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Bounded Functions in Möbius Invariant Dirichlet Spaces

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For $p \in (0, 1)$, let $Q_p(Q_{p,0})$ be the space of analytic functions f on the unit disk Δ with $\sup_{w \in \Delta} \|f \circ \varphi_w\|_{\mathscr{D}_p} < \infty$ $(\lim_{|w| \to 1} \|f \circ \varphi_w\|_{\mathscr{D}_p} = 0)$, where $\|\cdot\|_{\mathscr{D}_p}$ means the weighted Dirichlet norm and φ_w is the Möbius map of Δ onto itself with $\varphi_w(0) = w$. In this paper, we prove the Corona theorem for the algebra $Q_p \cap H^\infty$ $(Q_{p,0} \cap H^\infty)$; then we provide a Fefferman–Stein type decomposition for $Q_p(Q_{p,0})$, and finally we describe the interpolating sequences for $Q_p \cap H^\infty(Q_{p,0} \cap H^\infty)$.

1. INTRODUCTION

For $p \in (-1, \infty)$, let $\mathscr{D}_p$ be the space of analytic functions f on the unit disk Δ with

$$\|f\|^2_{\mathscr{D}_p} = \int_\Delta |f'(z)|^2 (1 - |z|^2)^p \, dm(z) < \infty,$$

where $dm(z)$ denotes the usual Lebesgue measure on Δ. These are called Dirichlet type spaces because for $p = 0$ one gets the classical Dirichlet space $\mathscr{D}$ of all analytic functions on Δ whose images have finite area, counting multiplicities. Also, observe that for $p = 1$, $\mathscr{D}_p$ is just the usual Hardy space H^2 and for $p > 1$ is the Bergman space with weight $(1 - |z|^2)^{p-2}$.

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Here, we are mainly interested in the conformally invariant version of these spaces. More precisely, for $p \in (-1, \infty)$, the space Q_p consists of all analytic functions f on Δ such that

$$\|f\|_{Q_p}^2 = \sup_{w \in \Delta} \|f \circ \varphi_w\|_{\mathscr{D}_p}^2 < \infty,$$

where φ_w is the Möbius transformation of Δ onto itself, sending the origin to w. An easy computation shows that a function f in $\mathscr{D}_p$ belongs to Q_p if and only if

$$\sup_{w \in \Delta} \int_\Delta |f'(z)|^2 \left[\log \frac{1}{|\varphi_w(z)|}\right]^p dm(z) < \infty.$$

Moreover, if $f \in Q_p$ and the above integrals tend to zero as $|w| \to 1$ then we say $f \in Q_{p,0}$.

If $-1 < p < 0$, the space Q_p only contains constant functions, while Q_0 is the classical Dirichlet space.

If $p \in (0, 1)$, then $Q_p = Q_p(\partial\Delta) \cap H^2$ ($Q_{p,0} = Q_{p,0}(\partial\Delta) \cap H^2$), where $\partial\Delta$ is the boundary of Δ and $Q_p(\partial\Delta)$ is defined as the space of all functions $f \in L^2(\partial\Delta)$ with

$$\|f\|_{Q_p(\partial\Delta)}^2 = \sup_{I \subseteq \partial\Delta} \frac{1}{|I|^p} \int_I \int_I \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^{2-p}} \, d\theta \, d\varphi < \infty.$$

As usual, the supremum is taken over all subarcs $I \subset \partial\Delta$ and $|I|$ denotes the normalized arc length of I, while $Q_{p,0}(\partial\Delta)$ is the set of functions $f \in Q_p(\partial\Delta)$, for which the above integrals tend to zero when $|I| \to 0$ [10]. In addition, if $0 < p_1 < p_2 < 1$ then $Q_{p_1} \subsetneq Q_{p_2}$ ($Q_{p_1,0} \subsetneq Q_{p_2,0}$) [3].

If $p = 1$ then $Q_1 = \mathrm{BMOA} = \mathrm{BMO}(\partial\Delta) \cap H^2$ ($Q_{1,0} = \mathrm{VMOA} = \mathrm{VMO}(\partial\Delta) \cap H^2$), where $\mathrm{BMO}(\partial\Delta)$ ($\mathrm{VMO}(\partial\Delta)$) is the usual space of functions in $L^2(\partial\Delta)$ with bounded (vanishing) mean oscillation on $\partial\Delta$, (see [11, Chapter VI]).

If $p \in (1, \infty)$, then $Q_p = B$ ($Q_{p,0} = B_0$), where B (B_0) is the classical Bloch space (little Bloch space), [1, 23, 24].

In this paper, we study the Banach algebra $Q_p \cap H^\infty$($Q_{p,0} \cap H^\infty$). To be precise, we first discuss the Corona property for $Q_p \cap H^\infty$($Q_{p,0} \cap H^\infty$); secondly consider a Fefferman–Stein type decomposition of $Q_p(\partial\Delta)$ ($Q_{p,0}(\partial\Delta)$) and finally investigate interpolating sequences for $Q_p \cap H^\infty$ ($Q_{p,0} \cap H^\infty$). Here H^∞ stands for the space of all bounded analytic functions on Δ and $\|f\|_\infty = \sup\{|f(z)|: z \in \Delta\}$. For convenience, we will now state our main results.

First of all, we will prove that the Corona theorem holds for the algebra $Q_p \cap H^\infty$ ($Q_{p,0} \cap H^\infty$), whenever $p \in (0, 1)$, that is to say, the unit disk Δ is dense in the maximal ideal space of $H^\infty \cap Q_p$ ($H^\infty \cap Q_{p,0}$), $p \in (0, 1)$. This fact can be reformulated in the following way.

THEOREM 1.1. *Let $p \in (0, 1)$. If $f_1, ..., f_n \in Q_p \cap H^\infty$ ($Q_{p,0} \cap H^\infty$) with*

$$\inf_{z \in \Delta} (|f_1(z)| + \cdots + |f_n(z)|) > 0,$$

then there exist $g_1, ..., g_n \in Q_p \cap H^\infty (Q_{p,0} \cap H^\infty)$ with

$$f_1 g_1 + \cdots + f_n g_n \equiv 1.$$

It is certainly well known that Theorem 1.1 holds for $p \geqslant 1$. If $p \geqslant 1$, the space $Q_p \cap H^\infty$ is just H^∞, and the Corona Theorem holds by a celebrated result of L. Carleson ([8], [11, Chapter VIII]). Also, Theorem 1.1 was proved for the algebra $Q_{1,0} \cap H^\infty = \mathrm{VMO} \cap H^\infty$ by C. Sundberg and T. Wolff ([19]). If $p \geqslant 1$, $Q_{p,0} \cap H^\infty = B_0 \cap H^\infty$ and the result is also known. However, since we have not found a reference in the literature, a proof is presented. The proof of Theorem 1.1 follows the usual method of solving some $\bar{\partial}$-equations with L^∞ and Q_p estimates simultaneously. For this, we use an explicit solution due to P. Jones [12].

As is well known, there is a close relation between $\bar{\partial}$-equations and the Fefferman–Stein decomposition asserting that any $f \in \mathrm{BMO}(\partial\Delta)$ ($\mathrm{VMO}(\partial\Delta)$) can be decomposed into $f = u + \tilde{v}$, where $u, v \in L^\infty(\partial\Delta)$ ($C(\partial\Delta)$) and $\tilde{v}$ means the conjugate function of v. So, it is not surprising that solving $\bar{\partial}$-equations with appropiate estimates leads to the following result.

THEOREM 1.2. *Let $p \in (0, 1)$ and $f \in L^2(\partial\Delta)$. Then $f \in Q_p(\partial\Delta)$ ($Q_{p,0}(\partial\Delta)$) if and only if $f = u + \tilde{v}$, where $u, v \in Q_p(\partial\Delta) \cap L^\infty(\partial\Delta)$ ($Q_{p,0}(\partial\Delta) \cap C(\partial\Delta)$). When $f \in Q_p(\partial\Delta)$* and $\int_{\partial\Delta} f(e^{i\theta})\, d\theta = 0$, *the functions can be chosen so that*

$$\|u\|_{L^\infty(\partial\Delta)} + \|v\|_{L^\infty(\partial\Delta)} + \|u\|_{Q_p(\partial\Delta)} + \|v\|_{Q_p(\partial\Delta)} \leqslant C \, \|f\|_{Q_p(\partial\Delta)},$$

where $C > 0$ is an absolute constant.

Finally, we will discuss the free interpolation problem in the algebra $Q_p \cap H^\infty$ ($Q_{p,0} \cap H^\infty$), $p \in (0, 1)$. A sequence $\{z_n\} \subset \Delta$ is called an interpolating sequence for $Q_p \cap H^\infty$ ($Q_{p,0} \cap H^\infty$) if for each bounded sequence $\{w_n\}$ of complex numbers there exists $f \in Q_p \cap H^\infty$ ($Q_{p,0} \cap H^\infty$) such that $f(z_n) = w_n$ for all n. Let ρ be the pseudohyperbolic distance on Δ, that is,

$$\rho(z, w) = |\varphi_w(z)| = \left| \frac{w - z}{1 - \bar{w}z} \right|, \qquad z, w \in \Delta.$$

THEOREM 1.3. *Let $p \in (0, 1)$. A sequence $\{z_n\}$ of points in the unit disk is an interpolating sequence for $Q_p \cap H^\infty$ if and only if the following conditions hold*:

(a) $\inf_{m \neq n} \rho(z_m, z_n) > 0$

(b) $\sup_{w \in \Delta} \sum_n (1 - |\varphi_w(z_n)|^2)^p < \infty$.

THEOREM 1.4. *Let $p \in (0, 1)$. A sequence $\{z_n\}$ of points in the unit disk is an interpolating sequence for $Q_{p,0} \cap H^\infty$ if and only if the following conditions hold*:

(a) $\lim_{n \to \infty} \inf_{m \neq n} \rho(z_m, z_n) = 1$

(b) $\lim_{r \to 1} \sup_{w \in \Delta} \sum_{n:\, |\varphi_w(z_n)| \geqslant r} (1 - |\varphi_w(z_n)|^2)^p = 0$.

It is worth remarking that Theorems 1.3 and 1.4 still hold true for $p = 1$. Since $Q_p \cap H^\infty = H^\infty$, $p \geqslant 1$, this is again a result of L. Carleson [7], while interpolating sequences for the algebra $H^\infty \cap Q_{1,0} = H^\infty \cap \mathrm{VMOA}$ were described in [19], as the ones satisfying (a), (b) in Theorem 1.4 with $p = 1$. As in [19], a sequence $\{z_n\}$ of points in the unit disc is called p-uniformly separated (p-thin) if (a) and (b) in Theorem 1.3 ((a) and (b) in Theorem 1.4) hold. We will also show that the interpolating sequences for $H^\infty \cap B_0$ are the 1-thin sequences.

The necessity in both results follows from an argument which combines the Khinchin's inequality and a reproducing formula due to R. Rochberg and Z. Wu [16]. The sufficiency for $Q_p \cap H^\infty$ is easy and it can be derived in different ways. However the sufficiency for $Q_{p,0} \cap H^\infty$ is more difficult as one can already see in the case of $\mathrm{VMOA} \cap H^\infty$ [19]. There are two main reasons. First, there are no inner functions in $Q_{p,0}$ except for Blaschke products with a finite number of zeros and, second, it is not simple to construct non-analytic functions solving the required interpolation problems. Our proof will take some ideas from [19] but our construction of such functions is quite different from theirs and, we believe, is more elementary.

The paper is organized as follows. In Section 2 we collect some basic facts about Q_p-spaces, including their analytic and non-analytic forms. Theorems 1.1 and 1.2 will be proved in Section 3, and Section 4 is devoted to the discussion of interpolating sequences.

Throught this paper some notations will be used repeatedly. The letters, C, C', C_1 etc will denote absolute constants, not necessarily the same at each occurrence. The notation $a \simeq b$ means that there are absolute constants $c_1, c_2 > 0$ satisfying $c_1 b \leqslant a \leqslant c_2 b$. Similarly, $a \lesssim b$ means that the second inequality holds. Also given an arc I of the unit circle, let $S(I)$ be the Carleson box based on I, i.e.,

$$S(I) = \{re^{i\theta}: 1 - |I| \leqslant r < 1,\ e^{i\theta} \in I\}\ ;$$

and $T(S(I))$ denotes the "top half" of $S(I)$, i.e.,

$$T(S(I))=\left\{re^{i\theta}\in S(I)\colon 1-|I|\leqslant r\leqslant 1-\frac{|I|}{2}\right\}.$$

For $0\neq z\in\Delta$ let I_z be the arc of the unit circle of length $(1-|z|)/2\pi$ centered at $z/|z|$, and further we denote by $S(z)$ and $T(z)$ respectively the sets $S(I_z)$ and $T(S(I_z))$. If I is an arc on $\partial\Delta$ then z_I means the point in Δ such that $I=I_z$, and for a positive integer n, nI means the arc with the same center as I and length $n\,|I|$.

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2. PRELIMINARY FACTS

In this section we will collect some known facts on Q_p which will be used in the following sections.

3.1. *Two Characterizations of* $Q_p(\partial\Delta)$

Let $p\in(0,\infty)$. A positive Borel measure μ defined on Δ is called a p-Carleson measure if

$$\|\mu\|_p=\sup_{I\subset\partial\Delta}\frac{\mu(S(I))}{|I|^p}<\infty,$$

where the supremum is taken over all subarcs $I\subset\partial\Delta$. If the right hand fractions tend to zero as $|I|\to 0$ then μ is said to be a p-vanishing Carleson measure. As in the case $p=1$, p-Carleson measures can be characterized in conformally invariant terms ([ASX]). Actually μ is a p-Carleson measure if and only if

$$\sup_{z\in\Delta}\int_\Delta\left(\frac{1-|z|^2}{|1-\bar{z}w|^2}\right)^p d\mu(w)<\infty.$$

Actually this supremum is comparable to $\|\mu\|_p$. Similarly, μ is a p-vanishing Carleson measure if and only if the above integral tends to 0 when $|z|\to 1$.

For $p\in(0,1)$, this notion has been used to characterize Q_p ($Q_{p,0}$)-functions [2, 10]. Given $f\in L^1(\partial\Delta)$ let $\hat{f}$ be its Poisson extension, that is

$$\hat{f}(z)=\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\,\frac{1-|z|^2}{|1-ze^{-i\theta}|^2}\,d\theta.$$

Then we will obtain the following theorem which may be viewed as an extension of Theorem 1.1 in [2], and Theorem 2.1, 5.3 and Corollary 5.2 in [10].

THEOREM 2.1. *Let* $p \in (0, 1)$ *and* $f \in L^2(\partial \Delta)$ *with* $f(e^{i\theta}) \sim \sum_{-\infty}^{\infty} a_k e^{ik\theta}$. *Then the following conditions are equivalent*:

(i) $f \in Q_p(\partial \Delta)(Q_{p,0}(\partial \Delta))$.

(ii) $|\nabla \hat{f}(z)|^2 (1-|z|^2)^p \, dm(z)$ *is a* p*-Carleson measure* (p*-vanishing Carleson measure*).

(iii) $\|\sigma_n\|_{Q_p(\partial \Delta)} = O(1)(o(1))$, *where*

$$\sigma_n(e^{i\theta}) = \frac{1}{2\pi} \sum_{-n}^{n} \left(1 - \frac{|k|}{n+1}\right) a_k e^{ik\theta}.$$

Proof. The equivalence between (i) and (ii) is derived easily from the proof of Theorem 2.1 in [10]. Nevertheless, we provide a new proof. Indeed, for $f \in L^2(\partial \Delta)$, $p \in (0, 1)$, one can check (see Lemma 2.6 of [17]), that

$$\int_\Delta |\nabla(\hat{f} \circ \varphi_w)(z)|^2 (1-|z|^2)^p \, dm(z)$$

$$\simeq \int_{\partial \Delta} \int_{\partial \Delta} \frac{|f(\varphi_w(e^{i\theta})) - f(\varphi_w(e^{i\varphi}))|^2}{|e^{i\theta} - e^{i\varphi}|^{2-p}} \, d\theta \, d\varphi$$

$$= \int_{\partial \Delta} \int_{\partial \Delta} \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^{2-p}} \left[\frac{1-|w|^2}{|1-\bar{w}e^{i\theta}| \, |1-\bar{w}e^{i\varphi}|}\right]^p d\theta \, d\varphi.$$

The first estimate follows expressing the double integral in terms of the Fourier coefficients of f and the area integral in terms of the power series of the analytic and antianalytic parts of $\hat{f}$. See Lemma 2 in [20]. Let $I \subset \partial \Delta$ be any arc on $\partial \Delta$ and let $w \in \Delta$ satisfy $S(w) = S(I)$. Then

$$\frac{1}{|I|} \lesssim \frac{1-|w|^2}{|1-\bar{w}e^{i\theta}| \, |1-\bar{w}e^{i\varphi}|}, \qquad e^{i\theta}, e^{i\varphi} \in I$$

and hence (i) follows whenever (ii) is true.

Conversely, let $f \in Q_p(\partial \Delta)$. For $w \in \Delta$, let $I \subset \partial \Delta$ be the arc such that $S(I) = S(w)$. Observe that if $e^{i\theta} \in 2^{k+1} I \backslash 2^k I$, $e^{i\varphi} \in 2^{j+1} I \backslash 2^j I$, $k, j = 0, 1, ...$ then we have

$$\frac{1-|w|^2}{|1-\bar{w}e^{i\theta}| \, |1-\bar{w}e^{i\varphi}|} \lesssim \frac{1}{2^{(j+k)} \, |I|}.$$

Thus

$$\int_{\Delta} |\nabla(\hat{f}\circ\varphi_w)(z)|^2 (1-|z|^2)^p\, dm(z)$$

$$\lesssim \sum_{j,k} \frac{1}{2^{(j+k)p}\, |I|^p} \int_{2^j I}\int_{2^k I} \frac{|f(e^{i\theta})-f(e^{i\varphi})|^2}{|e^{i\theta}-e^{i\varphi}|^{2-p}}\, d\theta\, d\varphi$$

$$\lesssim \sum_j \frac{1}{(2^{2j}\, |I|)^p} \int_{2^j I}\int_{2^j I} + \sum_j \sum_{k>0} \frac{1}{(2^{k+2j}\, |I|)^p} \int_{2^j I}\int_{2^{k+j} I\setminus 2^j I}.$$

The first term is already bounded by $\|f\|^2_{Q_p(\partial\Delta)}$. For the second, noting that $|e^{i\theta}-e^{i\varphi}| \geqslant 2^{k+j-1}\, |I|$ and then using a fact in [17] we get the bound

$$\sum_j \sum_{k\geqslant 0} \frac{1}{(2^{(k+j)}\, |I|)^2} \frac{1}{2^{pj}} \int_{2^j I}\int_{2^{k+j} I} |f(e^{i\theta})-f(e^{i\varphi})|^2\, d\theta\, d\varphi$$

$$\lesssim \left(\sum_j \frac{1}{2^{pj}}\right) \cdot \|f\|^2_{\mathrm{BMO}(\partial\Delta)}.$$

In the case that $d\mu_p$ is a p-vanishing Carleson measure or $f\in Q_{p,0}(\partial\Delta)$, the argument only requires minor modifications and the proof is omitted here.

Now, let's turn to (i) $\Leftrightarrow$ (iii), which is equivalent to saying that a complex sequence $\{a_k\}$ is the Fourier coefficients of a function in $Q_p(\partial\Delta)$ ($Q_{p,0}(\partial\Delta)$) if and only if $\|\sigma_n\|_{Q_p(\partial\Delta)} = O(1)$ ($o(1)$) as $n\to\infty$.

On the one hand, suppose that $f\in Q_p(\partial\Delta)$ with

$$a_k = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\, e^{-ik\theta}\, d\theta, \qquad k=0, \pm 1, \pm 2, ...$$

So

$$\sigma_n(e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i(\theta+\varphi)})\, K_n(e^{i\varphi})\, d\varphi,$$

where

$$K_n(e^{i\varphi}) = \sum_{-n}^{n} \left(1-\frac{|k|}{n+1}\right) e^{ik\varphi} \geqslant 0$$

is the Fejer kernel. The Minkowski inequality applied to σ_n yields

$$\|\sigma_n\|_{Q_p(\partial\Delta)} \leqslant \|f\|_{Q_p(\partial\Delta)}.$$

On the other hand, let $\|\sigma_n\|_{Q_p(\partial\Delta)} = O(1)$. Since $Q_p(\partial\Delta) \subset \mathrm{BMO}(\partial\Delta)$,

$$\sup_n \|\sigma_n\|_{\mathrm{BMO}(\partial\Delta)} < \infty$$

and hence for each n, σ_n determines a bounded linear functional on H^1_R, the real part of the Hardy space H^1, and so $\{\sigma_n\}$ can be regarded as a sequence in a weakly compact subset of $\mathrm{BMO}(\partial\Delta) \cong (H^1_R)^*$. Consequently, there is a $\sigma \in \mathrm{BMO}(\partial\Delta)$ which is a weak cluster point of $\{\sigma_n\}$ and thus, if $\varepsilon > 0$ and $g \in H^1_R$, then for infinitely many n, one has

$$\left| \int_0^{2\pi} \sigma_n(e^{i\theta})\, g(e^{i\theta})\, d\theta - \int_0^{2\pi} \sigma(e^{i\theta})\, g(e^{i\theta})\, d\theta \right| < \varepsilon.$$

In particular, from $g(e^{i\theta}) = \cos k\theta$, $\sin k\theta$ respectively, it follows that

$$\left| 2\pi \left(1 - \frac{|k|}{n+1}\right) a_k - \int_0^{2\pi} \sigma(e^{i\theta})\, e^{-ik\theta}\, d\theta \right| < \varepsilon$$

and then that $\{a_k\}$ are the Fourier coefficients of $\sigma \in \mathrm{BMO}(\partial\Delta)$. Next, we will further prove that $\sigma \in Q_p(\partial\Delta)$.

Again, using the assumption: $\|\sigma_n\|_{Q_p(\partial\Delta)} = O(1)$ we find a subsequence $\{\sigma_{n_k}\}$ which converges to σ^* in $L^2(\partial\Delta)$. Moreover, applying Fatou's Lemma to $\|\sigma_{n_k}\|_{Q_p(\partial\Delta)} = O(1)$ we have $\sigma^* \in Q_p(\partial\Delta)$. Now for any $g \in L^2(\partial\Delta)$, one has

$$\left| \int_0^{2\pi} [\sigma(e^{i\theta}) - \sigma^*(e^{i\theta})]\, g(e^{i\theta})\, d\theta \right| \leqslant \left| \int_0^{2\pi} [\sigma(e^{i\theta}) - \sigma_{n_k}(e^{i\theta})]\, g(e^{i\theta})\, d\theta \right| + \left| \int_0^{2\pi} [\sigma_{n_k}(e^{i\theta}) - \sigma^*(e^{i\theta})]\, g(e^{i\theta})\, d\theta \right| \to 0 \quad \text{as} \quad k \to \infty,$$

and so $\sigma = \sigma^* = f$ almost everywhere and then $f \in Q_p(\partial\Delta)$. The argument in the space $Q_{p,0}(\partial\Delta)$ is completely similar. ∎

Remark. Let $0 < p < 1$. Let f be a C^∞ function in a neighbourhood of the closed unit disk. One has

$$\|f\|^2_{Q_p(\partial\Delta)} \lesssim \| |\nabla f(z)|^2 (1 - |z|)^p\, dm(z) \|_p.$$

Following the proof of (ii) $\Rightarrow$ (i) in Theorem 2.1, one can show that it is sufficient to establish the following estimate

$$\int_{\Delta} |\nabla(f \circ \varphi_w)(z)|^2 (1-|z|^2)^p \, dm(z)$$
$$\geqslant C \int_{\partial\Delta}\int_{\partial\Delta} \frac{|f\circ\varphi_w(e^{i\theta}) - f\circ\varphi_w(e^{i\varphi})|^2}{|e^{i\theta}-e^{i\varphi}|^{2-p}}\, d\theta\, d\varphi, \qquad w \in \Delta$$

(where C is an absolute constant).

Writting $g = f \circ \varphi_w$, it sufficies to verify

$$\int_{\Delta} |\nabla g(z)|^2 (1-|z|^2)^p \, dm(z) \geqslant C \int_o^{1/2} \frac{1}{h^{2-p}} \int_0^{2\pi} |g(e^{i(\varphi+h)}) - g(e^{i\varphi})|^2 \, d\varphi\, dh.$$

Take $r = 1-h$. We have

$$\begin{aligned} |g(e^{i(\varphi+h)}) - g(e^{i\varphi})| &\leqslant |g(e^{i(\varphi+h)}) - g(re^{i(\varphi+h)})| + |g(re^{i(\varphi+h)}) - g(re^{i\varphi})| \\ &\quad + |g(re^{i\varphi}) - g(e^{i\varphi})| \\ &\leqslant \int_r^1 |\nabla g(te^{i(\varphi+h)})|\, dt + \int_0^h |\nabla g(re^{i(s+\varphi)})|\, ds \\ &\quad + \int_r^1 |\nabla g(te^{i\varphi})|\, dt. \end{aligned}$$

Apply Minkowski integral inequality [18, p. 271], to get

$$\begin{aligned} &\left(\int_0^{2\pi} |g(e^{i(\varphi+h)}) - g(e^{i\varphi})|^2 \, d\varphi\right)^{1/2} \\ &\quad \leqslant 2\int_r^1 \left(\int_0^{2\pi} |\nabla g(te^{i\varphi})|^2 \, d\varphi\right)^{1/2} dt + h\left(\int_0^{2\pi} |\nabla g(re^{i\varphi})|^2 \, d\varphi\right)^{1/2} \\ &\quad = (I) + (II). \end{aligned}$$

Changing to planar coordinates one gets

$$\begin{aligned} \int_0^{1/2} \frac{1}{h^{2-p}} (II)^2 \, dh &= \int_0^{1/2} h^p \int_o^{2\pi} |\nabla g(re^{i\varphi})|^2 \, d\varphi\, dh \\ &\leqslant \int_{\Delta} |\nabla g(z)|^2 (1-|z|)^p \, dm(z). \end{aligned}$$

For the term (I), put $x=1-t$ and apply Hardy's inequality ([18, p. 272]) to obtain

$$\begin{aligned}\int_0^{1/2}\frac{1}{h^{2-p}}(I)^2\,dh&=4\int_0^{1/2}\frac{1}{h^{2-p}}\left[\int_r^1\left(\int_o^{2\pi}|\nabla g(te^{i\varphi})|^2\,d\varphi\right)^{1/2}dt\right]^2dh\\&=4\int_0^{1/2}\frac{1}{h^{2-p}}\left[\int_o^h\left(\int_0^{2\pi}|\nabla g(1-x)\,e^{i\varphi})|^2\,d\varphi\right)^{1/2}dx\right]^2dh\\&\lesssim\int_o^{1/2}h^p\int_o^{2\pi}|\nabla g((1-h)\,e^{i\varphi})|^2\,d\varphi\,dh\\&\leqslant\int_\Delta|\nabla g(z)|^2\,(1-|z|)^p\,dm(z).\end{aligned}$$

This gives the Remark.

Also we remark that (i) $\Leftrightarrow$ (iii) still holds for BMO($\partial\Delta$) (VMO($\partial\Delta$)).

Theorem 2.1 and the conformally invariant characterization of p-Carleson measures give the following result.

Corollary 2.2. *Let $0<p<1$ and $f\in L^2(\partial\Delta)$. Then the quantities*

$$\|f\|^2_{Q_p(\partial\Delta)}=\sup_{I\subset\partial\Delta}\frac{1}{|I|^p}\int_I\int_I\frac{|f(e^{i\theta})-f(e^{i\varphi})|^2}{|e^{i\theta}-e^{i\varphi}|^{2-p}}\,d\theta\,d\varphi$$

$$A=\sup_{I\subset\partial\Delta}\frac{1}{|I|^p}\int_{S(I)}|\nabla\hat f(w)|^2\,(1-|w|^2)^p\,dm(w)$$

$$B=\sup_{z\in\Delta}\int_\Delta\left(\frac{(1-|z|^2)\,(1-|w|^2)}{|1-\bar wz|^2}\right)^p|\nabla\hat f(w)|^2\,dm(w)$$

are comparable, that is, there exist constants C_1, C_2, $C_3>0$ independent of f such that

$$\|f\|^2_{Q_p(\partial\Delta)}\leqslant C_1A\leqslant C_2B\leqslant C_3\,\|f\|^2_{Q_p(\partial\Delta)}.$$

Recall that VMO($\partial\Delta$) is the closure of trigonometric polynomials in BMO($\partial\Delta$). An analogous result holds for $Q_{p,0}(\partial\Delta)$.

Corollary 2.3. *Let $p\in(0,1)$ and $f\in Q_p(\partial\Delta)$. Then the following conditions are equivalent*:

(i) $f \in Q_{p,0}(\partial \Delta)$.

(ii) $\lim_{t \to 0} \|T_t f - f\|_{Q_p(\partial \Delta)} = 0$, *where* $T_t f(e^{i\theta}) = f(e^{i(\theta - t)})$.

(iii) $\lim_{r \to 1} \|f_r - f\|_{Q_p(\partial \Delta)} = 0$, *where* $f_r(e^{i\theta}) = \hat{f}(re^{i\theta})$.

Proof. (i) $\Rightarrow$ (ii). If $f \in Q_{p,0}(\partial \Delta)$ and $F_t = T_t f - f$ then for any $\varepsilon > 0$ there is a $\delta > 0$ so that for all subarcs $I \subset \partial \Delta$ with $|I| < \delta$, one has

$$\frac{1}{|I|^p} \int_I \int_I \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^{2-p}} \, d\theta \, d\varphi < \varepsilon,$$

and thus for any $I \subset \partial \Delta$ with $|I| < \delta$, one has

$$\begin{aligned}
&\frac{1}{|I|^p} \int_I \int_I \frac{|F_t(e^{i\theta}) - F_t(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^{2-p}} \, d\theta \, d\varphi \\
&\quad \lesssim \frac{1}{|I|^p} \int_I \int_I \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^{2-p}} \, d\theta \, d\varphi \\
&\qquad + \frac{1}{|I|^p} \int_I \int_I \frac{|f(e^{i(\theta - t)}) - f(e^{i(\varphi - t)})|^2}{|e^{i(\theta - t)}) - e^{i(\varphi - t)}|^{2-p}} \, d\theta \, d\varphi \lesssim \varepsilon.
\end{aligned}$$

However, for any arc $I \subset \partial \Delta$ with $|I| \geqslant \delta$, applying Theorem 2.1 one gets

$$\begin{aligned}
&\frac{1}{|I|^p} \int_I \int_I \frac{|F_t(e^{i\theta}) - F_t(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^{2-p}} \, d\theta \, d\varphi \\
&\quad \leqslant \frac{1}{\delta^p} \int_0^{2\pi} \int_0^{2\pi} \frac{|F_t(e^{i\theta}) - F_t(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^{2-p}} \, d\theta \, d\varphi \\
&\quad \lesssim \frac{1}{\delta^p} \int_\Delta |\nabla \hat{F}_t(z)|^2 \, (1 - |z|^2)^p \, dm(z) \\
&\quad \lesssim \frac{1}{\delta^p} \int_\Delta |\nabla(\hat{f}(ze^{it}) - \hat{f}(z))|^2 \, (1 - |z|^2)^p \, dm(z) \to 0 \qquad \text{as} \quad t \to 0.
\end{aligned}$$

In other words, (ii) follows.

(ii) $\Rightarrow$ (iii). Assuming that $f \in Q_p(\partial \Delta)$ satisfies (ii), we use Minkowski inequality in

$$f(e^{i\theta}) - f_r(e^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} [f(e^{i\theta}) - f(e^{i(\theta - \varphi)})] \, P_r(\varphi) \, d\varphi$$

to get that for any small $\varepsilon > 0$,

$$\|f - f_r\|_{Q_p(\partial\Delta)} \leqslant \frac{1}{2\pi}\int_0^{2\pi} \|f - T_\varphi f\|_{Q_p(\partial\Delta)} P_r(\varphi)\, d\varphi$$

$$\lesssim \int_{|\varphi| < \varepsilon} \|f - T_\varphi f\|_{Q_p(\partial\Delta)} P_r(\varphi)\, d\varphi$$

$$+ \int_{|\varphi| \geqslant \varepsilon} \|f\|_{Q_p(\partial\Delta)} P_r(\varphi)\, d\varphi,$$

which gives (iii)

(iii) $\Rightarrow$ (i). This implication is obvious since $Q_{p,0}(\partial\Delta)$ is closed in $Q_p(\partial\Delta)$. ∎

2.2. *Inner Functions in Q_p*

This paragraph is designed to discuss inner functions in $Q_p(Q_{p,0})$ for $p \in (0, 1)$. An inner function is a bounded analytic function on Δ whose radial limits have modulus 1 almost everywhere on $\partial\Delta$. Inner functions in Q_p, $p \in (0, 1)$ are described in [10] as the Blaschke products whose zeros $\{z_n\}$ have the property that

$$\sum_n (1 - |z_n|^2)^p\, \delta_n$$

is a p-Carleson measure, where δ_n is the Dirac measure at z_n. Moreover the only inner functions in $Q_{p,0} \subset \text{VMOA}$, $p \in (0, 1)$ are the finite Blaschke products. The later fact follows also from the equality

$$\frac{1}{2\pi}\int_0^{2\pi} |B(e^{i\theta}) - B(z_I)|^2\, P_{z_I}(\theta)\, d\theta = 1 - |B(z_I)|^2,$$

which holds for any inner function B and any subarc $I \subset \partial\Delta$. Besides, all inner functions B have small mean variation on many arcs I, namely, those such that $|B(z_I)|$ is close to 1. The next property presents an analogue of this phenomenon for Q_p, $p \in (0, 1)$.

THEOREM 2.4. *Let $p \in (0, 1)$ and $B \in Q_p$ be the Blaschke product with zeros $\{z_n\}$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $r > 1 - \delta$, one has*

$$\frac{1}{|I|^p}\int_{S(I)} |B'(z)|^2 (1 - |z|^2)^p\, \chi_r(z)\, dm(z) < \varepsilon,$$

where $S(I)$ *is the Carleson box based on the subarc* $I \subset \partial \Delta$ *and* χ_r *is the characteristic function of the set* $\{z \in \Delta : \inf_n \rho(z, z_n) \geqslant r\}$.

Proof. Let us first check the following fact.

Claim. Let B be a Blaschke product in Q_p, $p \in (0, 1)$. If $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\log |B(z)|^{-2} < \varepsilon \qquad \text{if} \quad \inf_n \rho(z, z_n) > 1 - \delta.$$

An elementary estimate gives that if $\inf_n \rho(z, z_n) \geqslant 1/2$ then there is a constant $C > 0$ such that

$$\sum_n \frac{(1-|z|^2)(1-|z_n|^2)}{|1-\bar{z}_n z|^2} \leqslant \log |B(z)|^{-2} \leqslant C \sum_n \frac{(1-|z|^2)(1-|z_n|^2)}{|1-\bar{z}_n z|^2}. \tag{2.1}$$

Then the claim follows as long as we prove that

$$\sum_n \frac{(1-|z|^2)(1-|z_n|^2)}{|1-\bar{z}_n z|^2} \to 0 \qquad \text{as} \quad r = \inf_n \rho(z, z_n) \to 1. \tag{2.2}$$

For this, it suffices to show that for any $\varepsilon > 0$ there exists $j(r) \to \infty$ (when $r \to 1$) so that

$$\frac{1}{|2^j I_z|} \sum_{z_n \in S(2^j I_z)} (1-|z_n|^2) < \varepsilon, \qquad j = 0, 1, ..., j(r). \tag{2.3}$$

Actually, since

$$\sum_n (1-|z_n|)\, \delta_{z_n}$$

is a 1-Carleson measure, (2.3) gives (2.2) after one breaks the sum into dyadic blocks.

Observe that $z_n \in S(2^j I_z)$ and $\rho(z, z_n) \geqslant r$, give

$$\frac{1-|z_n|^2}{4 \cdot 2^{2j}(1-|z|^2)} \leqslant \frac{(1-|z|^2)(1-|z_n|^2)}{|1-\bar{z}_n z|^2} \leqslant 1 - r^2,$$

that is,

$$1-|z_n|^2 \leqslant 4 \cdot 2^{2j}(1-r^2)(1-|z|^2).$$

Now fix $j(r)$ so that $\varepsilon(r) = 4 \cdot 2^{j(r)}(1-r^2) \to 0$ as $r \to 1$. Since $B \in Q_p$,

$$\sum_n (1-|z_n|^2)^p\, \delta_{z_n}$$

is a p-Carleson measure and consequently, for $0 \leqslant j \leqslant j(r)$, we have

$$\sum_{z_n \in S(2^j I_z)} (1-|z_n|^2) \leqslant [|2^j I_z|\, \varepsilon(r)]^{1-p} \sum_{z_n \in S(2^j I_z)} (1-|z_n|^2)^p$$
$$\lesssim [\varepsilon(r)]^{1-p}\, |2^j I_z|$$

which gives (2.3) and proves the claim.

A calculation with logarithmic derivatives gives

$$\frac{B'(z)}{B(z)} = \sum_n \frac{1-|z_n|^2}{(z-z_n)(1-\bar{z}_n z)}.$$

Hence

$$|B'(z)| \leqslant \sum_n \frac{1-|z_n|^2}{|1-\bar{z}_n z|^2}. \tag{2.4}$$

Using the claim we deduce that for any $\varepsilon > 0$, there is an r in $(0, 1)$ such that $1-|B(z)|^2 \leqslant \varepsilon$ if $\inf_n \rho(z, z_n) \geqslant r$. Hence, (2.4) gives

$$\int_{S(I)} |B'(z)|^2\, (1-|z|^2)^p\, \chi_r(z)\, dm(z)$$
$$\leqslant \int_{S(I)} (1-|B(z)|^2)\, |B'(z)|\, (1-|z|^2)^{p-1}\, \chi_r(z)\, dm(z)$$
$$\leqslant \varepsilon \sum_n (1-|z_n|^2) \int_{S(I)} \frac{(1-|z|)^{p-1}}{|1-\bar{z}_n z|^2}\, dm(z)$$
$$\leqslant \varepsilon \Bigg[\sum_{z_n \in S(2I)} (1-|z_n|^2) \int_{\Delta} \frac{(1-|z|^2)^{p-1}}{|1-\bar{z}_n z|^2}\, dm(z)$$
$$+ \Bigg(\sum_{z_n \in \Delta \setminus S(2I)} (1-|z_n|^2) \int_{S(I)} \frac{(1-|z|^2)^{p-1}}{|1-\bar{z}_n z|^2}\, dm(z) \Bigg) \Bigg]$$
$$\lesssim \varepsilon \Bigg[\sum_{z_n \in S(2I)} (1-|z_n|^2)^p + |I|^p \sum_n \frac{(1-|z_n|^2)(1-|z_I|^2)}{|1-\bar{z}_n z_I|^2} \Bigg] \lesssim \varepsilon\, |I|^p.$$

In the last line one may use the following formula. Let

$$I(\gamma) = \int_{\Delta} \frac{(1-|z|^2)^{\gamma}}{|1-\bar{w} z|^2}\, dm(z), \qquad \gamma > -1. \tag{2.5}$$

Then $I(\gamma) \lesssim 1$ if $\gamma > 0$, $I(0) \simeq \log(1-|w|)^{-1}$, $I(\gamma) \simeq (1-|w|)^{\gamma}$ if $-1 < \gamma < 0$.

Since $\varepsilon > 0$ is arbitrary, this completes the proof. ∎

The above Theorem 2.4 has an interesting consequence which is analogous to Lemma 3.3 in [19] and will be used to solve the interpolation problem in $Q_{p,0} \cap H^\infty$.

COROLLARY 2.5. *Let* $p \in (0, 1)$ *and* $B \in Q_p$ *be a Blaschke product with zeros* $\{z_n\}$. *If* $F \in Q_{p,0} \cap H^\infty$ *satisfies* $\lim_{n \to \infty} F(z_n) = 0$ *then* $BF \in Q_{p,0} \cap H^\infty$.

Proof. By the previous results, Theorems 2.1 and 2.4, we only have to show that

$$\frac{1}{|I|^p} \int_{S(I)} |F(z)|^2 \, |B'(z)|^2 \, (1 - |z|^2)^p \, [1 - \chi_r(z)] \, dm(z) \to 0 \qquad \text{as} \quad |I| \to 0,$$

where χ_r is still the function given in Theorem 2.4. Because $B \in Q_p$, the above integral can be bounded by

$$C \sup \left\{ |F(z)|^2 \colon z \in S(I), \inf_n \rho(z, z_n) \leqslant r \right\}$$

which tends to zero when $|I| \to 0$, because $\lim_{n \to \infty} F(z_n) = 0$ and $F \in Q_{p,0} \subseteq B_0$. Actually the little Bloch space consists on those analytic functions having vanishing oscillation on pseudohyperbolic disks with fixed radius [6]. ∎

Remark. Corollary 2.5 still holds for $p = 1$ if the sequence $\{z_n\}$ is 1-thin. Actually, since

$$\inf_{z \in \Delta} \left\{ |B(z)| \colon \inf_n \rho(z, z_n) \geqslant r \right\} \to 1 \qquad \text{as} \quad r \to 1,$$

one has

$$\sup_{I \subset \partial \Delta} \frac{1}{|I|} \int_{S(I)} |B'(z)|^2 \, (1 - |z|^2) \, \chi_r(z) \, dm(z) \to 0 \qquad \text{as} \quad r \to 1,$$

and one can mimic the last proof.

3. CORONA PROPERTY AND FEFFERMAN–STEIN DECOMPOSITION

In this section we use p-Carleson measures to solve $\bar{\partial}$-problems in $Q_p(\partial \Delta) \cap L^\infty(\partial \Delta)$. This leads to the Corona theorem in $Q_p \cap H^\infty$ and to the Fefferman–Stein decomposition in $Q_p(\partial \Delta)$.

3.1. $\bar{\partial}$-Problem in $Q_p(\partial\Delta)\cap L^\infty(\partial\Delta)(Q_{p,0}(\partial\Delta)\cap L^\infty(\partial\Delta))$

Given a Carleson measure μ on the unit disk, it is well known (see Chapter III of [11]) that the $\bar{\partial}$-problem,

$$\bar{\partial}F=\mu,$$

has a solution F, in the sense of distributions, satisfying

$$\|F\|_{L^\infty(\partial\Delta)}\leqslant C\,\|\mu\|_1.$$

P. Jones further found in [12] that such a solution F can be given by an extremely simple and flexible formula,

$$F(z)=\int_\Delta K\left(\frac{\mu}{\|\mu\|_1},z,\xi\right)d\mu(\xi), \tag{3.1}$$

where

$$K\left(\frac{\mu}{\|\mu\|_1},z,\xi\right)=\frac{2i}{\pi}\cdot\frac{1-|\xi|^2}{(1-\bar{\xi}z)(z-\xi)}\exp\left[\int_{|w|\geqslant|\xi|}\left(\frac{1+\bar{w}\xi}{1-\bar{w}\xi}-\frac{1+\bar{w}z}{1-\bar{w}z}\right)\frac{d\mu(w)}{\|\mu\|_1}\right].$$

The estimate

$$\int_\Delta\left|K\left(\frac{\mu}{\|\mu\|_1},e^{i\theta},\xi\right)\right|d\mu(\xi)\leqslant C_1\,\|\mu\|_1.$$

shows that $F\in L^\infty(\partial\Delta)$. As a consequence, if $|g(z)|\,dm(z)$ is a 1-Carleson measure then $\bar{\partial}F=g$, has a solution $F\in L^\infty(\partial\Delta)$. We want to find an analogous result for $Q_p(\partial\Delta)\cap L^\infty(\partial\Delta)$ $(Q_{p,0}(\partial\Delta)\cap L^\infty(\partial\Delta))$; that is to say, under which conditions on g does the equation $\bar{\partial}F=g$ have a solution $F\in Q_p(\partial\Delta)\cap L^\infty(\partial\Delta)$ $(Q_{p,0}(\partial\Delta)\cap L^\infty(\partial\Delta))$? It is surely reasonable to observe in advance that $L^\infty(\partial\Delta)\not\subset Q_p(\partial\Delta)$ $(C(\partial\Delta)\not\subset Q_{p,0}(\partial\Delta))$ for $p\in(0,1)$, in fact, for example,

$$f(e^{i\theta})=\sum_{n=0}^\infty 2^{n(p-1)/2}e^{i2^n\theta}$$

is in $C(\partial\Delta)\backslash Q_p(\partial\Delta)$. See [3] where the power series with Hadamard gaps which are in $Q_p(\partial\Delta)$ are characterized.

Our answer to the above question is:

THEOREM 3.1. *Let* $p \in (0,1)$. *If* $d\lambda(z) = |g(z)|^2 (1-|z|^2)^p\, dm(z)$ *is a p-Carleson measure (p-vanishing Carleson measure) then* $\bar{\partial} F = g$ *has a solution* $F \in Q_p(\partial\Delta) \cap L^\infty(\partial\Delta)$ $(Q_{p,0}(\partial\Delta) \cap L^\infty(\partial\Delta))$. *Actually,*

$$\|F\|_{Q_p(\partial\Delta)} + \|F\|_{L^\infty(\partial\Delta)} \leqslant C\, \|\lambda\|_p^{1/2},$$

where C is an absolute constant.

Proof. We only give a proof for $Q_p(\partial\Delta) \cap L^\infty(\partial\Delta)$, because the proof for $Q_{p,0}(\partial\Delta) \cap L^\infty(\partial\Delta)$ is similar. Let $d\mu(z) = |g(z)|\, dm(z)$; then μ is a 1-Carleson measure. Actually,

$$\begin{aligned}\mu(S(I)) &= \int_{S(I)} d\mu(z)\\ &\leqslant \left[\int_{S(I)} d\lambda(z)\right]^{1/2} \left[\int_{S(I)} (1-|z|^2)^{-p}\, dm(z)\right]^{1/2} \lesssim \|\lambda\|_p^{1/2}\, |I|.\end{aligned}$$

Thus the function F given by (3.1) is in $L^\infty(\partial\Delta)$ and $\bar{\partial} F = g$. Next, our hope is to show that $F \in Q_p(\partial\Delta)$. For this purpose, consider a new function G on Δ which has the same boundary values on $\partial\Delta$ as zF,

$$\begin{aligned}G(z) = \frac{2i}{\pi} \int_\Delta \frac{1-|\xi|^2}{|1-\bar{\xi}z|^2} \exp\Bigg[\int_{|w| \geqslant |\xi|} &\left(\frac{1+\bar{w}\xi}{1-\bar{w}\xi} - \frac{1+\bar{w}z}{1-\bar{w}z}\right)\\ &\times \frac{|g(w)|}{\|\mu\|_1}\, dm(w)\Bigg]\, g(\xi)\, dm(\xi).\end{aligned} \tag{3.2}$$

By the Remark after Theorem 2.1, one only has to show that G satisfies

$$\sup_{I \subset \partial\Delta} \frac{1}{|I|^p} \int_{S(I)} |\nabla G(z)|^2 (1-|z|^2)^p\, dm(z) < \infty. \tag{3.3}$$

Without loss of generality, let $g(z) \geqslant 0$ and $\|\mu\|_1 = 1$. Then,

$$\operatorname{Re}\left(\int_{|w| \geqslant |\xi|} \frac{1+\bar{w}\xi}{1-\bar{w}\xi}\, g(w)\, dm(w)\right) \leqslant 2 \int_\Delta \frac{1-|\xi|^2}{|1-\bar{w}\xi|^2}\, g(w)\, dm(w) \leqslant C_2, \tag{3.4}$$

where $C_2 > 0$ is a constant independent of $\xi \in \Delta$. Moreover,

$$\int_\Delta \frac{1-|\xi z|^2}{|1-\bar{\xi}z|^2} \exp\left[-\int_{|w| \geqslant |\xi|} \frac{1-|wz|^2}{|1-\bar{w}z|^2}\, g(w)\, dm(w)\right] g(\xi)\, dm(\xi) \leqslant 1 \tag{3.5}$$

(see the proof of Lemma 2.1 in [12]). Using (3.4) and (3.5) one can show that

$$|\nabla G(z)| \lesssim \int_{\Delta} \frac{g(w)}{|1-\bar{w}z|^2}\, dm(w)$$

and hence that

$$\begin{aligned}
&\int_{S(I)} |\nabla G(z)|^2 (1-|z|^2)^p \, dm(z) \\
&\quad \lesssim \int_{S(I)} (1-|z|^2)^p \left[\left(\int_{S(2I)} + \int_{\Delta \setminus S(2I)} \right) \frac{g(w)}{|1-\bar{w}z|^2}\, dm(w) \right]^2 dm(z) \\
&\quad \lesssim \int_{S(I)} (1-|z|^2)^p \left[\int_{S(2I)} \frac{g(w)}{|1-\bar{w}z|^2}\, dm(w) \right]^2 dm(z) \\
&\qquad + \int_{S(I)} (1-|z|^2)^p \left[\int_{\Delta \setminus S(2I)} \frac{g(w)}{|1-\bar{w}z|^2}\, dm(w) \right]^2 dm(z) = (A) + (B).
\end{aligned}$$

For (A), we use Schur Lemma [25, p. 42]. Indeed, consider

$$k(z, w) = \frac{(1-|z|^2)^{p/2} (1-|w|^2)^{-p/2}}{|1-\bar{w}z|^2}$$

and the integral operator T induced by the kernel $k(z, w)$,

$$(Tf)(z) = \int_{\Delta} f(w)\, k(z, w)\, dm(w).$$

Taking $\alpha \in (-1, -p/2)$ and applying the formula (2.5), one gets

$$\int_{\Delta} k(z, w)(1-|w|^2)^{\alpha}\, dm(w) \lesssim (1-|z|^2)^{\alpha}$$

and

$$\int_{\Delta} k(z, w)(1-|z|^2)^{\alpha}\, dm(z) \lesssim (1-|w|^2)^{\alpha}.$$

Therefore the operator T is bounded from $L^2(\Delta)$ to $L^2(\Delta)$. Once letting

$$f(w) = (1-|w|^2)^{p/2}\, g(w)\, \chi_{S(2I)}(w),$$

where $\chi_{S(2I)}$ denotes the characteristic function of $S(2I)$, we have

$$(A)=\int_{\Delta}\left[\int_{\Delta} f(w)\,k(z,w)\,dm(w)\right]^2 dm(z) \lesssim \int_{\Delta} |f(z)|^2\,dm(z)$$

$$=\int_{S(2I)} |g(z)|^2\,(1-|z|^2)^p\,dm(z) \lesssim |I|^p.$$

We estimate (B) using dyadic blocks,

$$(B) \lesssim \int_{S(I)} (1-|z|^2)^p \left[\sum_{n=1}^{\infty} \int_{S(2^{n+1}I)\setminus S(2^n I)} \frac{g(w)}{|1-\bar{w}z|^2}\,dm(w)\right]^2 dm(z)$$

$$\lesssim \int_{S(I)} (1-|z|^2)^p \left[\sum_{n=1}^{\infty} \frac{\mu(S(2^{n+1}I))}{(2^n\,|I|)^2}\right]^2 dm(z) \lesssim |I|^p.$$

These estimates on (A) and (B) imply (3.3). ▮

3.2. *Corona Theorem for* $Q_p \cap H^\infty (Q_{p,0} \cap H^\infty)$

The first application of Theorem 3.1 is the Corona theorem for $Q_p \cap H^\infty$ $(Q_{p,0} \cap H^\infty)$.

Proof of Theorem 1.1. From a normal family argument, we can assume that the given functions $f_1 \cdots f_n$ are analytic on a neighbourhood of the closed unit disk and we are forced to find functions $g_1, ..., g_n \in Q_p \cap H^\infty(Q_{p,0} \cap H^\infty)$ satisfying

$$\sum_{k=1}^{n} f_k g_k \equiv 1.$$

It is clear that

$$h_j(z) = \overline{f_j(z)} \Big/ \sum_{k=1}^{n} |f_k(z)|^2$$

are nonanalytic functions making

$$\sum_{k=1}^{n} f_k h_k \equiv 1.$$

As in the case of H^∞ (see Chapter VIII of [11]), to replace h_k by functions in $Q_p \cap H^\infty$ one needs to solve the equations below,

$$\bar{\partial} b_{j,k} = h_j \bar{\partial} h_k, \qquad 1 \leqslant j, \quad k \leqslant n$$

in $Q_p(\partial\Delta)\cap L^\infty(\partial\Delta)$ $(Q_{p,0}(\partial\Delta)\cap L^\infty(\partial\Delta))$. It is sufficient to deal with an equation $\bar\partial b = h$ where $h = h_j \bar\partial h_k$. An easy calculation shows

$$|h(z)| \lesssim \sum_{j=1}^{n} |f_j'(z)|.$$

So by Theorem 2.1, $|h(z)|^2 (1-|z|^2)^p\, dm(z)$ is a p-Carleson measure (p-vanishing Carleson measure). Using Theorem 3.1 there is a solution $b \in Q_p(\partial\Delta)\cap L^\infty(\partial\Delta)$ $(Q_{p,0}(\partial\Delta)\cap L^\infty(\partial\Delta))$, $\bar\partial b = h$ such that

$$\|b\|_{Q_p(\partial\Delta)} + \|b\|_{L^\infty(\partial\Delta)} \leqslant C \sum_{j=1}^{n} \|f_j\|_{Q_p}. \quad ∎$$

Remark. As mentioned in the introduction, the Corona Theorem for $Q_{1,0}\cap H^\infty = \mathrm{VMO}\cap H^\infty$ was proved in [19]. It also holds for $Q_{p,0}\cap H^\infty$, $p>1$, that is, for the algebra of bounded analytic functions in the little Bloch space. Actually, given $f_1, ..., f_n \in B_0 \cap H^\infty$ satisfying

$$\inf_{z\in\Delta} (|f_1(z)| + \cdots + |f_n(z)|) = \delta > 0,$$

one can find $\varepsilon = \varepsilon(\delta) > 0$ and $\Psi_j \in C^\infty(\Delta)$, $j = 1, ..., n$ such that

$$\Psi_j(z) = 1 \qquad \text{if} \quad |f_j(z)| > \delta/n,$$

$$\Psi_j(z) = 0 \qquad \text{if} \quad |f_j(z)| < \varepsilon,$$

and $|\nabla \Psi_j(z)|\, dm(z)$ is a Carleson measure (see [11, p. 342]). Moreover since $f_j \in B_0$, one can also assume

$$(1-|z|)\, |\nabla \Psi_j(z)| \to 0 \qquad \text{as} \quad |z| \to 1.$$

Set

$$\varphi_j = \frac{\Psi_j}{f_j \sum \Psi_k}$$

and one has

$$f_1\varphi_1 + \cdots + f_n\varphi_n \equiv 1.$$

Observe that φ_j are bounded, $|\nabla \varphi_j(z)|\, dm(z)$ is a Carleson measure and

$$(1-|z|)\, |\nabla \varphi_j(z)| \to 0 \qquad \text{as} \quad |z| \to 1.$$

To replace φ_j by functions in $H^\infty \cap B_0$ one has to solve the equations

$$\frac{\partial b_{j,k}}{\partial \bar{z}} = \varphi_j \frac{\partial \varphi_k}{\partial z}, \qquad 1 \leqslant j \leqslant n$$

$$\|b_{j,k}\|_{L^\infty(\partial \Delta)} < +\infty$$

$$\sup_{|z| \geqslant r} \{|b_{j,k}(z) - b_{j,k}(w)| : \rho(z, w) \leqslant 1/2\} \to 0 \qquad \text{as} \quad r \to 1.$$

For this one can consider the P. Jones solution (3.1) and observe

$$\int_{\{\xi:\, \rho(\xi, z) \leqslant 1/2\}} \frac{1 - |\xi|^2}{|\xi - z|\, |1 - \bar{\xi} z|} K(\xi, z) \left| \varphi_j(\xi) \frac{\partial \varphi_k}{\partial \bar{z}}(\xi) \right| dm(\xi) \to 0 \qquad \text{as } |z| \to 1$$

$$\int_\Delta \frac{(1 - |z|^2)(1 - |\xi|^2)}{|1 - \bar{\xi} z|^3} \left| \varphi_j(\xi) \frac{\partial \varphi_k}{\partial \bar{z}}(\xi) \right| dm(\xi) \to 0 \qquad \text{as } |z| \to 1.$$

3.3. *Fefferman–Stein Decomposition for* $Q_p(\partial \Delta)(Q_{p,0}(\partial \Delta))$

The second application of Theorem 3.1 is a decomposition of $Q_p(\partial \Delta)$ $(Q_{p,0}(\partial \Delta))$ similar to the Fefferman–Stein decomposition of $\mathrm{BMO}(\partial \Delta)$ $(\mathrm{VMO}(\partial \Delta))$.

Proof of Theorem 1.2. Denoting by $\tilde{X}$ the conjugate space of a given space X, we will show that

(a) $Q_p(\partial \Delta) = Q_p(\partial \Delta) \cap L^\infty(\partial \Delta) + [Q_p(\partial \Delta) \cap L^\infty(\partial \Delta)]^\sim$

(b) $Q_{p,0}(\partial \Delta) = Q_{p,0}(\partial \Delta) \cap C(\partial \Delta) + [Q_{p,0}(\partial \Delta) \cap C(\partial \Delta)]^\sim$

First of all, we show (a). On the one hand, if $f = u + \tilde{v}$, u, $v \in Q_p(\partial \Delta) \cap L^\infty(\partial \Delta)$, then from Corollary 3.2 in [10] it follows that $\tilde{v} \in Q_p(\partial \Delta)$ and hence $f \in Q_p(\partial \Delta)$.

On the other hand, suppose that $f \in Q_p(\partial \Delta)$, is real-valued and $\hat{f}(0) = 0$. We find immediately that $F = f + i\tilde{f} \in Q_p(\partial \Delta)$ and its Poisson extension $\hat{F} \in Q_p$. From Theorem 2.1, one has that $|\nabla \hat{F}(z)|^2 (1 - |z|^2)^p\, dm(z)$, and then $|\bar{\partial} f(z)|^2 (1 - |z|^2)^p\, dm(z)$ are p-Carleson measures. Let $d\mu(z) = \bar{\partial} f(z)\, dm(z)$ and let $f_\mu(z)$ be the function given by Theorem 3.1; then $\bar{\partial} f_\mu = \mu$ and $f_\mu \in Q_p(\partial \Delta) \cap L^\infty(\partial \Delta)$. Hence $g = f - f_\mu$ is analytic and $g \in Q_p$. Put $u = \mathrm{Re}\, f_\mu$, then $f - u = -\widetilde{\mathrm{Im}\, g}$. So $f = u + \tilde{v}$, where $u = \mathrm{Re}\, f_\mu$ and $v = -\mathrm{Im}\, g$ belong to $Q_p(\partial \Delta) \cap L^\infty(\partial \Delta)$. The estimates on u, v in Theorem 1.2 follow from the bound on f_μ in Theorem 3.1.

Next we show (b). This part may be seen as a by-product of (a) and Corollary 2.3. It suffices to check that $Q_{p,0}(\partial \Delta)$ is a subset of the righthand set in (b). To this end, let $f \in Q_{p,0}(\partial \Delta)$ with $\hat{f}(0)=0$. From (a) it follows that there are $g_1, g_2 \in Q_p(\partial \Delta) \cap L^\infty(\partial \Delta)$ satisfying $f=g_1+\tilde{g}_2$ with

$$\|g_j\| = \|g_j\|_{L^\infty(\partial \Delta)} + \|g_j\|_{Q_p(\partial \Delta)} \leqslant C_3 \|f\|_{Q_p(\partial \Delta)}, \qquad j=1,2,$$

where $C_3>0$ is a constant independent of f and g_j. Since $f \in Q_{p,0}(\partial \Delta)$, by Corollary 2.3 it follows that there is an $r \in (0,1)$ satisfying

$$\|f-f_r\|_{Q_p(\partial \Delta)} \leqslant \frac{\|f\|_{Q_p(\partial \Delta)}}{2}.$$

Also, let $f_j^{(1)}=(g_j)_r$, $(j=1,2)$, which obviously are in $Q_{p,0}(\partial \Delta) \cap C(\partial \Delta)$, and then let $f_r = f_1^{(1)} + \widetilde{f_2^{(1)}}$. So

$$\|f-(f_1^{(1)}+\widetilde{f_2^{(1)}})\|_{Q_p(\partial \Delta)} \leqslant \frac{\|f\|_{Q_p(\partial \Delta)}}{2};$$

consequently, $f_1 = f-(f_1^{(1)}+\widetilde{f_2^{(1)}}) = g_1 - f_1^{(1)} + \widetilde{g_2 - f_2^{(1)}} \in Q_{p,0}(\partial \Delta)$ with

$$\|f_1\|_{Q_p(\partial \Delta)} \leqslant \frac{\|f\|_{Q_p(\partial \Delta)}}{2} \qquad \text{and} \qquad \|g_j - f_j^{(1)}\| \leqslant C_3 \|f\|_{Q_p(\partial \Delta)}.$$

Repeating the above argument with f_1 and iterating, we have $f=u+\tilde{v}$ where

$$u = \sum_{k=1}^{\infty} f_1^{(k)} \qquad \text{and} \qquad v = \sum_{k=1}^{\infty} f_2^{(k)}$$

belong to $Q_{p,0}(\partial \Delta) \cap C(\partial \Delta)$. Hence (b) is proved. ∎

Remarks. (i) In the above proof, we use Theorem 3.1 to deduce Theorem 1.2. Conversely, Theorem 1.2 can be used to deduce Theorem 3.1 as well. For instance, assume that

$$Q_{p,0}(\partial \Delta) = Q_{p,0}(\partial \Delta) \cap L^\infty(\partial \Delta) + [Q_{p,0}(\partial \Delta) \cap L^\infty(\partial \Delta)]^\sim.$$

If $d\lambda(z) = |g(z)|^2 (1-|z|^2)^p \, dm(z)$ is a p-vanishing Carleson measure then we wish to show that $\partial F = g$ has a solution F in $Q_{p,0}(\partial \Delta) \cap L^\infty(\partial \Delta)$. For this, let

$$F_1(z) = \frac{1}{\pi} \int_\Delta \frac{g(w)}{z-w} \, dm(w).$$

Then $\bar{\partial} F_1 = g$ and $\partial F_1 = B(g)$ is the Beurling transform of g. We now argue as in [14]. Since $|(1-|z|^2)^p|$ is an A_2-weight for $p \in (0,1)$, $B(f)$ is a bounded linear operator from $L^2(\Delta, (1-|z|^2)^p\, dm(z))$ to itself, [11, Chapter VI], and consequently,

$$\begin{aligned}
&\int_{S(I)} |\partial F_1(z)|^2\, (1-|z|^2)^p\, dm(z)\\
&\quad\lesssim \int_{S(I)} |B(g\chi_{S(2I)}(z)|^2\, (1-|z|^2)^p\, dm(z)\\
&\qquad + \int_{\Delta} |B(g\cdot(1-\chi_{S(2I)})(z)|^2\, (1-|z|^2)^p\, dm(z)\\
&\quad\lesssim \int_{S(I)} |g(z)\,\chi_{S(2I)}(z)|^2\, (1-|z|^2)^p\, dm(z)\\
&\qquad + \int_{S(I)} \left[\int_{\Delta\setminus S(2I)} \frac{|g(w)|}{|w-z|^2}\, dm(w)\right]^2 (1-|z|^2)^p\, dm(z)\\
&\quad\lesssim \int_{S(2I)} |g(z)|^2\, (1-|z|^2)^p\, dm(z)\\
&\qquad + \int_{S(I)} \left[\left(\sum_{k=1}^{N} + \sum_{k=N+1}^{\infty}\right) \frac{\mu(S(2^{k+1}I))}{(2^k\,|I|)^2}\right]^2 (1-|z|^2)^p\, dm(z)\\
&\quad = o(|I|^p) \qquad \text{as} \quad |I| \to 0,
\end{aligned}$$

where $\chi_{S(2I)}$ is the characteristic function on $S(2I)$, and $d\mu(z) = |g(z)|\, dm(z)$. Together with $\bar\partial F = g$, the estimate of ∂F_1 gives that $F_1 \in Q_{p,0}(\partial\Delta)$ and also by Theorem 1.2 it yields $u, v \in Q_{p,0}(\partial\Delta) \cap L^\infty(\partial\Delta)$ such that $F_1 = u + \tilde v$. Setting $F = F_1 + i(v + i\tilde v) = u + iv$, we obtain $\bar\partial F = f$ with $F \in Q_{p,0}(\partial\Delta) \cap L^\infty(\partial\Delta)$.

(ii) Let P be the Riesz projection from $L^2(\partial\Delta)$ onto H^2. Then Theorem 1.2 is equivalent to $P(Q_p(\partial\Delta) \cap L^\infty(\partial\Delta)) \equiv Q_p$, $P(Q_{p,0}(\partial\Delta) \cap C(\partial\Delta)) = Q_{p,0}$. The corresponding fact for BMO$(\partial\Delta)$ (VMO$(\partial\Delta)$) can be found in [25, p. 79–186].

(iii) It follows easily that an analytic function f in the unit disk belongs to $Q_p(Q_{p,0})$ if and only if $f = f_1 + if_2$, where f_j are analytic functions such that $\mathrm{Re} f_j \in Q_p(\partial\Delta) \cap L^\infty(\partial\Delta)$ $(Q_{p,0}(\partial\Delta) \cap \mathscr{C}(\partial\Delta))$ $j = 1, 2$.

4. INTERPOLATING SEQUENCES

In this section, which will occupy the rest part of this paper, we will prove Theorems 1.3 and 1.4.

4.1. *Necessary Conditions*

The necessity parts in both cases will follow from the same argument, which combines Khinchin's inequality and a reproducing formula for $\mathcal{D}_p$, $p>0$. To be precise, given finitely many complex numbers $w_1, ..., w_n$, consider the 2^n possible sums

$$\sum_{j=1}^{n} \pm w_j$$

obtained as the plus-minus signs vary in the 2^n possible ways. For $q>0$ we use

$$\mathcal{E}\left(\left|\sum_{j=1}^{n} \pm w_j\right|^q\right)$$

to denote the average value of

$$\left|\sum_{j=1}^{n} \pm w_j\right|^q$$

over the 2^n choices of sign. Khinchin's inequality states an estimate on the expectation below,

$$\mathcal{E}\left(\left|\sum_{j=1}^{n} \pm w_j\right|^q\right) \leqslant C(q)\left(\sum_{j=1}^{n} |w_j|^2\right)^{q/2} \tag{4.1}$$

(see [11, p. 302]). Actually $C_q=1$ if $q\leqslant 2$. This inequality will be used in the reproducing formula for $\mathcal{D}_p$. The reproducing formula in [16] asserts that for $f\in\mathcal{D}_p$, one has

$$f(z)=f(0)+\int_{\Delta} f'(w)\, K(z,w)(1-|w|^2)^p\, dm(w), \qquad z\in\Delta, \tag{4.2}$$

where

$$K(z,w)=\frac{(1-z\bar{w})^{1+p}-1}{\bar{w}(1-z\bar{w})^{1+p}}.$$

Proof of the Necessity in Theorems 1.3 *and* 1.4. Let $0<p\leqslant 1$ and assume that $\{z_n\}$ is an interpolating sequence for $Q_p\cap H^\infty$. Then for

$\varepsilon_k^{(j)} = \pm 1$, $j = 1, ..., 2^n$, $k = 1, ..., n$, there are $f_j \in Q_p \cap H^\infty$ such that $f_j(z_k) = \varepsilon_k^{(j)}$, $k = 1, ..., n$ and

$$\|f_j\|_{H^\infty} + \|f_j\|_{Q_p} \leqslant C_4,$$

where $C_4 > 0$ is an absolute constant. Applying (4.2) to $f_j \circ \varphi_w$ at $\varphi_w(z_k)$ we get

$$f_j(z_k) = f_j(w) + \int_\Delta (f_j \circ \varphi_w)'(\xi)\, K(\varphi_w(z_k), \xi)(1 - |\xi|^2)^p\, dm(\xi).$$

We have

$$\begin{aligned}
&\sum_{k=1}^n (1 - |\varphi_w(z_k)|^2)^p \\
&= \sum_{k=1}^n \varepsilon_k^{(j)} f_j(z_k)(1 - |\varphi_w(z_k)|^2)^p \\
&= f_j(w) \sum_{k=1}^n \varepsilon_k^{(j)} (1 - |\varphi_w(z_k)|^2)^p \\
&\quad + \int_\Delta (f_j \circ \varphi_w)'(\xi) \left[\sum_{k=1}^n \varepsilon_k^{(j)} K(\varphi_w(z_k), \xi)(1 - |\varphi_w(z_k)|^2)^p \right] (1 - |\xi|^2)^p\, dm(\xi) \\
&= (A) + (B).
\end{aligned}$$

We will compute the expectation of both sides of this equality. Observe that by (4.1) with $q = 1$ we find

$$\mathscr{E}(A) \leqslant C_4 \left[\sum_{k=1}^n (1 - |\varphi_w(z_k)|^2)^{2p} \right]^{1/2} \leqslant C_4 \left[\sum_{k=1}^n (1 - |\varphi_w(z_k)|^2)^p \right]^{1/2}. \tag{4.4}$$

Also, applying Hölder's inequality and (4.1) with $q = 2$, we get

$$\begin{aligned}
\mathscr{E}(B) &\leqslant \sup_j \|f \circ \varphi_w\|_{\mathscr{D}_p} \\
&\quad \cdot \left[\int_\Delta \sum_{k=1}^n |K(\varphi_w(z_k), \xi)|^2 (1 - |\varphi_w(z_k)|^2)^{2p} (1 - |\xi|^2)^p\, dm(\xi) \right]^{1/2} \\
&\leqslant C_5 \left(\sum_{k=1}^n [1 - |\varphi_w(z_k)|^2]^{2p} \int_\Delta \frac{(1 - |\xi|^2)^p}{||1 - \overline{\varphi_w(z_k)}\xi|^{2+2p}}\, dm(\xi) \right)^{1/2} \\
&\leqslant C_5 \left(\sum_{k=1}^n (1 - |\varphi_w(z_k)|^2)^p \right)^{1/p}.
\end{aligned} \tag{4.5}$$

So, putting (4.4) and (4.5) in (4.3) we see that (b) of Theorem 1.3 holds. Since $\{z_n\}$ is an interpolating sequence for $[H^\infty]$, (a) also holds.

If $\{z_n\}$ is an interpolating sequence for $Q_{p,0} \cap H^\infty \subset \mathrm{VMO} \cap H^\infty$, it is easy to show (a) in Theorem 1.4 holds (see [19] or the Remark below). For (b), we may suppose that $\lim_{|w| \to 1} \sup_j \|f_j \circ \varphi_w\|_{\mathscr{D}_p} = 0$; then (4.3)–(4.5) yield

$$\sum_{k:\, |\varphi_w(z_k)| \geq r} (1 - |\varphi_w(z_k)|^2)^p \leq C_6[(1-r^2)^p + \sup_j \|f_j \circ \varphi_w\|_{\mathscr{D}_p}]^{1/2} \tag{4.6}$$

which implies (b).

Remark. For $Q_{p,0} \cap H^\infty$, $p \in (1, \infty)$, we can obtain a better necessary condition, that is, $\{z_n\}$ is 1-thin. Here, we only have to check that $\{z_n\}$ is 1-thin when $\{z_n\}$ is an interpolating sequence for $B_0 \cap H^\infty$. Using Bloch's theorem we can easily show that $f \in \mathscr{B}_0$ iff it has vanishing variation in pseudohyperbolic disks of fixed radius, that is, for fixed $r \in (0, 1)$,

$$\sup\{|f(z) - f(w)| : \rho(z, w) \leq r, |z| > s\} \to 0 \qquad \text{as} \quad s \to 1$$

(see [6]). So if $f \in Q_{p,0} = B_0$ and $r \in (0, 1)$ is fixed we have

$$|f(z_n) - f(z_m)| \leq \varepsilon(n, m), \qquad \rho(z_n, z_m) < r,$$

where $\varepsilon(n, m) \to 0$ as $n, m \to \infty$. Hence, if one wants to interpolate any bounded sequence at the points $\{z_n\}$, it follows that

$$\inf_{m \neq n} \rho(z_m, z_n) \to 1 \qquad \text{as} \quad n \to \infty.$$

It remains to check (b). It is not difficult to show (see [19]) that $\{z_n\}$ is 1-thin if and only if given any integer $N > 0$, one has

$$\frac{1}{(1 - |z_n|)} \sum_{z_m \in S(NI_{z_n});\, m \neq n} (1 - |z_m|) \to 0 \qquad \text{as} \quad n \to \infty. \tag{4.7}$$

Let $f = f_n \in Q_{p,0} \cap H^\infty$ with $f(z_n) = 1$, $f(z_m) = 0$, $m \neq n$. Given $k > 0$, consider the arcs

$$L_k = \{re^{i\theta} : r = |z_k|, |\theta - \arg z_k| \leq 1 - |z_k|\}.$$

Since $f \in B_0$, given $\varepsilon > 0$ there exists $n_0 > 0$ such that

$$\xi_k = \xi_k(n) = \sup_{z \in L_k} \frac{|f(z)|}{\|f\|_{H^\infty}} < \varepsilon \qquad \text{if} \quad k \neq n, \quad k > n_0.$$

From harmonic majorization [22, p. 302] it follows that

$$\log\left(\frac{|f(z_n)|}{\|f\|_{H^\infty}}\right)\leqslant \sum_{k\neq n} \omega(z_n, L_k, \Delta\backslash L_k)\log \xi_k, \tag{4.8}$$

where $\omega(z, L, \Delta\backslash L)$ means the harmonic measure at z of L in the domain $\Delta\backslash L$. If (4.7) is not satisfied then there would exist $\eta>0$ such that

$$\omega\left(z_n, \bigcup_{k\neq n} L_k, \Delta\backslash \bigcup_{k\neq n} L_k\right)\geqslant \eta$$

for infinitely many n. So (4.8) would imply

$$\frac{|f(z_n)|}{\|f\|_{H^\infty}}<\varepsilon^\eta,$$

which is a contradiction.

Although we have handled the necessity for $Q_p\cap H^\infty$ and $Q_{p,0}\cap H^\infty$ with the same idea, we are forced to prove the sufficiency individually.

4.2. *Sufficient Condition for Interpolation in* $Q_p\cap H^\infty$

Proof of sufficiency in Theorem 1.3. Assume that $\{z_n\}$ is p-uniformly separated and let

$$0<\delta=\inf_{m\neq n}\rho(z_m, z_n).$$

Given a bounded sequence $\{w_n\}$ of complex numbers, take a function $\varphi\in C^\infty(\Delta)$ with the following properties:

(i) $\varphi(z)\equiv w_n$ if $\rho(z, z_n)\leqslant \delta/4$;

(ii) $\varphi(z)\equiv 0$ if $\inf_n \rho(z, z_n)\geqslant \delta/2$.

(iii) $\sup_{z\in\Delta}(1-|z|^2)\,|\nabla\varphi(z)|<\infty$.

Let B be the Blaschke product with zeros $\{z_n\}$. So,

$$B\in Q_p \qquad\text{and}\qquad [\,|\nabla\varphi(z)/|B(z)|\,]^2\,(1-|z|^2)^p\,dm(z)$$

must be a p-Carleson measure. Hence from Theorem 3.1 it follows that there exists $b\in Q_p(\partial\Delta)\cap L^\infty(\partial\Delta)$ solving the equation: $\bar\partial b=\bar\partial\varphi/B$. So, $f=\varphi-Bb\in Q_p\cap H^\infty$ and $f(z_n)=w_n$ for $n=1, 2, \ldots$. ▮

Remarks. Here we would like to point out that J. Earl's [9] and P. Jones' [12] constructive proofs for H^∞-interpolation are suitable for our situation, that is to say, their solutions are in $Q_p\cap H^\infty$ if $\{z_n\}$ is p-uniformly separated.

In a similar way one can prove that $\{z_n\}$ is an interpolating sequence for $\mathcal{D}_p \cap H^\infty$ if and only if $\{z_n\}$ is uniformly separated and $\sum (1-|z_n|)^p < \infty$.

In order to give an argument for sufficiency in the case of $Q_{p,0} \cap H^\infty$, we require some further information on p-thin sequences and non-analytic interpolation solutions.

4.3. *p-Vanishing Carleson Measured and p-Thin Sequences*

The following three auxiliary lemmas are similar to results given in [19] but they are proved by different methods.

LEMMA 4.1. *Let $d\mu$ be a p-vanishing Carleson measure, $p \in (0, 1]$. Then there exists a positive function f on Δ so that $\lim_{|z| \to 1} f(z) = \infty$ and $f(z)\, d\mu(z)$ is a p-vanishing Carleson measure.*

Proof. For $z, w \in \Delta$, define

$$F(z, w) = \left[\int_{|\xi| \geq |z|} \left(\frac{1-|w|^2}{|1-\bar{w}\xi|^2} \right)^p d\mu(\xi) \right]^{-1/2}$$

and

$$f(z) = \inf_{w \in \Delta} F(z, w).$$

Since

$$\lim_{|w| \to 1} \int_\Delta \left(\frac{1-|w|^2}{|1-wz|^2} \right)^p d\mu(z) = 0,$$

one has $f(z) \to \infty$ as $|z| \to 1$. Using the identity

$$\int_0^1 g(x) \left[\int_x^1 g(t)\, dt \right]^{-1/2} dx = 2 \left[\int_0^1 g(t)\, dt \right]^{1/2}, \qquad 0 \leqslant g \in L^1(0, 1),$$

one deduces

$$\int_\Delta \left(\frac{1-|w|^2}{|1-\bar{w}z|^2} \right)^p f(z)\, d\mu(z) \leqslant 2 \left[\int_\Delta \left(\frac{1-|w|^2}{|1-\bar{z}w|^2} \right)^p d\mu(z) \right]^{1/2} \to 0 \qquad \text{as} \quad |w| \to 1.$$

Hence, Lemma 4.1 is proved. ∎

The next result provides a characterization of p-thin sequences in terms of p-Carleson measures.

LEMMA 4.2. *Let $p \in (0, 1]$ and $\{z_n\}$ be a sequence of points in the unit disc. Then the following conditions are equivalent:*

(i) $\{z_n\}$ *is p-thin*;

(ii) $\lim_{n\to\infty} \inf_{m\neq n} \rho(z_n, z_n) = 1$ *and*

$$\lim_{r\to 1} \sup_{I\subset\partial\Delta} \frac{1}{|I|^p} \sum_{z_n\in S(I),\, \rho(z_n, z_I)\geqslant r} (1-|z_n|^2)^p = 0.$$

Proof. (i) $\Rightarrow$ (ii). Let I be an arc of the unit circle and $w = z_I$. Given $\varepsilon > 0$, if $0 < 1 - r$ is sufficiently small, one has

$$\begin{aligned}\varepsilon &> \sum_{n:\, \rho(z_n, w)\geqslant r} (1 - |\varphi_w(z_n)|^2)^p \\ &\geqslant \sum_{\substack{z_n\in S(I)\\ \rho(z_n, w)\geqslant r}} \left(\frac{(1-|w|^2)(1-|z_n|^2}{|1-\bar{w}z_n|^2}\right)^p \\ &\geqslant C\,|I|^{-p} \sum_{\substack{z_n\in S(I)\\ \rho(z_n, w)\geqslant r}} (1-|z_n|^2)^p.\end{aligned}$$

(ii) $\Rightarrow$ (i). Given $w \in \Delta$ let $I \subset \partial\Delta$ be the arc such that $w = z_I$ and λ_k be the point associated to the arc $2^k I$. Observe that $\rho(z_n, w) \geqslant r$ implies that there exists $j = j(r) \to \infty$ (as $r \to 1$) such that

$$\inf_{k\leqslant j} \rho(z_n, \lambda_k) = \rho(r) \to 1 \qquad \text{as} \quad r\to 1.$$

Since $\sum (1-|z_n|^2)^p\, \delta_{z_n}$ is a p-Carleson measure, we get

$$\begin{aligned}\sum_{n:\, \rho(w, z_n)\geqslant r} (1-|\varphi_w(z_n)|^2)^p &\leqslant 2\,|I|^{-p} \sum_{\substack{z_n\in S(I),\\ \rho(w, z_n)\geqslant r}} (1-|z_n|^2)^p \\ &\quad + 2\sum_{k=1}^{j} (2^{2k}\,|I|)^{-p} \sum_{\substack{z_n\in S(2^k I),\\ \rho(z_n, \lambda_k)\geqslant \rho(r)}} (1-|z_n|^2)^p \\ &\quad + 2\sum_{k=j+1}^{\infty} (2^{2k}\,|I|)^{-p} \sum_{z_n\in S(2^k I)} (1-|z_n|^2)^p \to 0 \\ &\quad \text{as} \quad r\to 1.\end{aligned}$$

So, (i) holds. ∎

Remarks. (i) In the case $p = 1$, the conditions of Lemma 4.2 are equivalent to

$$\sum_{m\neq n} \frac{(1-|z_m|^2)(1-|z_n|^2)}{|1-\bar{z}_m z_n|^2} \to 0 \qquad \text{as} \quad n\to\infty \tag{4.9}$$

(see [19]). However the corresponding condition, in the case $p \in (0, 1)$,

$$\sum_{m \neq n} \left[\frac{(1-|z_m|^2)(1-|z_n|^2)}{|1-\bar{z}_m z_n|^2} \right]^p \to 0 \qquad \text{as} \quad n \to \infty, \tag{4.10}$$

is necessary but not sufficient for $\{z_n\}$ to be p-thin. The necessity is easily seen taking $w = z_n$ in the definition of p-thin sequence. To check that it is not sufficient we construct counterexamples.

Case 1. $p \in (0, 1/2)$. Choose integers $k_n \to \infty$ such that $2^{-np} k_n \to \infty$ and $2^{-2np} k_n \to 0$. Let $z_j = (1 - 2^{-n}) \exp(i\theta_j)$, $\theta_j = j k_n^{-1}$, $j = 1, ..., k_n$. Then

$$\sum_{j=1}^{k_n} (1 - |z_j|)^p = 2^{-np} k_n \to \infty$$

while

$$\sum_{j \neq m} \left[\frac{(1-|z_j|^2)(1-|z_m|^2)}{|1-\bar{z}_m z_j|^2} \right]^p \lesssim 2^{-2np} \sum_{j=1}^{k_n} \frac{1}{(j k_n^{-1})^{2p}} \lesssim 2^{-2np} k_n \to 0.$$

Case 2. $p = 1/2$. Choose integers $k_n \to \infty$ such that $2^{-n/2} k_n \to \infty$ and $2^{-n} k_n \log k_n \to 0$, and let z_j be as above.

Case 3. $p \in (1/2, 1)$. Likewise, choose integers k_n such that $2^{-n} k_n \to 0$ and $2^{-np} k_n \to \infty$, also take z_j as in case 1, then we still find that $\{z_j\}$ satisfies (4.10) but $\{z_j\}$ is not p-thin.

The next result allows us to make *big* hyperbolic perturbations of p-thin sequences.

LEMMA 4.3. *Let $p \in (0, 1]$. If a sequence $\{z_n\} \subset \Delta$ is p-thin, then there exists a sequence $\{\rho_n\}$ of positive numbers, with $\lim_{n \to \infty} \rho_n = 1$, such that whenever $\{\xi_n\} \subset \Delta$ satisfies $\rho(z_n, \xi_n) \leqslant \rho_n$, $n = 1, 2, ...$, $\{\xi_n\}$ is also p-thin.*

Proof. Since $\{z_n\} \subset \Delta$ is p-thin, for any sequence r_n tending to 1 one has

$$\sup_{|w| \geqslant r} \sum_{n:\, \rho(w, z_n) \geqslant r_n} (1 - |\varphi_w(z_n)|^2)^p \to 0 \qquad \text{as} \quad r \to 1.$$

Arguing as in Lemma 4.1, one can find a sequence of positive numbers a_n tending to infinity such that

$$\sup_{|w| \geqslant r} \sum_{n:\, \rho(w, z_n) \geqslant r_n} a_n (1 - |\varphi_w(z_n)|^2)^p \to 0 \qquad \text{as} \quad r \to 1.$$

Here, we may, for example, take

$$a_n = \inf_{w \in \Delta} \left[\sum_{\substack{|z_m| \geqslant |z_n| \\ \rho(z_m, w) \geqslant r_m}} (1 - |\varphi_w(z_m)|^2)^p \right]^{-1/2}.$$

Now, fix numbers r_n tending to 1 satisfying

$$\frac{1 - \inf_{m \neq n} \rho(z_m, z_n)}{1 - r_n} \to 0 \qquad \text{as} \quad n \to \infty. \tag{4.11}$$

Then, consider the corresponding sequence $\{a_n\}$ and pick numbers ρ_n tending to 1 satisfying

$$\frac{1 - r_n}{1 - \rho_n} \to 0 \qquad \text{as} \quad n \to \infty. \tag{4.12}$$

Also, by taking $\rho_n \to 1$ sufficiently slow one can assume that $\rho(z_n, \xi_n) \leqslant \rho_n$ implies $(1 - |\xi_n|^2)^p \leqslant a_n(1 - |z_n|^2)^p$ and that $\xi_n \in S(I)$, $\rho(\xi_n, z_I) \geqslant (r_n + \rho_n)/(1 + r_n \rho_n)$ implies that $z_n \in S(2I)$.

Let ξ_n be a sequence of points in the unit disk, $\rho(\xi_n, z_n) \leqslant \rho_n$, $n = 1, 2, \ldots$. We are going to show that $\{\xi_n\}$ is p-thin using (ii) of Lemma 4.2. Given an arc I of the unit circle let $\lambda = z_I$ and set

$$\Omega = \left\{ \xi_n \in S(I) : \rho(\xi_n, \lambda) \geqslant \frac{r_n + \rho_n}{1 + \rho_n r_n} \right\}.$$

Given $\varepsilon > 0$, if $|I|$ is small enough we have

$$\begin{aligned} \frac{1}{|I|^p} \sum_{\xi_n \in \Omega} (1 - |\xi_n|^2)^p &\leqslant \frac{1}{|I|^p} \sum_{\substack{z_n \in S(2I), \\ \rho(z_n, \lambda) \geqslant r_n}} a_n (1 - |z_n|^2)^p \\ &\leqslant \sum_{n:\, \rho(z_n, \lambda) \geqslant r_n} a_n (1 - |\varphi_\lambda(z_n)|^2)^p < \varepsilon. \end{aligned}$$

On the other hand, if $\xi_n \in S(I)$ but $\xi_n \notin \Omega$, that is,

$$\rho(\xi_n, \lambda) \leqslant (r_n + \rho_n)/(1 + \rho_n r_n),$$

then we deduce that

$$1 - \rho(z_n, \lambda) \geqslant C_n.$$

From (4.11) it follows that there is at most one $\xi_m \in S(I) \backslash \Omega$ and hence

$$\frac{1}{|I|^p} \sum_{\substack{\xi_n \in S(I) \\ \rho(\xi_n, \lambda) \geqslant r}} (1-|\xi_n|^2)^p \leqslant \frac{1}{|I|^p} \sum_{\xi_n \in \Omega} (1-|\xi_n|^2)^p + (1-r^2)^p.$$

An easy calculation using the triangle inequality for the pseudohyperbolic distance [11, p. 2–6] and (4.11), (4.12) shows that

$$\inf_{m \neq n} \rho(\xi_m, \xi_n) \to 1 \qquad \text{as} \quad n \to \infty. \quad \blacksquare$$

4.4. *Interpolation by Non-analytic Functions*

As in [19], to interpolate by $Q_{p,0} \cap H^\infty$ functions we first construct non-analytic interpolating functions.

It is clear that a Blaschke sequence $\{z_n\}$, that is, a sequence satisfying

$$\sum_n (1-|z_n|^2) < \infty,$$

can at most accumulate non-tangentially in a set of points (on $\partial \Delta$) of length zero. We will refine this fact.

LEMMA 4.4. *Let $\{z_n\} \subset \Delta$ be a Blaschke sequence. Then there exists an increasing function $h(t)$ on $[0, \infty)$ depending on $\{z_n\}$ and satisfying $h(0)=0$, $\lim_{t\to 0} t^{-1} h(t) = \infty$ and $|\{e^{i\theta} \in \partial\Delta : \mathrm{Card}(\Gamma_h(e^{i\theta}) \cap \{z_n\}) = \infty\}| = 0$, where $\mathrm{Card}(E)$ means the cardinal number of the set E, and*

$$\Gamma_h(e^{i\theta}) = \{z \in \Delta : |z - e^{i\theta}| < h(1-|z|)\}.$$

Proof. Assume that $|z_n| \leqslant |z_{n+1}|$ for all n. Let h_n be an increasing sequence, $h_n \to \infty$ as $n \to \infty$ so that

$$\sum_n (1-|z_n|)\, h_n < \infty.$$

Define $h(1-|z_n|) = (1-|z_n|)\, h_n$ and extend linearly between $1-|z_{n+1}|$ and $1-|z_n|$. It is clear that $h(0)=0$ and

$$\lim_{t \to 0} h(t)/t = \infty.$$

In addition, the set $E = \{e^{i\theta} \in \partial\Delta : \mathrm{Card}(\Gamma_h(e^{i\theta}) \cap \{z_n\}) = \infty\}$ can be covered by

$$\bigcup_{n \geqslant n_0} [h(1-|z_n|)/(1-|z_n|)]\, I_{z_n},$$

where $n_0 > 0$ is any given integer. So $|E|$ can be bounded by

$$\sum_{n \geqslant n_0} h(1-|z_n|) = \sum_{n \geqslant n_0} (1-|z_n|)\, h_n \to 0 \qquad \text{as} \quad n_0 \to \infty$$

and hence $|E| = 0$. ▮

In order to construct a non-analytic interpolating function, we use the idea of *Generations* in [11, p. 299–300] to split p-thin sequences. Let $\{\xi_n\}$ be a p-thin sequence of points of the unit disk, $p \in (0, 1)$. Given an arc $I \subset \partial \Delta$, consider the dyadic subarcs of I, that is, for $n = 1, 2, ...$

$$I = \bigcup_{k=1}^{2^n} I_k^{(n)},$$

where $I_k^{(n)}$ are 2^n disjoint subarcs of I with length $2^{-n}\,|I|$. Now, we will divide $\{\xi_n\}$ into *generations*. For $I = \partial \Delta$, consider the maximal dyadic subarcs J of $\partial \Delta$ such that $T(S(J))$ contains some points in $\{\xi_n\}$ and then the first generation G_1 of $\{\xi_n\}$ consists of these points in $\{\xi_n\}$. Since $\{\xi_n\}$ satisfies

$$\inf_{m \neq n} \rho(\xi_m, \xi_n) \to 1 \qquad \text{as} \quad n \to \infty,$$

$T(S(J))$ can contain at most one point of $\{\xi_n\}$ whenever $|J|$ is sufficiently small. Next, for each $\xi_n \in G_1$ one repeats the above selection replacing $\partial \Delta$ by the dyadic subarc $J \subsetneq \partial \Delta$ such that $\xi_n \in T(S(J))$. In this way one obtains a new subcollection of the sequence which will be denoted by $G_1(\xi_n)$-the first generation corresponding to ξ_n. The second generation of $\{\xi_n\}$ is defined as

$$G_2 = \bigcup_{\xi_n \in G_1} G_1(\xi_n).$$

The later generations, $G_3, G_4, ...$ are defined recursively. See Figure 1.

For n large enough, if $\xi_n \in G_j$ for some j, then there exists a unique $\xi_m \in G_{j-1}$ such that $S(\xi_n) \subset S(\xi_m)$. Moreover,

$$\sum_{\xi_n \in G_j,\, \xi_n \in S(I)} (1-|\xi_n|^2)^p \leqslant C\,|I|^p \tag{4.13}$$

holds for any arc $I \subset \partial \Delta$, where $C > 0$ is a constant independent of j and I.

We can now state a non-analytic interpolation theorem which is the main difference between our proof and the one for $\mathrm{VMOA} \cap H^\infty$ due to C. Sundberg and T. Wolff [19].

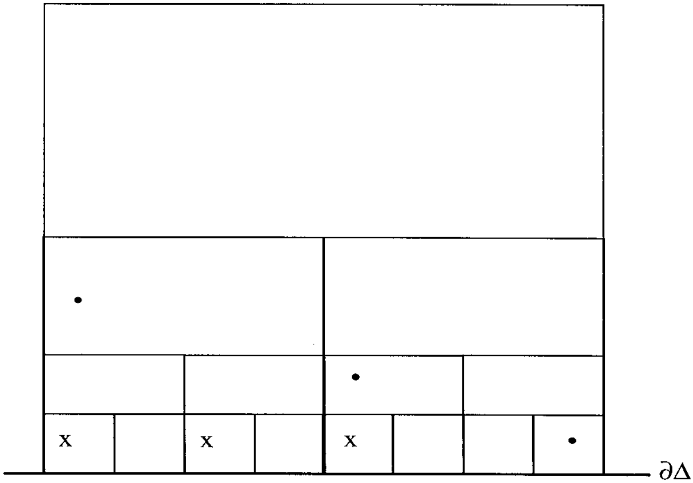

FIG. 1. The points marked with dots are in the first generation and those with crosses are in the second.

THEOREM 4.5. *Let $p \in (0, 1)$. Let $\{z_n\} \subset \Delta$ be a p-thin sequence with $|z_n| \leqslant |z_{n+1}|$ for all n. Then there exists an increasing sequence $\{K_n\}$ of positive numbers, $K_n \to \infty$ as $n \to \infty$, and $c > 0$, such that whenever $\{w_n\}$ is a sequence of complex numbers satisfying*

$$\sup_{m,\, n > k} \frac{|w_n - w_m|^2}{\max\{K_n, K_m\}} \to 0 \qquad as \quad k \to \infty, \tag{4.14}$$

there exists $\varphi \in C^\infty(\Delta)$ with $\varphi(z) \equiv w_n$ in $\{z\colon \rho(z, z_n) \leqslant c\}$ satisfying

(i) $\nabla\varphi(z) \equiv 0$ *in* $\Delta \backslash \bigcup_n R_n$, *where* $\{R_n\}$ *are some pairwise disjoint regions on* Δ *and* $z_n \in R_n$;

(ii) $|\nabla\varphi(z)|^2\,(1-|z|^2)^p\,dm(z)$ *is a p-vanishing Carleson measure*;

(ii) $|\Delta\varphi(z)|\,(1-|z|) \lesssim |\nabla\varphi(z)|$ *and* $(1-|z|^2)\,|\nabla\varphi(z)| \lesssim 1$, *where* $\Delta\varphi$ *means the Laplacian of* φ;

(iv) $\sup_{z \in \Delta} |\varphi(z)| \leqslant 2 \sup_n |w_n|$, *if* $\sup_n |w_n| < \infty$;

(v) *If* $w_n > 0$ *increases when* $(1-|z_n|)$ *decrease, one has* $0 \leqslant \varphi(z) \leqslant 2w_n$ *for* $z \in R_n$.

Proof. Observe that one only has to do the construction when n is large. Hence one may assume that $\{z_n\}$ are very close to $\partial\Delta$.

Let $I_n = I_{z_n}$ (i.e., $S(z_n) = S(I_n)$) and ξ_n be the point associated to the arc $2^{K_n} I_n$. By Lemma 4.3, K_n can be chosen tending to ∞ sufficiently slowly so that $\{\xi_n\}$ is also p-thin.

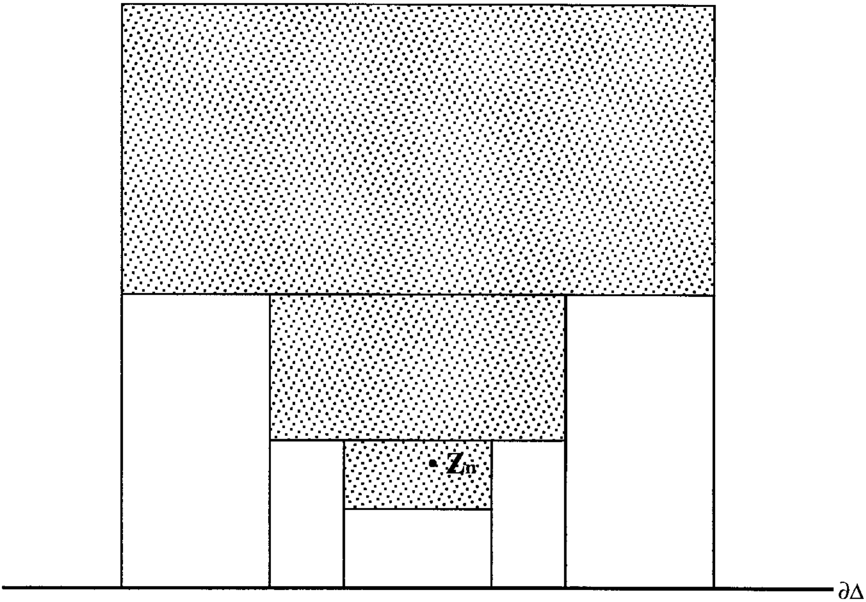

FIG. 2. The shadowed region is the tower over z_n when $K_n = 2$.

For $n = 1, 2, ...,$ we consider a region $T(z_n)$ which will be called the tower over z_n,

$$T(z_n) = T(S(I_n)) \cup T(S(2I_n)) \cup \cdots \cup T(S(2^{K_n} I_n)).$$

See Figure 2. Since $S(2^{k_n} I_n) = S(\xi_n)$, the decomposition of $\{\xi_n\}$ into generations gives a corresponding decomposition for the towers, that is,

$$\{T(z_n)\} = \bigcup_j \{T(z_n) : \xi_n \in G_j\}.$$

If $K_n \to \infty$ sufficiently slowly, then the following properties hold:

(1) $\inf\{\rho(T(z_n), T(z_m)) : n \neq m \text{ and } \xi_n, \xi_m \in G_j\} \to 1$ as $n \to \infty$, where $\rho(E, F) = \inf\{\rho(z, w) : z \in E,\ w \in F\}$ stands for the pseudohyperbolic distance between $E \subset \Delta$ and $F \subset \Delta$.

(2) For any arc $I \subset \partial\Delta$ one has

$$\sum_{T(z_n) \subset S(I)} (1 - |\xi_n|^2)^p \lesssim |I|^p.$$

Moreover, using a slightly bigger Carleson box as a substitute for $S(z_n)$ one can also assume

(3) Given a tower $T(z_n)$ of the j-generation, that is $\xi_n \in G_j$, there is a unique z_m in the $(j-1)$ generation, that is $\xi_m \in G_{j-1}$, such that $T(z_n) \subset S(z_m)$.

Next, we will determine a neighborhood $R(z_n)$, the so-called extended tower over z_n, of $T(z_n)$. For $j=1, ..., K_n$ consider the *brothers* of $T(S(2^j I_n))$ as

$$S^{(j)}_{-(K_n-j)}, \dots S^{(j)}_0, \dots S^{(j)}_{K_n-j},$$

which are the $2(K_n-j)+1$ adjacent (from both sides) truncated Carleson boxes to $T(S(2^j I_n))$ of the same size. So, $S^{(j)}_0 = T(S(2^j I_n))$. Also, for $j=0$, consider

$$S^{(0)}_{-K_n}, ..., S^{(0)}_0, ..., S^{(0)}_{K_n}$$

the $2K_n+1$ adjacent (from both sides) Carleson boxes to $S(I_n) = S^{(0)}_0$. Then consider

$$R(z_n) = \left(\bigcup_{j=0}^{K_n} \bigcup_{l=-(K_n-j)}^{K_n-j} S^{(j)}_l \right).$$

See Figure 3. If $K_n \to \infty$ sufficiently slowly, and again replacing $S(z_n)$ by a slightly bigger Carleson box if necessary, one can assume that properties (2) and (3) hold for $R(z_n)$ instead of $T(z_n)$.

Finally, for $l = -K_n, ..., K_n$, consider a point $\xi = \xi(l)$ in the radial projection of $S^{(0)}_l$ onto $\partial \Delta$ so that $S^{(0)}_l \cap \Gamma_h(\xi)$ do not intersect any extended tower $R(z_m)$, $m \neq n$, where h is the function given by Lemma 4.4. One may also assume that the usual truncated Stolz angles

$$\Gamma(\xi(l)) = \{ z \in \overline{S^{(0)}_l} : |z - \xi(l)| < 2(1-|z|) \}, \qquad l = -K_n, ..., K_n$$

are pairwise disjoint. Now, define

$$R_n = \left[R(z_n) \Big\backslash \bigcup_{-K_n}^{K_n} S^{(0)}_l \right] \cup \left[\bigcup_{-K_n}^{K_n} (\Gamma_h(\zeta(l)) \cap S^{(0)}_l) \right].$$

It follows from properties (1) and (3) that $\{R_n\}$ are pairwise disjoint.

We will define the function φ as the limit of some $\varphi_j \in C^\infty(\Delta)$ which will be constructed using generations. The function φ_j will be constant outside the sets R_n corresponding to points z_n in generations G_k, $k \leqslant j$. The key estimate will be

$$(1-|z|)\, |\nabla \varphi_j(z)| \leqslant \frac{\max_{k<n} \{ |w_n - w_k| : z_k \in S(z_m), R(z_n) \subset S(z_m) \}}{K_n},$$

$$z \in R(z_n).$$

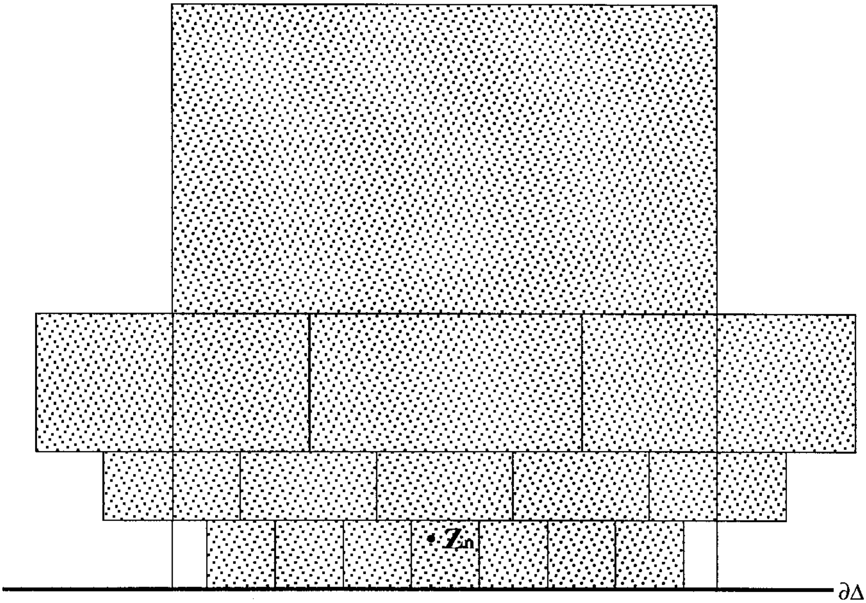

FIG. 3. The shadowed region is the extended tower over z_n when $K_n = 3$.

It will be clear from the construction below how to define φ_1. Assume we have defined φ_{j-1} with the following properties.

(a) $\varphi_{j-1} \equiv w_n$ in $\{z: \rho(z, z_n) \leqslant c\}$ if $z_n \in \bigcup_{k=1}^{j-1} G_k$.

(b) $\operatorname{supp}(\nabla \varphi_{j-1}) \subset \bigcup_{k=1}^{j-1} \bigcup_{z_n \in G_k} R_n$.

(c) $(1-|z|)\, |\nabla \varphi_{j-1}(z)| \leqslant (\max_{m \leqslant n} |w_n - w_m|)/K_n$, $z \in R(z_n)$ if $R(z_n) \subset S(z_m)$.

After that, let us construct φ_j. Given a point $z_m \in G_{j-1}$ let $R = R(z_n)$ be the largest extended tower of G_j contained in $S(z_m)$, namely,

$$1 - |\xi_n| = \sup\{1 - |\xi_k|: \zeta_k \in G_j, R(z_k) \subset S(z_m)\}.$$

We want to find a function $\psi = \psi_R$ in $S(z_m)$, which satisfies the analogues of (a)–(c). Let w_n be the value that one wants to interpolate at z_n.

Take $w^{(0)} = w_n, w^{(1)}, ..., w^{(K_n)} = w_m$ with

$$|w^{(k)} - w^{(k+1)}| = K_n^{-1}\, |w_n - w_m|$$

and define

$$\psi \equiv \psi_R = w^{(j+l)} \quad \text{in} \quad S_l^{(j)}, \qquad l = -(K_n - j), ..., K_n - j, \quad j = 0, ..., K_n$$

and $\psi = \varphi_{j-1}$ in $\Delta \backslash R(z_n)$. So, after a regularization we can assume that $\psi \in C^\infty(\Delta)$ with $\psi \equiv w_n$ in $\{z\colon \rho(z, z_n) \leqslant c\}$, satisfies

$$(1-|z|)\,|\Delta\psi(z)| \lesssim |\nabla\psi(z)|, \qquad z \in \Delta, \tag{4.15}$$

$$(1-|z|)\,|\nabla\psi(z)| \lesssim 1, \qquad z \in \Delta, \tag{4.16}$$

and

$$(1-|z|)\,|\nabla\psi(z)| \lesssim \frac{|w_n - w_m|}{K_n}, \qquad z \in R(z_n). \tag{4.17}$$

Furthermore, we can do the regularization in such a way that

$$\nabla\psi(z) \equiv 0, \qquad z \in S(z_m) \backslash R_n. \tag{4.18}$$

This is just the place where the cones $\Gamma_h(\xi(l))$ are used. Actually, we can do the regularization in such a way that

$$\nabla\psi(z) \equiv 0, \qquad z \in \bigcup_l (S_l^{(0)} \backslash \Gamma(\xi(l))), \qquad 1-|z| > h^{-1}(|I_n|),$$

where $\Gamma(\xi(l)) = \{z \in \Delta : |z - \xi(l)| < 2(1-|z|)\}$ are the usual Stölz angles and

$$h(1-|z|)\,|\nabla\psi(z)| \lesssim \frac{|w_n - w_m|}{K_n}, \qquad z \in S_l{}^{(0)} \cap \Gamma_h(\xi(l)), \quad 1-|z| < \frac{1}{2} h^{-1}(|I_n|). \tag{4.19}$$

In the sequel, we shall take care of the remaining extended towers in $S(z_m)$ and select the maximal one, say, $R^* = R(z_k)$ for some k. If R^* does not intersect the preceding one $R = R(z_n)$, then repeat the same construction. Otherwise, just repeat the construction with φ_{j-1} replaced by $\psi \equiv \psi_R$ and $S(z_m)$ by the Carleson box $S_l^{(0)} \subset R$ which contains z_k. In this way we get a function $\psi^* \equiv \psi_{R^*}$ satisfying (4.15) and (4.16). Note that even if φ_{j-1} has been substituted by ψ_R, the estimates (4.17) and (4.19) still hold if we replace the right-hand term by

$$\max_{m \leqslant n} |w_n - w_m| / K_n.$$

Continue this process. If it ends after finitely many steps, that is, if there is a finite number of points of the sequence $\{z_n\}$ in $S(z_m)$, then we take φ_j to be the last ψ_R. If there are infinitely many points of $\{z_n\}$ in $S(z_m)$, then the corresponding ψ_R converges pointwise to a function which will be our φ_j. It is clear from (4.15)–(4.19) that φ_j satisfies (a)–(c). Finally, we define φ as the pointwise limit of φ_j and it is clear from the construction that φ satisfies (i) (iii) (iv) and (v) in Theorem 4.5. To show (ii) we will use the estimates

$$\nabla\varphi(z)=0, \qquad z\in\bigcup (S_l^{(0)}\setminus\Gamma(\xi(l))), \qquad 1-|z|>h^{-1}(|I_n|),$$

$$(1-|z|)\,|\nabla\varphi(z)|\lesssim\frac{\max_{m\leqslant n}|w_n-w_m|}{K_n}, \qquad z\in R(z_n), \tag{4.20}$$

$$h(1-|z|)\,|\nabla\varphi(z)|\lesssim\frac{\max_{m\leqslant n}|w_n-w_m|}{K_n},$$

$$z\in S_l^{(0)}\cap\Gamma_h(\xi(l)), \qquad 1-|z|<\frac{1}{2}h^{-1}(|I_n|), \tag{4.21}$$

which follow from the similar inequalities (4.17) and (4.19) for φ_j.

Put $\Gamma_n=\bigcup_l S_l^{(0)}\cap\Gamma_h(\xi(l))\cap\{z\colon 1-|z|<\frac{1}{2}h^{-1}(|I_n|)\}$, where $S_l^{(0)}$ are the Carleson sectors in $R(z_n)$. Estimates (4.20) and (4.21) give respectively

$$\begin{aligned}\int_{R(z_n)\setminus\Gamma_n}&|\nabla\varphi(z)|^2\,(1-|z|^2)^p\,dm(z)\\&\lesssim\left[\frac{\max_{m\leqslant n}|w_n-w_m|}{K_n}\right]^2\sum_{j=0}^{K_n}\sum_{-(K_n-j)}^{K_n-j}(2^j(1-|z_n|^2)^p\\&\lesssim\frac{\max_{m\leqslant n}|w_n-w_m|^2}{K_n}\,[2^{K_n}(1-|z_n|^2)]^p\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}\int_{\Gamma_n\cap S_l^{(0)}}&|\nabla\varphi(z)|^2\,(1-|z|^2)^p\,dm(z)\\&\leqslant\left[\max_{t\leqslant 1-|z_n|}\frac{t}{h(t)}\right]^2\left[\frac{\max_{m\leqslant n}|w_n-w_m|}{K_n}\right]^2|S_l^{(0)}|^p.\end{aligned} \tag{4.23}$$

Now, let $S(I)$ be a Carleson box and let n be such that $R_n \cap S(I) \neq \emptyset$. Then

$$\int_{R_n \cap S(I)} |\nabla\varphi(z)|^2 \, (1-|z|^2)^p \, dm(z)$$
$$= \left(\int_{R(z_n) \setminus \Gamma_n \cap S(I)} + \int_{\Gamma_n \cap S(I)} \right) |\nabla\varphi(z)|^2 \, (1-|z|^2)^p \, dm(z)$$
$$= (A) + (B).$$

If $2^{K_n} |I_n| \leqslant |I|$, then as in (4.22) we have

$$(A) \lesssim \frac{\max_{m \leqslant n} |w_n - w_m|^2}{K_n} \, [2^{K_n}(1-|z_n|^2)]^p.$$

Otherwise, that is, if $2^{K_n} |I_n| > |I|$, we get

$$(A) \lesssim \frac{\max_{m \leqslant n} |w_n - w_m|^2}{K_n} \, |I \cap (2^{K_n} I_n)|^p.$$

On the other hand, if $|I| \geqslant |I_n|$, then using (4.23) we obtain

$$(B) \lesssim \left[\max_{t \leqslant 1-|z_n|} \frac{t}{h(t)} \right]^2 \left[\frac{\max_{m \leqslant n} |w_n - w_m|}{K_n} \right]^2 \min \left\{ \frac{|I|}{|I_n|}, K_n \right\} \cdot |I_n|^p$$

and if $|I| < |I_n|$,

$$(B) \lesssim \left[\max_{t \leqslant |I|} \frac{t}{h(t)} \right]^2 \left[\frac{\max_{m \leqslant n} |w_n - w_m|^2}{K_n} \right]^2 |I|^p.$$

So,

$$\int_{R_n \cap S(I)} |\nabla\varphi(z)|^2 \, (1-|z|^2)^p \, dm(z)$$
$$\lesssim \frac{\max_{m \leqslant n} |w_n - w_m|^2}{K_n} \, |(2^{K_n} I_n) \cap (2I)|^p$$
$$+ \left[\max_{t \leqslant \max\{|I|, |I_n|\}} \frac{t}{h(t)} \right] \left[\frac{\max_{m \leqslant n} |w_n - w_m|^2}{K_n} \right]$$
$$\left[1 + \min \left\{ \frac{|I|}{|I_n|}, K_n \right\} \right] [\min\{|I|, |I_n|\}]^p.$$

Applying the assumption (4.14) and the fact that

$$\sum_n (1-|\xi_n|)^p \, \delta_{\xi_n} = \sum_n (2^{K_n} \, |I_n|)^p \, \delta_{\xi_n}$$

is a p-Carleson measure we arrive at (ii). Hence, the proof is completed. ∎

Remarks. (i) In the case $p=1$, Theorem 4.5 still holds with property (ii) in place of the weaker condition that $|\nabla\varphi(z)| \, dm(z)$ is a 1-vanishing Carleson measure. This fact can be shown in a similar way.

(ii) Any bounded sequence $\{w_n\}$ satisfies (4.14), but one can also take unbounded sequences $\{w_n\}$ tending to infinity sufficiently slowly.

(iii) If φ satisfies (ii) and (iii) of Theorem 4.5, there exists $b \in Q_{p,0}(\partial\Delta) \cap L^\infty(\Delta)$ such that $\bar\partial b = \bar\partial\varphi$. This follows from the fact that when $|\bar\partial\varphi(z)| \, dm(z)$ and $|\Delta\varphi(z)|(1-|z|) \, dm(z)$ are 1-Carleson measures and $\sup_{z\in\Delta} |\tilde\Delta\varphi(z)| \, (1-|z|) < \infty$, any smooth solution b of $\bar\partial b = \bar\partial\varphi$ which is bounded on the unit circle is already bounded on the unit disk [21] [5, p. 53].

4.5. *Sufficient Condition for Interpolation in* $Q_{p,0} \cap H^\infty$

After this long preparation, we can give a proof for the sufficient condition for $Q_{p,0} \cap H^\infty$-interpolation.

Proof of sufficiency in Theorem 1.4. Without loss of generality, let $\{z_n\}$ be a p-thin sequence of points in the unit disk, $|z_n| \leqslant |z_{n+1}|$ for all n, and let $\{w_n\} \subset \mathbb{C}$ be a bounded sequence of complex numbers. Choose $K_n \to \infty$ verifying Theorem 4.5. Also let φ be the function given by Theorem 4.5.

Since $|\nabla\varphi(z)|^2 \, (1-|z|^2)^p \, dm(z)$ is a p-vanishing Carleson measure, Lemma 4.1 provides a function $k(z) \geqslant 0$ increasing to infinity (as $|z| \to 1$) such that $[k(z) \, |\nabla\varphi(z)|]^2 \, (1-|z|^2)^p \, dm(z)$ is still a p-vanishing Carleson measure. Setting

$$k_n = \inf_{z \in R_n} k(z)^{1/2},$$

where R_n are given by Theorem 4.5, and replacing k_n by smaller numbers, increasing to infinity as well, one can assume that

$$\sup_{n,m \geqslant k} \frac{|\log k_n - \log k_m|^2}{\max\{K_n, K_m\}} \to 0 \qquad \text{as} \quad k \to \infty.$$

Another application of Theorem 4.5 with the values $\{\log k_n\}$ gives a function ψ satisfying (i)–(v) in Theorem 4.5. So, $\psi(z_n) \to \infty$ as $n \to \infty$ and $0 \leqslant \psi(z) \leqslant 2 \log k_n$ for $z \in R_n$. Using the third remark after Theorem 4.5, we can find a function $a \in L^\infty(\Delta) \cap Q_{p,0}(\partial\Delta))$ so that $\bar{\partial} a = \bar{\partial}\psi$. Hence $F = \exp(a - \psi) \in Q_{p,0} \cap H^\infty$, $F(z_n) \to 0$ as $n \to \infty$ and $|F(z)|^{-1} \lesssim k(z)$ for $z \in \bigcup_n R_n$.

Let B be the Blaschke product with zeros $\{z_n\}$. Then the measure

$$\left|\frac{\nabla\varphi(z)}{B(z)\,F(z)}\right|^2 (1-|z|^2)^p \, dm(z)$$

is a p-vanishing Carleson measure. Now, from Theorem 3.1 it follows that there is a function $b \in Q_{p,0}(\partial\Delta) \cap L^\infty(\partial\Delta)$ with $\bar{\partial} b = \bar{\partial}\varphi / BF$, and then that $f = \varphi - BFb \in Q_{p,0} \cap H^\infty$ because here Corollary 2.5 is applied to $F \in Q_{p,0} \cap H^\infty$ with $F(z_n) \to 0$ and so $FB \in Q_{p,0} \cap H^\infty$. It is obvious that this function f interpolates w_n at z_n for $n = 1, 2, \ldots$. ∎

Remark. If $\{z_n\} \subset \Delta$ is 1-thin, using Remark (i) after Theorem 4.5, one can assume $|\nabla\varphi(z)|\, dm(z)$ is a 1-vanishing Carleson measure. Repeating the same argument we also get that

$$\left|\frac{\nabla\varphi(z)}{B(z)\,F(z)}\right| dm(z)$$

is a 1-vanishing Carleson measure. So, it remains to check that the equation

$$\bar{\partial} b = \frac{\bar{\partial}\varphi}{BF} = g$$

can be solved by $b \in \mathrm{VMO}(\partial\Delta) \cap L^\infty(\partial\Delta)$. This is well-known and follows, for instance, from the Fefferman-Stein decomposition. Actually the function

$$b_0(z) = \frac{1}{\pi} \int_\Delta \frac{g(\xi)}{z - \xi} \, dm(\xi)$$

is in $\mathrm{VMO}(\partial\Delta)$ and $\bar{\partial} b_0 = g$. Hence $b_0 = u + \tilde{v}$, where $u, v \in C(\partial\Delta)$ and one can take $b = b_0 + i(v + i\tilde{v}) \in \mathrm{VMO}(\partial\Delta) \cap L^\infty(\partial\Delta)$. Finally, the interpolating function $\varphi - bBF$ is in $\mathrm{VMOA} \cap H^\infty \subset B_0 \cap H^\infty$.

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