f^{kT+n}, \quad k = 0, 1, 2, \ldots$$

Next split G_{2k} into successive blocks of length T' and pick the subblock for which the sum of the modulus of the coefficients is the least. In other words, pick $\mathcal{G}_k$ a subset of T' consecutive integers in the interval $[2kT, (2k+1)T)$ such that

$$\sum_{n\in\mathcal{G}_k} |a_n| \leq \sum_{\ell\in\mathcal{G}} |a_\ell|$$

for any subset $\mathcal{G}$ of T' consecutive integers in $[2kT, (2k+1)T)$. Since the number of disjoint subblocks of length T' in $[2kT, (2k+1)T)$ is T/T' we deduce

$$\sum_{n\in\mathcal{G}_k} |a_n| \leq \frac{T'}{T}\sum_{n=0}^{T-1} |a_{2kT+n}|. \tag{3.27}$$

Each set of indices $\mathcal{G}_k$, $k \geq 1$, has T' elements and the corresponding block

$$S_k = \sum_{n\in\mathcal{G}_k} a_n f^n$$

will be called a short block. The long blocks are defined as the blocks between two short blocks. More concretely

$$L_1 = \sum_{n=0}^{N_1} a_n f^n,$$

where $N_1 + 1$ is the first index in $\mathcal{G}_1$ and

$$L_k = \sum_{n=M_k}^{N_k} a_n f^n, \quad k > 1,$$

where $M_k - 1$ is the largest index in $\mathcal{G}_{k-1}$ and $N_k + 1$ is the smallest index in $\mathcal{G}_k$. Note that each short block has T' terms while the number of terms in a long block is between T and $3T$. Note also that if $\mathcal{L}_k = \{n \in \mathbb{Z} : M_k \leq n \leq N_k\}$ denotes the set of indices appearing in the long block L_k, then (3.27) implies

$$\sum_{k=1}^{L}\sum_{n\in\mathcal{L}_k} |a_n| \geq \left(1 - \frac{T'}{T}\right)\sum_{n=1}^{N_L} |a_n|, \quad L = 1, 2, \ldots \tag{3.28}$$

The idea is that (3.28) will imply that the short blocks are irrelevant and the construction of the arcs $\{I_k\}$ and the sequence of integers $\{M_k\}$ verifying (3.26) will depend on the long blocks. Moreover, the fact that two long blocks are separated by a short block will provide a sort of independence between the long blocks.

Let s_k denote the ℓ^2-norm of the coefficients in the long block L_k, that is,

$$s_k^2 = \sum_{n \in \mathcal{L}_k} |a_n|^2, \quad k = 1, 2, \dots$$

Let $\varepsilon = \varepsilon(f) > 0$ and $c = c(f) > 0$ be the constants given by Lemma 3.6. Let $0 < \gamma = \gamma(f) < 1$ be a constant to be fixed later satisfying $1 - \gamma < \varepsilon$. Let $D(0, R)$ denote the disc centered at the origin of radius R. By the Denjoy–Wolff Theorem, f^n tends to 0 uniformly on compact sets of $\mathbb{D}$. The integer T' will be taken large enough so that

$$f^n(D(0, 1 - c(1-\gamma))) \subset D(0, \varepsilon), \quad n \geq T'. \tag{3.29}$$

Fix an arc $I^* \subset I$ such that $c^{-1}I^* \subset I$. The construction starts with the first long block L_{k_0} such that $|f^{M_{k_0}}(z(I^*))| < \varepsilon$. Without loss of generality we can assume that the constant c is smaller than the constants appearing in Lemma 3.5. Apply Lemma 3.5 to find an arc I_{k_0} with $c^{-1}I_{k_0} \subset c^{-1}I^* \subset I$ such that

$$c(1-\gamma) \leq 1 - |f^{N_{k_0}}(z(I_{k_0}))| \leq 1 - \gamma,$$
$$\frac{1}{m(I_{k_0})} \int_{I_{k_0}} \operatorname{Re} L_{k_0}\, dm \geq c s_{k_0}.$$

Assume by induction that we have constructed arcs $I_{k_0}, I_{k_0+1}, \dots, I_k$, $k \geq k_0$, with $c^{-1}I_j \subset c^{-1}I_{j-1}$, $j = k_0+1, \dots, k$, such that the following two conditions hold

$$c(1-\gamma) \leq 1 - |f^{N_k}(z(I_k))| \leq 1 - \gamma, \tag{3.30}$$

$$\frac{1}{m(I_k)} \int_{I_k} \operatorname{Re}\left(\sum_{j=k_0}^{k} L_j\right) dm \geq c \sum_{j=k_0}^{k} s_j - c^{-1}(1-\gamma)^{1/2} \sum_{\ell=k_0}^{k-1} \left(\sum_{j=k_0}^{\ell} t^{N_\ell - N_j} s_j^2\right)^{1/2}. \tag{3.31}$$

Here $0 < t < 1$ is the constant appearing in Lemma 3.6 and when $k = k_0$, we replace the right hand side term of (3.31) by $c s_{k_0}$. Recall that two different long blocks are separated by a short block which has T' terms. Hence the estimates (3.29) and (3.30) give that $|f^{M_{k+1}}(z(I_k))| \leq \varepsilon$. Apply Lemma 3.6 to $F_1 = \sum_{j=k_0}^{k} L_j$ and $F_2 = L_{k+1}$, to find an arc I_{k+1} with $c^{-1}I_{k+1} \subset c^{-1}I_k$ such that

$$c(1-\gamma) \leq 1 - |f^{N_{k+1}}(z(I_{k+1}))| \leq 1 - \gamma,$$
$$\frac{1}{m(I_{k+1})} \int_{I_{k+1}} \operatorname{Re} L_{k+1}\, dm \geq c s_{k+1}, \tag{3.32}$$

(3.33)
$$\frac{1}{m(I_{k+1})} \int_{I_{k+1}} \operatorname{Re} \sum_{j=k_0}^{k} L_j\, dm \geq \frac{1}{m(I_k)} \int_{I_k} \operatorname{Re} \sum_{j=k_0}^{k} L_j\, dm - c^{-1}(1-\gamma)^{1/2} \left(\sum_{j=k_0}^{k} t^{N_k - N_j} s_j^2\right)^{1/2}.$$

Then (3.33) and the induction assumption (3.31) give

$$\frac{1}{m(I_{k+1})} \int_{I_{k+1}} \operatorname{Re}\left(\sum_{j=k_0}^{k} L_j\right) dm \geq c \sum_{j=k_0}^{k} s_j - c^{-1}(1-\gamma)^{1/2} \sum_{\ell=k_0}^{k} \left(\sum_{j=k_0}^{\ell} t^{N_\ell - N_j} s_j^2\right)^{1/2}.$$

Applying (3.32) we deduce

$$\frac{1}{m(I_{k+1})}\int_{I_{k+1}} \operatorname{Re}\left(\sum_{j=k_0}^{k+1} L_j\right) dm \geq c\sum_{j=k_0}^{k+1} s_j - c^{-1}(1-\gamma)^{1/2}\sum_{\ell=k_0}^{k}\left(\sum_{j=k_0}^{\ell} t^{N_\ell - N_j} s_j^2\right)^{1/2}.$$

This concludes the inductive step.

Note that for any $k > k_0$, we have

$$\sum_{\ell=k_0}^{k-1}\left(\sum_{j=k_0}^{\ell} t^{N_\ell-N_j}s_j^2\right)^{1/2} \leq \sum_{\ell=k_0}^{k-1}\sum_{j=k_0}^{\ell} t^{(N_\ell-N_j)/2}s_j = \sum_{j=k_0}^{k-1} s_j \sum_{\ell=j}^{k-1} t^{(N_\ell-N_j)/2} \leq (1-t^{1/2})^{-1}\sum_{j=k_0}^{k-1} s_j.$$

Pick $0<\gamma<1$ sufficiently close to 1 so that $c^{-1}(1-\gamma)^{1/2}(1-t^{1/2})^{-1} \leq c/2$. The estimate (3.31) gives

$$\frac{1}{m(I_k)}\int_{I_k} \operatorname{Re}\left(\sum_{j=k_0}^{k} L_j\right) dm \geq \frac{c}{2}\sum_{j=k_0}^{k} s_j. \tag{3.34}$$

Recall that each long block has at most $3T$ terms. Hence Cauchy-Schwarz's inequality gives

$$\sum_{n\in\mathcal{L}_j} |a_n| \leq s_j (3T)^{1/2}, \quad j = 1,2,\dots$$

Then apply (3.34) to deduce

$$\frac{1}{m(I_k)}\int_{I_k} \operatorname{Re}\sum_{j=k_0}^{k}(L_j+S_j)\, dm \geq \frac{c}{2}(3T)^{-1/2}\sum_{j=k_0}^{k}\sum_{n\in\mathcal{L}_j}|a_n| - \sum_{j=k_0}^{k}\sum_{n\in\mathcal{G}_j}|a_n|, \quad k\geq k_0. \tag{3.35}$$

Let $\mathcal{A}_k$ be the set of indices appearing in $\sum_{j=k_0}^{k}(L_j+S_j)$, that is, $\mathcal{A}_k = \bigcup_{j=k_0}^{k}(\mathcal{L}_j\cup\mathcal{G}_j)$, $k\geq k_0$. Applying (3.35), (3.27) and (3.28) we obtain

$$\frac{1}{m(I_k)}\int_{I_k} \operatorname{Re}\sum_{j=k_0}^{k}(L_j+S_j)\, dm \geq \frac{c}{2}(3T)^{-1/2}\left(1-\frac{2T'}{T}\right)\sum_{n\in\mathcal{A}_k}|a_n| - \frac{2T'}{T}\sum_{n\in\mathcal{A}_k}|a_n|.$$

We now choose T so that $T'T^{-1/2}$ is sufficiently small so that

$$\frac{c}{2}(3T)^{-1/2}\left(1-\frac{2T'}{T}\right) - \frac{2T'}{T} \geq \frac{c}{4}T^{-1/2}.$$

Then

$$\frac{1}{m(I_k)}\int_{I_k} \operatorname{Re}\sum_{j=k_0}^{k}(L_j+S_j)\, dm \geq \frac{c}{4}T^{-1/2}\sum_{n\in\mathcal{A}_k}|a_n|. \tag{3.36}$$

As observed previously, $|f^{M_{k+1}}(z(I_k))| \leq \varepsilon$. By Corollary 2.2 there exists a constant $0<c_1<1$ such that $|f^n(z(I_k))| \leq c_1^{n-M_{k+1}}$, $n\geq M_{k+1}$. Then

$$\sum_{n\geq M_{k+1}} |a_n||f^n(z(I_k))| \leq (1-c_1^2)^{-1/2}\left(\sum_{n\geq M_{k+1}}|a_n|^2\right)^{1/2}.$$

Since there exists a constant $c_2 = c_2(f) > 0$ such that

$$\left| \frac{1}{m(I_k)} \int_{I_k} \sum_{n \geq M_{k+1}} a_n f^n \, dm - \sum_{n \geq M_{k+1}} a_n f^n(z(I_k)) \right| \leq c_2 \left(\sum_{n \geq M_{k+1}} |a_n|^2 \right)^{1/2},$$

the estimate (3.36) gives

$$\frac{1}{m(I_k)} \int_{I_k} \mathrm{Re} \sum_{n=1}^{\infty} a_n f^n \, dm \geq \frac{c}{4} T^{-1/2} \sum_{n \in \mathcal{A}_k} |a_n| - (c_2 + (1 - c_1^2)^{-1/2}) \left(\sum_{n \geq 1} |a_n|^2 \right)^{1/2}.$$

This gives (3.26) and finishes the proof. □

4 Other function spaces

Let $A(\mathbb{D})$ denote the disc algebra, that is, the space of analytic functions in $\mathbb{D}$ which extend continuously to $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$. The only inner functions which belong to $A(\mathbb{D})$ are the finite Blaschke products. See [Ga, p. 72]. Our next result is an easy consequence of Theorem 1.5.

Theorem 4.1. *Let f be an inner function with $f(0) = 0$ which is not a rotation and let $\{a_n\}$ be a sequence of complex numbers not identically zero. Then $\sum_{n=1}^{\infty} a_n f^n \in A(\mathbb{D})$ if and only if f is a finite Blaschke product and $\sum_{n=1}^{\infty} |a_n| < \infty$.*

Proof. Write $F = \sum_{n=1}^{\infty} a_n f^n$. The sufficiency follows from the uniform convergence of the partial sums $\sum_{n=1}^{N} a_n f^n$. We now prove the necessity. Assume $F \in A(\mathbb{D})$. Since $\sup\{|F(\xi)| : \xi \in \partial\mathbb{D}\} < \infty$, Theorem 1.5 gives that $\sum_{n=1}^{\infty} |a_n| < \infty$. We now show that f is a finite Blaschke product arguing by contradiction. Assume that f does not extend analytically at any neighbourhood of the point $\xi \in \partial\mathbb{D}$. By Frostman's Theorem (see [Ga, p. 77]) there exists a set $E \subset \mathbb{D}$ of logarithmic capacity zero such that for any $\alpha \in \mathbb{D} \backslash E$ one can find a sequence $\{z_k\} = \{z_k(\alpha)\}$, of points in $\mathbb{D}$ converging to ξ, such that $f(z_k) = \alpha$ for any k. Then

$$F(z_k) = \sum_{n=1}^{\infty} a_n f^{n-1}(\alpha), \quad k = 1, 2, \ldots$$

Here $f^0(\alpha) = \alpha, \alpha \in \mathbb{D}$. Since $\{z_k\}$ converges to ξ and $F \in A(\mathbb{D})$, for any $\alpha \in \mathbb{D} \backslash E$ we have

$$\sum_{n=1}^{\infty} a_n f^{n-1}(\alpha) = F(\xi).$$

Hence the function $\sum_{n=1}^{\infty} a_n f^{n-1}$ is constant in $\mathbb{D}$. Since it vanishes at the origin, we deduce $\sum_{n=1}^{\infty} a_n f^{n-1} \equiv 0$. Hence $\sum_{n=1}^{\infty} a_n f^n \equiv 0$. By Lemma 2.3, $a_n = 0$ for any $n \geq 1$, which gives the contradiction. □

Let $\mathcal{D}$ denote the Dirichlet space of analytic functions f in $\mathbb{D}$ such that

$$\|f\|_{\mathcal{D}}^2 = \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < \infty,$$

where $dA(z)$ denotes the area measure. Note that $\|f\|_{\mathcal{D}}^2$ can be understood as the area of the image domain $f(\mathbb{D})$, counting multiplicities. Hence the only inner functions which belong to the Dirichlet space are finite Blaschke products. Our next result is a description of linear combinations of iterates of an inner function which belong to the Dirichlet space.

Theorem 4.2. *Let f be a finite Blaschke product with $N > 1$ zeros and with $f(0) = 0$. Let $\{a_n\}$ be a sequence of complex numbers with $\sum_{n=1}^{\infty} |a_n|^2 < \infty$. Then $F = \sum_{n=1}^{\infty} a_n f^n$ converges in the Dirichlet space if and only if $\sum_{n=1}^{\infty} |a_n|^2 N^n < \infty$. Moreover there exists a universal constant $C = C(f) > 0$ such that*

$$C^{-1} \sum_{n=1}^{\infty} |a_n|^2 N^n \leq \int_{\mathbb{D}} |F'(z)|^2 \, dA(z) \leq C \sum_{n=1}^{\infty} |a_n|^2 N^n.$$

Proof. Note that

$$\int_{\mathbb{D}} |F'(z)|^2 \, dA(z) = \sum_{n=1}^{\infty} |a_n|^2 b_{n,n} + 2 \operatorname{Re} \sum_{n=1}^{\infty} \overline{a}_n \sum_{k>n} a_k b_{k,n},$$

where

$$b_{k,n} = \int_{\mathbb{D}} \overline{(f^n)'(z)} (f^k)'(z) \, dA(z), \quad k, n = 1, 2, \ldots$$

Since f^n is a Blaschke product with N^n zeros, the area counting multiplicities, of $f^n(\mathbb{D})$ is πN^n. Then

$$\int_{\mathbb{D}} |(f^n)'(z)|^2 \, dA(z) = \pi N^n.$$

On the other hand, the change of variables formula with multiplicities (see [EG, p. 122]) gives

$$b_{k,n} = \int_{\mathbb{D}} |(f^n)'(z)|^2 (f^{k-n})'(f^n(z)) \, dA(z) = \int_{\mathbb{D}} N^n (f^{k-n})'(w) \, dA(w), \quad k > n.$$

The mean value property gives $b_{k,n} = \pi N^n f'(0)^{k-n}$, $k > n$. Hence

$$\frac{1}{\pi} \int_{\mathbb{D}} |F'(z)|^2 \, dA(z) = \sum_{n=1}^{\infty} |a_n|^2 N^n + 2 \operatorname{Re} \sum_{n=1}^{\infty} \overline{a}_n N^n \sum_{k>n} a_k f'(0)^{k-n} = \vec{a}^t T \vec{a},$$

where $\vec{a}$ denotes the vector $(a_n N^{n/2})_{n \geq 1}$ and T is the Toeplitz matrix whose entries are

$$t_{n,k} = (f'(0) N^{-1/2})^{k-n}, \quad k \geq n; \qquad t_{n,k} = (\overline{f'(0)} N^{-1/2})^{n-k}, \quad n \geq k.$$

Consider the symbol

$$t(\xi) = \sum_{n=-\infty}^{\infty} t_{n,0} \xi^n = \frac{1 - |f'(0)|^2 N^{-1}}{|1 - f'(0) N^{-1/2} \xi|^2}, \quad \xi \in \partial\mathbb{D}.$$

It is well known that T diagonalizes and its eigenvalues are between the minimum and the maximum of t. See [BG]. Since $C(f,N)^{-1} \leq t(\xi) \leq C(f,N)$, a.e. $\xi \in \partial\mathbb{D}$, where

$$C(f,N) = \frac{1 + |f'(0)| N^{-1/2}}{1 - |f'(0)| N^{-1/2}}, \quad \text{a.e. } \xi \in \partial\mathbb{D},$$

we deduce that

$$C(f,N)^{-1} \sum_{n=1}^{\infty} |a_n|^2 N^n \leq \frac{1}{\pi} \int_{\mathbb{D}} |F'(z)|^2 \, dA(z) \leq C(f,N) \sum_{n=1}^{\infty} |a_n|^2 N^n.$$

□

Let $\mathcal{B}$ denote the Bloch space of analytic functions f in $\mathbb{D}$ such that

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

It is well known that a lacunary series is in the Bloch space if and only if its coefficients are uniformly bounded. See [ACP]. In our setting this condition is still sufficient but we will see that it is not necessary.

Theorem 4.3. *Let f be an inner function with $f(0)=0$ which is not a rotation and let $\{a_n\}$ be a bounded sequence of complex numbers. Then $F=\sum_{n=1}^{\infty} a_n f^n \in \mathcal{B}$ and there exists a constant $C=C(f)>0$ such that $\|F\|_{\mathcal{B}} \le C \sup_{n\ge 1}|a_n|$.*

Proof. Let $0<r_0=r_0(f)<1$ and $0<c_0=c_0(f)<1$ be the constants given by Corollary 2.2. Since f^n tends to 0 uniformly on compacts of $\mathbb{D}$, for any $z\in\mathbb{D}$ we can pick $N(z)$ to be the minimum positive integer n such that $|f^n(w)|\le r_0$ for any $w\in\mathbb{D}$ with $\rho(w,z)\le 1/2$. Corollary 2.2 gives

$$\sup\{|f^n(w)| : \rho(w,z)\le r_0\} \le c_0^{n-N(z)}, \quad n\ge N(z), \quad z\in\mathbb{D}. \tag{4.1}$$

By Cauchy's estimate, there exist a universal constant $c_1>0$ such that

$$(1-|z|)|(f^n)'(z)| \le c_1 c_0^{n-N(z)}, \quad n\ge N(z), \quad z\in\mathbb{D}.$$

Hence

$$(1-|z|)\sum_{n\ge N(z)} |(f^n)'(z)| \le c_1(1-c_0)^{-1}, \quad z\in\mathbb{D}. \tag{4.2}$$

Let $M=N(z)-1$. Note that there exists a point w with $\rho(w,z)\le 1/2$ such that $|f^M(w)|\ge r_0$. According to part (a) of Lemma 2.1, there exists a constant $0<c_2<1$ such that $1-|f^n(w)|\le c_2^{M-n}$ for any $n<N(z)$. By Schwarz's Lemma there exists a constant $c_3>0$ such that $1-|f^n(z)|\le c_3 c_2^{M-n}$ for any $n<N(z)$. Since $(1-|z|^2)|(f^n)'(z)|\le 1-|f^n(z)|^2$, $z\in\mathbb{D}$, we deduce

$$(1-|z|^2)\sum_{n=1}^{M} |(f^n)'(z)| \le c_3(1-c_2)^{-1}, \quad z\in\mathbb{D}. \tag{4.3}$$

The estimates (4.2) and (4.3) give that $\|F\|_{\mathcal{B}} \le c\sup_n |a_n|$ with $c=c_1(1-c_0)^{-1}+c_3(1-c_2)^{-1}$. $\square$

Next we will show that the converse estimate in Theorem 4.3 does not hold, that is, there exist an inner function f with $f(0)=0$ and unbounded sequence $\{a_n\}$ of complex numbers such that $\sum a_n f^n\in\mathcal{B}$. We start with an auxiliary result on the hyperbolic derivative $D_h f$ defined in (2.1). Denote $f^0(z)=z, z\in\mathbb{D}$.

Lemma 4.4. *Let $f\in H^\infty(\mathbb{D})$, $\|f\|_\infty\le 1$. Then for any $n=1,2,\dots,$ we have*

$$D_h(f^n)(z) = \prod_{k=0}^{n-1} D_h(f)(f^k(z)), \quad z\in\mathbb{D}.$$

Proof. Note that $(f^n)'(z)=\prod_{k=0}^{n-1} f'(f^k(z))$, $n\ge 1$. Then

$$(1-|z|^2)\frac{(f^n)'(z)}{1-|f^n(z)|^2} = \prod_{k=0}^{n-1} \frac{(1-|f^k(z)|^2 f'(f^k(z))}{1-|f^{k+1}(z)|^2}.$$

$\square$

Let $f \in H^\infty$ with $\|f\|_\infty \leq 1$. Schwarz's Lemma gives that $D_h f(z) \leq 1$, $z \in \mathbb{D}$. Given any positive gauge function $w\colon [0,1] \to (0,+\infty)$ satisfying a mild regularity condition, such that

$$\int_0^1 \frac{w^2(t)}{t}\, dt = \infty,$$

there exists an inner function f with $f(0) = 0$ such that $D_h f(z) \leq w(1-|z|)$, $z \in \mathbb{D}$. See [AAN] or [Sm]. In particular, given any $0 < \tau < 1$, there exists an inner function f with $f(0) = 0$ such that $D_h f(z) \leq \tau$. Let $\{a_n\}$ be any sequence of complex numbers such that

$$\sum_{n=1}^{\infty} |a_n| \tau^n < \infty. \tag{4.4}$$

We next show that the function $F = \sum_{n=1}^{\infty} a_n f^n$ is in the Bloch space. Note that

$$(1-|z|^2)|F'(z)| \leq \sum_{n=1}^{\infty} |a_n||(f^n)'(z)|(1-|z|^2) = \sum_{n=1}^{\infty} |a_n|(1-|f^n(z)|^2) D_h(f^n)(z), \quad z \in \mathbb{D}.$$

Apply Lemma 4.4 to deduce $D_h(f^n)(z) \leq \tau^n$, $n \geq 1$, $z \in \mathbb{D}$. Then

$$(1-|z|^2)|F'(z)| \leq \sum_{n=1}^{\infty} |a_n| \tau^n, \quad z \in \mathbb{D}.$$

Hence if condition (4.4) holds, we have $F \in \mathcal{B}$.

We finish this section showing the following converse of part (b) of Corollary 1.3.

Corollary 4.5. *Let f be an inner function with $f(0) = 0$ which is not a rotation and let $\{a_n\}$ be a non-identically zero sequence of complex numbers with $\sum_{n=1}^{\infty} |a_n|^2 < \infty$. Assume $F = \sum_{n=1}^{\infty} a_n f^n \in$ VMOA($\mathbb{D}$). Then f is a finite Blaschke product.*

Proof. Assume f is not a finite Blaschke product. then there exists a sequence of points $\{z_k\}$ in $\mathbb{D}$ with $|z_k| \to 1$ such that $f(z_k) \to 0$, as $k \to \infty$. Lemma 3.2 gives that

$$\int_{\partial\mathbb{D}} |F(\xi) - F(z_k)|^2 \, P(z_k, \xi)\, dm(\xi) \geq \frac{1}{2} \frac{1-|f'(0)|}{1+|f'(0)|} \sum_{n=1}^{\infty} |a_n|^2,$$

if k is sufficiently large. Since the integral above tends to 0 as k tends to ∞, we deduce that $a_n = 0$ for any n. This finishes the proof. □

5 Open problems

We close the paper mentioning some open problems we have not explored.

Analytic continuation and smoothness. Hadamard's Theorem says that if a lacunary power series can be extended analytically across an arc of the unit circle, it can actually be extended analytically to a neighbourhood of the closed unit disc (see [Zy, p. 208]). It is natural to ask if the following analogous result holds.

Problem 1. Let f be an inner function with $f(0) = 0$ which is not a rotation. Let $\{a_n\}$ be a sequence of complex numbers whose modulus are square summable and consider $F = \sum_{n=1}^{\infty} a_n f^n$. Assume that F extends analytically across an arc $I \subset \partial\mathbb{D}$. Is it true that there exist $n_0 > 0$ such that $a_n = 0$ for any $n > n_0$ and that f^n extends analytically across I for any $n \leq n_0$?

For $0 < \alpha < 1$, let $\mathrm{Lip}_\alpha(\overline{\mathbb{D}})$ denote the space of continuous functions g in $\overline{\mathbb{D}}$ for which there exists a constant $C = C(g) > 0$ such that $|g(z) - g(w)| \leq C|z - w|^\alpha$, for any pair of points $z, w \in \overline{\mathbb{D}}$. Lacunary series which belong to $\mathrm{Lip}_\alpha(\overline{\mathbb{D}})$ can be described in terms of their coefficients. It is then natural to ask for the corresponding result in our context.

Problem 2. Let f be a finite Blaschke product with $f(0) = 0$ which is not a rotation. Describe the sequences of complex numbers $\{a_n\}$ with square summable modulus, such that $F = \sum_{n=1}^{\infty} a_n f^n \in \mathrm{Lip}_\alpha(\overline{\mathbb{D}})$.

Peano curves. Let $\sum a_k z^{n_k}$ be a lacunary series with $\sum |a_k| = \infty$ and $\lim a_k = 0$. Then for any $w \in \mathbb{C}$, there exists $\xi \in \partial\mathbb{D}$ such that $\sum a_k \xi^{n_k}$ converges to w ([We]). It is then natural to ask for the following analogous result.

Problem 3. Let f be an inner function with $f(0) = 0$ which is not a rotation. Let $\{a_n\}$ be a sequence of complex numbers with $\sum_{n=1}^{\infty} |a_n| = \infty$, but $\lim_{n\to\infty} a_n = 0$. Is it true that for any $w \in \mathbb{C}$, there exists $\xi \in \partial\mathbb{D}$ such that $\sum_{n=1}^{\infty} a_n f^n(\xi)$ converges to w?

There is also a version of the question above in the interior of the unit disc. Murai proved that any lacunary series $\sum a_k z^{n_k}$ convergent in $\mathbb{D}$ with $\sum |a_k| = \infty$, takes any complex value infinitely often. See [Mu]. It is natural to ask for the following analogous result.

Problem 4. Let f be an inner function with $f(0) = 0$ which is not a rotation. Let $\{a_n\}$ be any sequence of complex numbers such that $F(z) = \sum_{n=1}^{\infty} a_n f^n(z)$ is analytic in $\mathbb{D}$. Assume $\sum_{n=1}^{\infty} |a_n| = \infty$. Is true that for any $w \in \mathbb{C}$ there exists $z \in \mathbb{D}$ such that $F(z) = w$?

Abel's Theorem. Let $F(z) = \sum a_n z^n$ be a power series with radius of convergence 1. Let $\xi \in \partial\mathbb{D}$ such that $\sum a_n \xi^n$ converges. The classical Abel's Theorem says that $F(z)$ tends to $\sum a_n \xi^n$, as z approaches non-tangentially to ξ. The classical Hardy-Littlewood High Indices Theorem asserts that for lacunary series, the converse holds. In other words, if $F(z) = \sum a_k z^{n_k}$ is a lacunary series which has limit L when z approaches non-tangentially a point $\xi \in \partial\mathbb{D}$, then $\sum a_k \xi^{n_k}$ converges to L. It is then natural to ask for the following analogous results.

Problem 5. Let f be an inner function with $f(0) = 0$ which is not a rotation. Let $\{a_n\}$ be any sequence of complex numbers such that $F(z) = \sum_{n=1}^{\infty} a_n f^n(z)$ is analytic in $\mathbb{D}$. Let $\xi \in \partial\mathbb{D}$ such that $f^n(\xi) = \lim_{r\to 1} f^n(r\xi)$ exists for any positive integer n. Assume $\sum_{n=1}^{\infty} a_n f^n(\xi)$ converges. Is it true that $\lim_{r\to 1} F(r\xi)$ exists?

Problem 6. Let f be an inner function with $f(0) = 0$ which is not a rotation. Let $\{a_n\}$ be any sequence of complex numbers such that $F(z) = \sum_{n=1}^{\infty} a_n f^n(z)$ is analytic in $\mathbb{D}$. Assume $F(z)$ has limit when z approaches non-tangentially a point $\xi \in \partial\mathbb{D}$. Is it true that $\sum_{n=1}^{\infty} a_n f^n(\xi)$ converges?

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