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REGULARITY PROPERTIES OF MEASURES, ENTROPY AND THE LAW OF THE ITERATED LOGARITHM

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Introduction

In many problems in analysis one often encounters the following situation: we are given some positive measure intrinsically associated to the problem and we ask how large must a set of positive measure be, in terms of the geometry of the ambient space for instance, and what can be said about its Hausdorff dimension.

Let μ be a positive Borel measure on $\mathbb{R}^N$. We say that μ is *absolutely continuous* with respect to $\mathcal{H}_\phi$, the Hausdorff measure associated to the measure function ϕ (hereafter, denoted $\mu \ll \mathcal{H}_\phi$) if

$$\mu(E) > 0 \implies M_\phi(E) > 0$$

where M_ϕ is the Hausdorff ϕ-content. Specially important are the choices $\phi(t) = t^\alpha$, for some $0 < \alpha \leqslant N$, and in this case the corresponding Hausdorff measure (called α-dimensional Hausdorff measure) will be simply denoted by $\mathcal{H}_\alpha$. (See § 1.6 for the definition and basic properties of Hausdorff contents and measures.) Let us exhibit three important examples where this comparison question arises.

(a) *Zygmund measures*. A positive measure μ on $\mathbb{R}$ is a *Zygmund measure* if there is $C > 0$ such that

$$|\mu(I) - \mu(I')| \leqslant C|I|$$

for any two adjacent intervals $I, I' \subset \mathbb{R}$ of the same length. Zygmund measures have been extensively studied in harmonic analysis and they are also closely related to some questions in geometric function theory [**21**, **19**]. From the definition it is easy to get the global estimate $\mu(I) \leqslant C_1|I|\log(1/|I|)$ which implies that $\mu \ll \mathcal{H}_{\phi_1}$, where $\phi_1(t) = t\log t^{-1}$. However, the optimal result [**18**] is $\mu \ll \mathcal{H}_\phi$, where $\phi(t) = t\sqrt{\log t^{-1} \log\log\log t^{-1}}$. See [**7**].

(b) *Harmonic measure*. Consider a domain $\Omega \subset \mathbb{R}^N$ and fix $a \in \Omega$. For $E \subset \partial\Omega$, let $\omega(E, a, \Omega)$ be the value at a of the harmonic function in Ω with boundary values 1 on E and 0 on $\partial\Omega \setminus E$ (we assume that such a function is well defined). Then $\omega(\,\cdot\,, a, \Omega)$ is a Borel probability measure on $\partial\Omega$, called the *harmonic measure* with base point a. One of the most challenging problems in geometric function theory during the last thirty years has been to understand the metric

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properties of harmonic measure. Namely, for which ϕs is it true that $\omega \ll \mathcal{H}_\phi$? Suppose that $N=2$ and Ω is simply connected. Then it is not difficult to prove, using some standard harmonic measure estimates, a global inequality of the type $\omega(E) \leqslant C\sqrt{\operatorname{diam} E}$, which gives $\omega \ll \mathcal{H}_{1/2}$. An important advance was obtained by Carleson, who proved that $\omega \ll \mathcal{H}_{1/2+\epsilon}$ for some $\epsilon > 0$. The final answer is due to Makarov [**16**]: $\omega \ll \mathcal{H}_\psi$, where $\psi(t) = t\exp\{C\sqrt{\log t^{-1} \log\log\log t^{-1}}\}$ for some absolute constant $C>0$. Furthermore, ψ is sharp, up to the value of C. The situation is far from being well understood in higher dimensions or for other elliptic operators.

(c) *Distortion of homeomorphisms of the real line.* Let $h:\mathbb{R}\to\mathbb{R}$ be an increasing homeomorphism and μ_h its associated Lebesgue–Stieltjes measure, that is $\mu_h(E) = |h(E)|$. Increasing versions of the Cantor function show that μ_h can be singular in general. Suppose now that we have control of the distortion of h for all the different scales in the sense that

$$(M(t))^{-1} \leqslant \frac{h(x+t)-h(x)}{h(x)-h(x-t)} \leqslant M(t)$$

for some monotone function $M(t) \geqslant 1$ defined in $(0,\infty)$. In [**3**] it was shown that μ_h can be singular even if it is a doubling measure, that is, $\sup M(t) \leqslant M < \infty$. (We say that h is *M-quasisymmetric* in this case.) This suggests the question of how large must a set $E\subset\mathbb{R}$ be if we know that $|h(E)|>0$. In terms of Hausdorff measures this is equivalent to asking when $\mu_h \ll \mathcal{H}_\phi$ for some ϕ depending on $M(t)$. We will see that our methods give new answers to this question. See [**8**] for related results.

Let $Q_0 = [0,1)^N$ be the (half-open) unit cube in $\mathbb{R}^N$. We consider its dyadic decomposition, that is, for $k=1,2,\ldots$ let $\mathcal{F}_k$ be the collection of the 2^{kN} pairwise disjoint (half-open) dyadic subcubes of Q_0 of sidelength 2^{-k}. Given $Q_{k-1}\in\mathcal{F}_{k-1}$, we denote by $\{Q_k^i : i=1,\ldots,2^N\}$ the 2^N cubes of $\mathcal{F}_k$ contained in Q_{k-1}, so $Q_{k-1} = \bigcup_i Q_k^i$. Finally, for $x\in Q_0$, we denote by $Q_k(x)$ the unique cube in $\mathcal{F}_k$ which contains x.

The main purpose of the paper is to study regularity properties of a positive Borel measure μ in $\mathbb{R}^N$, such as being absolutely continuous with respect to various Hausdorff measures, in terms of its doubling behaviour, that is, in terms of the size of

$$\delta_k(x) = \min\left\{1 - \frac{\mu(Q_k^i)}{\mu(Q_{k-1}(x))}\right\},$$

where $Q_{k-1}(x)$ denotes the only dyadic cube of the generation $k-1$ containing x and $\{Q_k^i\}_{i=1}^{2^N}$ is its decomposition in dyadic cubes of the generation k. Observe that by additivity $0\leqslant\delta_k\leqslant 1-2^{-N}$. A positive Borel measure μ in the unit cube Q_0 of $\mathbb{R}^N$ is called *dyadic doubling* if there exists a constant $c=c(\mu)>0$ such that $\mu(Q_k^i) > c\mu(Q_{k-1}(x))$ for any $x\in Q_0$, $i=1,\ldots,2^N$ and $k=1,2,\ldots$.

Y. Heurteaux has proven the following nice result in this direction [**11**]. For simplicity we state his result in dimension 1. Let μ be a positive Borel measure in $[0,1]$ and assume that $\delta_k(x)\geqslant\delta>0$ for any $x\in[0,1]$. Then μ is absolutely continuous with respect to $\mathcal{H}_\beta$ for any

$$\beta < \frac{\delta \log \delta^{-1} + (1-\delta)\log(1-\delta)^{-1}}{\log 2}.$$

On the other hand, if $\delta_k(x) \leqslant \frac{1}{2} - \varepsilon$ for any $x \in [0,1]$, where $\varepsilon > 0$, then Y. Heurteaux also showed that μ is singular with respect to $\mathcal{H}_\beta$ for any

$$\beta > \frac{(\frac{1}{2}+\varepsilon)\log(\frac{1}{2}+\varepsilon)^{-1} + (\frac{1}{2}-\varepsilon)\log(\frac{1}{2}-\varepsilon)^{-1}}{\log 2}.$$

See [**2**, **12**] for other related results.

In our approach, the doubling behaviour of a probability measure μ in Q_0 will be described by means of an entropy type quantity. Recall that if $p_1, \dots, p_n$ is a probability distribution, its entropy is defined as $\sum p_i \log p_i^{-1}$. Similarly, for $x \in Q_0$, we define

$$h_k(x) = \sum_i \frac{\mu(Q_k^i)}{\mu(Q_{k-1}(x))} \log \frac{\mu(Q_{k-1}(x))}{\mu(Q_k^i)}, \quad \text{for } k = 1, 2, \dots,$$

where the sum is taken over the 2^N dyadic subcubes $Q_k^i \in \mathcal{F}_k$ contained in $Q_{k-1}(x)$. We will be interested in $h_k(x)$, μ-a.e. $x \in Q_0$, but for the sake of completeness, we write $h_k(x) \equiv 0$ if $\mu(Q_{k-1}(x)) = 0$, and if $\mu(Q_k^i) = 0$ for some i we interpret the corresponding term in the sum to be 0. Observe that $0 \leqslant h_k(x) \leqslant N \log 2$ and the extreme cases $h_k(x) = 0$ and $h_k(x) = N\log 2$ correspond respectively, to the situations where μ gives all the mass $\mu(Q_{k-1}(x))$ to one of the subcubes $\{Q_k^i : i = 1, \dots, 2^N\}$ and where μ fairly distributes its mass $\mu(Q_{k-1}(x))$ among all $\{Q_k^i : i = 1, \dots, 2^N\}$. Hence, $h_k(x)$ tells how μ distributes the mass $\mu(Q_{k-1}(x))$ among $\{Q_k^i : i = 1, \dots, 2^N\}$ and so the function

$$H_n(x) = \sum_{k=1}^n h_k(x), \quad \text{for } x \in Q_0,$$

gives information on the doubling behaviour of μ among all dyadic cubes containing x of generation smaller than n. Since $0 \leqslant h_k(x) \leqslant N\log 2$ and $H_n(x) \leqslant Nn\log 2$, both limits

$$H(x) = \lim_{n\to\infty} H_n(x) \quad \text{and} \quad \lim_{n\to\infty} (Nn\log 2 - H_n(x))$$

exist and are non-negative but may be infinite. The behaviour of a positive measure with respect to Hausdorff contents can be described in terms of the entropy H_n in the following way.

COROLLARY 6.2. (a) *Let μ be a probability measure in Q_0 and $0 < \beta \leqslant N$. Assume that for μ almost every point x there exists $n_0(x) > 0$ such that*

$$H_n(x) \geqslant \beta n \log 2, \quad \text{for } n > n_0(x).$$

Then μ is absolutely continuous with respect to $\mathcal{H}_{\phi_\beta}$, where

$$\phi_\beta(t) = t^\beta \exp(C(\log t^{-1} \log\log\log t^{-1})^{1/2})$$

and $C = C(N) > 0$ is a constant only depending on the dimension N.

(b) *Let μ be a dyadic doubling probability measure in Q_0 which is singular with respect to Lebesgue measure. Let $0 < \beta < N$ and assume that for μ almost every point x there exists $n_0(x) > 0$ such that*

$$H_n(x) \leqslant \beta n \log 2, \quad \text{for } n > n_0(x).$$

Then μ is singular with respect to $\mathcal{H}_{\phi_\beta}$, where

$$\phi_\beta(t) = t^\beta \exp(C_1(\log t^{-1} \log\log\log t^{-1})^{1/2})$$

and $C_1 = C_1(N) > 0$ is a constant only depending on the dimension N.

Since the entropy $H_n(x)$ gives information on the doubling behaviour of the measure μ, it is natural to relate $H_n(x)$ with the uniform quantity

$$\delta_k = \min_x \delta_k(x). \tag{0.1}$$

Here, we interpret $\delta_k(x) = 0$ if $\mu(Q_{k-1}(x)) = 0$. In terms of this uniform quantity our result is as follows.

Corollary 6.4. (a) *Let μ be a probability measure in Q_0. Assume $\delta = \inf_k \delta_k > 0$. Let p be the integer part of $(1-\delta)^{-1}$ and*

$$\beta = \frac{p(1-\delta)\log(1-\delta)^{-1} + (1-p(1-\delta))\log(1-p(1-\delta))^{-1}}{\log 2}.$$

Then μ is absolutely continuous with respect to $\mathcal{H}_{\phi_\beta}$, where

$$\phi_\beta(t) = t^\beta \exp(C(\log t^{-1} \log\log\log t^{-1})^{1/2})$$

and $C = C(N) > 0$ is a constant only depending on the dimension N.

(b) *Let μ be a dyadic doubling probability measure in Q_0 which is singular with respect to Lebesgue measure. Assume that for μ almost every point x, there exists $k_0(x) > 0$ such that $\delta_k(x) \leqslant \delta < 1 - 2^{-N}$ if $k > k_0(x)$. Then μ is singular with respect to $\mathcal{H}_{\phi_\gamma}$, where*

$$\phi_\gamma(t) = t^\gamma \exp(C_1(\log t^{-1} \log\log\log t^{-1})^{1/2}),$$
$$\gamma = \frac{(1-\delta)\log(1-\delta)^{-1} + \delta\log(\delta/(2^N-1))^{-1}}{\log 2}$$

and $C_1 = C_1(N) > 0$ is a constant only depending on the dimension N.

Observe that if $N = 1$, then $p = 1$ and $\beta = \gamma$. In [**11**], Y. Heurteaux proved that under the hypothesis of (a), the measure μ must be absolutely continuous with respect to $\mathcal{H}_{\beta+\varepsilon}$, for any $\varepsilon > 0$. Concerning part (b), he showed that μ must be singular with respect to $\mathcal{H}_{\beta-\varepsilon}$ for any $\varepsilon > 0$, without the doubling and singularity assumptions on the measure μ. Our methods also give this result, using the representation (0.4), Doob's theorem in § 1.4 and the theorem of Levy (0.7). Similar remarks apply to Corollaries 6.2 and 6.5. Moreover our arguments are quite flexible and can be used in situations where $\delta_k(x)$ are not uniformly bounded below (see Corollary 6.3). Actually in many special cases one can compute the right Hausdorff measure governing the regularity of the measure in terms of the sequence δ_k. A concrete list is given in § 8. In addition, our arguments can also be used in situations where we only have good doubling behaviour of the measure for some scales (see Corollaries 6.5 and 6.6). Also an analogue of Corollary 6.2 where we have weaker assumptions on the entropy H_n is stated in Theorem 6.1.

Our method can also be applied in the other extreme case, that is, when studying regularity properties of measures which double nicely for small scales. A positive Borel measure μ in Q_0 is called *dyadic symmetric* if $\lim \delta_k = 1 - 2^{-N}$ as $k \to \infty$, or equivalently if

$$\varepsilon_k = \inf \left\{ \frac{\mu(Q_k^i)}{\mu(Q_{k-1}(x))} : x \in Q_0, i = 1, \dots, 2^N \right\} \to 2^{-N} \tag{0.2}$$

as $k \to \infty$. A well-known result of L. Carleson [**5**] (see also [**9**]) states that μ is absolutely continuous with respect to N-dimensional Lebesgue measure if

$$\sum_{k=1}^{\infty} (2^{-N} - \varepsilon_k)^2 < \infty.$$

Moreover this result is sharp. Our techniques give the following quantitative version of Carleson's result.

THEOREM 8.2. (a) *Let μ be a dyadic symmetric measure in Q_0 and let ε_k be the quantities defined in* (0.2). *Then, there exists a constant $C = C(N)$ such that μ is absolutely continuous with respect to $\mathcal{H}_\phi$ for any measure function ϕ satisfying*

$$\phi(2^{-n}) = 2^{-nN} \exp\left(C \sum_{k=1}^{n} (2^{-N} - \varepsilon_k)^2 \right).$$

(b) *Let μ be a dyadic symmetric measure in Q_0 which is singular with respect to Lebesgue measure. Let ε_k be the quantities defined in* (0.2). *Then, there exists a constant $C_1 = C_1(N)$ only depending on the dimension N such that μ is singular with respect to $\mathcal{H}_\phi$ for any measure function ϕ satisfying*

$$\phi(2^{-n}) = 2^{-nN} \exp\left(C_1 \sum_{k=1}^{n} (2^{-N} - \varepsilon_k)^2 \right).$$

Again, concrete examples are given in § 8.

Our results have the following application to the distortion of homeomorphisms of the real line. (We recall that $M(t)$ stands for the function controlling the distortion of the homeomorphism, as in example (c) at the beginning of the introduction.)

COROLLARY 9.1. (a) *If $M(t) \leqslant M < \infty$, that is, if h is M-quasi-symmetric, then μ_h is absolutely continuous with respect to $\mathcal{H}_{\phi_M}$, where*

$$\phi_M(t) = t^{\beta(M)} \exp\{C(\log t^{-1} \log\log\log t^{-1})^{1/2}\}.$$

Here

$$\beta(M) = \frac{1}{M+1} \log_2(M+1) + \frac{M}{M+1} \log_2 \left(\frac{M+1}{M} \right)$$

and C is some fixed absolute constant.

(b) *Assume that*

$$\lim_{t\to 0} \frac{\int_t^1 \frac{\log M(s)}{M(s)} \frac{ds}{s}}{(\log\log t^{-1})(\log\log\log\log t^{-1})} = \infty.$$

Then μ *is absolutely continuous with respect to* $\mathcal{H}_\phi$, *where*

$$\phi(t) = \exp\left\{ -C\int_t^1 \frac{\log M(s)}{M(s)} \frac{ds}{s} \right\}$$

and C *is some fixed absolute constant.*

Essentially the same kind of results can be restated in terms of the harmonic measure for the elliptic operator obtained as a pull-back of the Laplacian with the Beurling–Ahlfors extension of h to the upper half-space. We refer to § 9 for further details.

We now explain our methods. Our analysis is modelled on the following well-known example which is extremal in Heurteaux's result. For fixed $0 < \lambda \leqslant \frac{1}{2}$, we will define a probability measure μ on $[0,1]$ by induction. Put $\mu[0,1] = 1$ and assume that the mass of μ has been defined on all dyadic intervals of generation k. If I is such an interval and $I = I_+ \cup I_-$ its decomposition into its right and left halves, we define

$$\mu(I_+) = \lambda\mu(I), \qquad \mu(I_-) = (1-\lambda)\mu(I).$$

So, the mass of μ on intervals of generation $k+1$ is defined. Iterating this construction, we define a Borel probability measure μ on $[0,1]$ so that

$$\mu(I_n(x)) = \lambda^{\nu(n,x)}(1-\lambda)^{n-\nu(n,x)}, \tag{0.3}$$

where $\nu(n,x)$ is the number of dyadic intervals of length bigger than 2^{-n} containing x which are at right position. Observe that in this case the entropy $H_n(x) = nh(\lambda)$, where $h(\lambda) = \lambda \log \lambda^{-1} + (1-\lambda)\log(1-\lambda)^{-1}$. Observe also that the pointwise estimate $\mu(I_n(x)) \leqslant (1-\lambda)^n$, for $x \in [0,1]$, shows that μ is absolutely continuous with respect to $\mathcal{H}_\alpha$ for $\alpha < \log_2(1-\lambda)^{-1}$. However this result can be substantially improved using the law of large numbers which says that

$$\lim_{n\to\infty} \frac{\nu(n,x)}{n} = \lambda \quad \mu\text{-a.e. } x \in [0,1].$$

So,

$$\lim_{n\to\infty} \frac{\log \mu(I_n(x))^{-1}}{n} = h(\lambda) \quad \mu\text{-a.e. } x \in [0,1],$$

and we deduce that μ is absolutely continuous with respect to $\mathcal{H}_\beta$,

$$\beta < h(\lambda)/\log 2$$

and is singular with respect to $\mathcal{H}_\gamma$, and

$$\gamma > h(\lambda)/\log 2.$$

This can also be deduced from the ergodic theorem once it is observed that μ is ergodic with respect to the shift transformation.

Our analysis uses martingales with respect to a probability measure on Q_0 and the law of the iterated logarithm relating the size of the martingale with the size of its quadratic characteristic. These notions and results are reviewed in § 1.

Given a positive Borel measure μ in Q_0, our method consists of proving the following analogue of identity (0.3):

$$\mu(Q_n(x)) = \exp(S_n(x) - H_n(x)), \quad \mu\text{-a.e. } x \in Q_0, \tag{0.4}$$

where (S_n, μ) is a dyadic martingale. This is just Doob's decomposition for the μ-submartingale $\log \mu(Q_n(x))/|Q_n(x)|$. In particular,

$$\frac{\mu(Q_n(x))}{|Q_n(x)|} = \exp\{S_n(x) + Nn \log 2 - H_n(x)\}. \tag{0.5}$$

It turns out that S_n is completely explicit and one can prove that its quadratic characteristic $\langle S\rangle_n$ is pointwise estimated as follows:

$$\langle S\rangle_n(x) \leqslant 2(nN \log 2 - H_n(x)). \tag{0.6}$$

Actually, the two terms are comparable if μ is a dyadic doubling measure, that is, if $\delta(x) = \inf_k \delta_k(x)$ is uniformly bounded below. In this general situation, some growth estimate of S_n, playing the role of the law of large numbers in the model case described above, is needed. From a theorem of Levy [**23**, p. 157; **20**, p. 519], it follows that

$$\lim_{n\to\infty} \frac{|S_n(x)|}{\langle S\rangle_n(x)} = 0 \tag{0.7}$$

at μ-a.e. $x \in \{x \in Q_0 : \langle S\rangle_\infty(x) = \infty\}$.

Now, (0.5), (0.6) and (0.7) tell us that at μ-almost every point

$$x \in \{x \in Q_0 : \langle S\rangle_\infty(x) = \infty\}$$

the densities $\mu(Q_n(x))/|Q_n(x)|$ behave as $\exp\{Nn \log 2 - H_n(x)\}$ if n is sufficiently large.

Theorem 4.3. *Let μ be a probability measure on the unit cube Q_0 of $\mathbb{R}^N$. Then:*

(a) *μ is singular with respect to Lebesgue measure if and only if*

$$\lim_{n\to\infty} (nN \log 2 - H_n(x)) = \infty, \quad \mu\text{-a.e. } x \in Q_0;$$

(b) *μ is absolutely continuous with respect to Lebesgue measure if and only if*

$$\lim_{n\to\infty} (nN \log 2 - H_n(x)) < \infty, \quad \mu\text{-a.e. } x \in Q_0.$$

However, the law of the iterated logarithm (LIL) shows that the estimate (0.7) can be typically improved. We have the following LIL.

Corollary 3.8. *Let μ be a probability measure in Q_0 and (S_n, μ) the martingale arising in* (0.4). *Then*

(a)
$$\varlimsup_{n\to\infty} \frac{S_n(x)}{\sqrt{\langle S\rangle_n(x)\log\log\langle S\rangle_n(x)}} \leqslant C_1 \tag{0.8}$$

μ-a.e. at $\{x \in Q_0 : \langle S\rangle_\infty(x) = \infty\}$; here C_1 is a constant depending on the dimension;

(b) *if μ is a dyadic doubling measure, we also have*
$$\varlimsup_{n\to\infty} \frac{S_n(x)}{\sqrt{\langle S\rangle_n(x)\log\log\langle S\rangle_n(x)}} \geqslant C_2 \tag{0.9}$$

μ-a.e. at $\{x \in Q_0 : \langle S\rangle_\infty(x) = \infty\}$; here C_2 is a constant depending on the dimension and on the doubling behaviour of μ.

For general dyadic martingales, the LIL only holds under certain size restrictions on the increments of the martingale. Actually, it turns out that the versions of the LIL that one can find in the literature require either the boundedness of the increments or some strong growth assumptions which would be quite restrictive in our setting [**22**]. Nevertheless, Corollary 3.8(a) states that the upper bound of the LIL holds for the special martingale S_n without any assumption on the increments. Therefore, even though we start from a general measure μ, the fact that the martingale S_n is constructed in a very special way seems to play an essential role in our arguments. We do not know whether this is a sign of a more general fact.

From the LIL, it is natural to expect that if the total entropy is large, then the iterated logarithm term should be small compared to the total entropy. In Theorem 5.6, we will show that if
$$\frac{H_n(x)}{(\log n)(\log\log\log n)} \to \infty \quad \text{as } n \to \infty \tag{0.10}$$
then
$$\sqrt{\langle S\rangle_n(x)\log\log\langle S\rangle_n(x)} = o(H_n(x)).$$
Therefore, the LIL can be used to show that $S_n(x) = o(H_n(x))$ and hence
$$\mu(Q_n(x)) = O(\exp(-(1+o(1))H_n(x)))$$
at μ-a.e. $x \in Q_0$ satisfying (0.10).

Corollary 6.2(a) follows from these considerations. Part (b) follows from similar arguments using the lower bound in the law of the iterated logarithm.

The paper is organized as follows. Section 1 contains background about martingales, quadratic characteristic and the law of the iterated logarithm. In § 2, the martingale S_n for which the main identity (0.4) holds is introduced. Also, its quadratic characteristic is computed. In § 3 we show that the law of the iterated logarithm holds for S_n, without any additional hypothesis on its growth. Section 4 is devoted to the proof of Theorem 4.3. In § 5, the quadratic characteristic $\langle S\rangle_n$, the entropy H_n and the uniform quantity δ_n are compared. The corresponding estimates are applied in § 6, which contains the proof of Theorem 6.1 and its corollaries. In § 7 a continuous version of the results is presented. Section 8 contains examples and the proof of Theorem 8.2. Finally, in § 9 we give an application of the preceding results to distortion of homeomorphisms of the real line and to the size of harmonic measure of certain degenerate elliptic equations, which are typically non-doubling.

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1. *Notation and background about martingales and Hausdorff measures*

1.1. *Notation*

Let $Q_0 = [0,1)^N \subset \mathbb{R}^N$ be the (half-open) unit cube in $\mathbb{R}^N$ and denote by $\mathcal{F}_k$ the family of all (half-open) dyadic subcubes of Q_0 of the generation k, that is, all cubes of the form

$$\prod_{i=1}^{2^N} [(m_i - 1)2^{-k}, m_i 2^{-k}) \quad (1 \leqslant m_i \leqslant 2^k)$$

for $i = 1, \dots, 2^N$. Given $Q_{k-1} \in \mathcal{F}_{k-1}$, there is a natural dyadic decomposition $Q_{k-1} = \bigcup_{i=1}^{2^N} Q_k^i$, where $\{Q_k^i\}_{i=1}^{2^N}$ are disjoint and $Q_k^i \in \mathcal{F}_k$. We will refer to such $\{Q_k^i\}$ as the 'dyadic sons' of the generation k of Q_{k-1}. Also, we remind the reader that, for $x \in Q_0$, we will denote by $Q_k(x)$ the unique $Q_k \in \mathcal{F}_k$ which contains x.

Finally, from now on, $|\ |$ will denote N-dimensional Lebesgue measure.

1.2. *Conditional expectation*

Note that the measurable functions with respect to the σ-algebra $\widetilde{\mathcal{F}}_k$ generated by $\mathcal{F}_k$ are just the step functions in Q_0 that are constant on each cube of $\mathcal{F}_k$. Now, given a probability measure μ in Q_0 and $f \in L^1(Q_0, \mu)$, define the conditional expectation of f with respect to $\mathcal{F}_k$, denoted by $E[f/\mathcal{F}_k, \mu]$, as the unique (up to sets of μ-measure zero) $\widetilde{\mathcal{F}}_k$-measurable function such that

$$\int_Q f \, d\mu = \int_Q E[f/\mathcal{F}_k, \mu] \, d\mu$$

for any $Q \in \mathcal{F}_k$. Note that $E[f/\mathcal{F}_k, \mu]$ is a step function, constant on the cubes of the generation k such that

$$E[f/\mathcal{F}_k, \mu]|_Q = \frac{1}{\mu(Q)} \int_Q f \, d\mu \quad (Q \in \mathcal{F}_k)$$

whenever $\mu(Q) \neq 0$ (otherwise any value would work).

1.3. *Martingales*

Given a real-valued sequence of functions $\{S_n\}_{n=0}^{\infty}$ in Q_0, we say that (S_n, μ) is a *dyadic martingale* if the following two conditions hold:

(i) each S_n is constant on any dyadic cube of the generation n, that is, S_n is $\widetilde{\mathcal{F}}_n$-measurable for any n;

(ii) $E[S_n/\mathcal{F}_{n-1},\mu] = S_{n-1}$ $(n = 1, 2, \ldots)$ or, equivalently,

$$S_{n-1}|_{Q_{n-1}}\mu(Q_{n-1}) = \sum_{i=1}^{2^N} S_n|_{Q_n^i}\mu(Q_n^i)$$

whenever $Q_{n-1} \in \mathcal{F}_{n-1}$ and $\{Q_n^i\}_{i=1}^{2^N}$ are its dyadic sons.

The differences $X_n = S_n - S_{n-1}$ will be called the *increments* of the martingale (S_n).

1.4. *Quadratic characteristic*

If (S_n, μ) is a martingale in Q_0, its quadratic characteristic $\langle S\rangle_n$ is defined to be the following sequence of functions:

$$\langle S\rangle_n = \sum_{k=1}^{n} E[(S_k - S_{k-1})^2/\mathcal{F}_{k-1},\mu].$$

Note that each $\langle S\rangle_k$ is $\widetilde{\mathcal{F}}_{k-1}$-measurable, that is, constant on the dyadic cubes of the generation $k-1$. Sequences with such a property are called *predictable*.

If $Q_{k-1} \in \mathcal{F}_{k-1}$, then

$$E[(S_k - S_{k-1})^2/\mathcal{F}_{k-1},\mu] = \sum_{i=1}^{2^N} \frac{\mu(Q_k^i)}{\mu(Q_{k-1})}(S_k^i - S_{k-1})^2$$

provided $\mu(Q_{k-1}) \neq 0$ (otherwise, the value of the conditional expectation is irrelevant). Therefore

$$\langle S\rangle_n(x) = \sum_{k=1}^{n}\sum_{i=1}^{2^N} \frac{\mu(Q_k^i)}{\mu(Q_{k-1})}(S_k^i - S_{k-1})^2 \quad \mu\text{-a.e. } x \in Q_0$$

where $\{Q_k\}$ is the dyadic tower of cubes containing x, $\{Q_k^i\}$ are the dyadic sons of Q_{k-1}, $S_{k-1} = S_{k-1}|_{Q_{k-1}}$ and $S_{k-1}^i = S_k|_{Q_k^i}$. In fact, $\langle S\rangle_n$ is the unique non-decreasing predictable sequence such that $S_n^2 = \langle S\rangle_n + M_n$ for some martingale M_n and all $n \geqslant 0$ (Doob's decomposition). We will put, hereafter, $\langle S\rangle_\infty = \lim_{n\to\infty}\langle S\rangle_n$.

It is well known that, in many aspects, the quadratic characteristic $\langle S\rangle_n$ determines the structure and properties of the martingale S_n. For instance we have the following result [**23**, p. 65].

THEOREM (Doob). *Let (S_n, μ) be a martingale in Q_0. Then*

$$\{x \in Q_0 : \langle S\rangle_\infty(x) < \infty\} \subset \{x \in Q_0 : \exists \lim_{n\to\infty} S_n(x) < \infty\} \quad \mu\text{-a.e.},$$

that is, $\lim_{n\to\infty} S_n(x)$ *exists at μ-almost every point x where $\langle S\rangle_\infty(x) < \infty$.*

1.5. *Law of the iterated logarithm*

The precise asymptotic growth of a martingale is closely related to its quadratic characteristic. Given a martingale (S_n, μ), we will say that it obeys the *upper bound* (respectively *lower bound*) of the law of the iterated logarithm (LIL) if there is $0 < C < \infty$ such that

$$\overline{\lim_{n\to\infty}} \frac{S_n(x)}{\sqrt{\langle S\rangle_n(x)\log\log\langle S\rangle_n(x)}} \leqslant C$$

(respectively $\geqslant C$) for μ-a.e. x in the set $\{\langle S\rangle_\infty = \infty\}$. The lower bound is usually harder to establish, whereas we will be mainly concerned with the upper bound.

The law of the iterated logarithm has a long history. It was first proved by Khintchine [**13**] for sums of independent, identically distributed, uniformly bounded random variables. Kolmogorov [**14**] and Hartman and Wintner [**10**] generalized it to wider classes of independent variables. The first extension to a martingale setting was due to Levy [**15**]. More recent extensions to martingales have been obtained by Strassen [**24**] and Stout [**22**, **23**]. For general martingales, some boundedness condition on the differences is required to prove the LIL, even the upper bound (see [**22**, **23**]). However, we will only be interested in the upper bound of the LIL for a special class of dyadic martingales. In this particular situation, we will see that a certain elementary exponential inequality allows one to drop any boundedness restriction on the differences, which is much more convenient for our purposes. The argument follows Makarov [**17**] and we will give the details in § 3.

1.6. *Some elementary facts about Hausdorff measures*

A measure function is a non-decreasing, positive, continuous function

$$\phi : [0, +\infty) \to [0, +\infty)$$

with $\phi(0) = 0$. Given a measure function ϕ, the Hausdorff ϕ-content of $E \subset Q_0$ is defined as

$$M_\phi(E) = \inf\left\{ \sum_j \phi(r(Q_j)) \right\}$$

where the infimum is taken over all coverings of E by cubes $\{Q_j\}$, each $r(Q_j)$ being the sidelength of Q_j. With the same notation, the Hausdorff ϕ-measure of E is defined as

$$\mathcal{H}_\phi(E) = \lim_{\delta\to 0} \inf\left\{ \sum_j \phi(r(Q_j)) : r(Q_j) \leqslant \delta \right\}.$$

It is easy to check that

$$M_\phi(E) = 0 \iff \mathcal{H}_\phi(E) = 0.$$

The advantage of the Hausdorff content is that it is always finite. This is the reason for which, throughout this paper, we will restrict our attention to Hausdorff contents instead of Hausdorff measures.

If $\phi(t) = t^\alpha$ ($\alpha > 0$), then we simply write $M_\alpha(E)$ (respectively $\mathcal{H}_\alpha$) and we refer to it as the *Hausdorff α-content* of E (respectively α-dimensional Hausdorff measure). Note that $M_N(\,\cdot\,)$ is just a multiple of the usual N-dimensional Lebesgue outer measure. The *Hausdorff dimension* of E is defined as

$$\dim E = \inf\{\alpha \geqslant 0 : M_\alpha(E) = 0\},$$

so $M_\alpha(E) > 0$ implies $\dim E \geqslant \alpha$. Given two measure functions ϕ_1 and ϕ_2 it is easy to check that $M_{\phi_1}(E) \asymp M_{\phi_2}(E)$ if $\phi_1(t) \asymp \phi_2(t)$ near $t = 0$. Also, $M_{\phi_1}(E) > 0$ implies $M_{\phi_2}(E) > 0$ if $\underline{\lim}_{\,t\to 0}(\phi_2(t)/\phi_1(t)) > 0$ and $M_{\phi_1}(E) = 0$ implies $M_{\phi_2}(E) = 0$ if $\overline{\lim}_{t\to 0}(\phi_2(t)/\phi_1(t)) < +\infty$.

As mentioned in the introduction, we will be interested in determining when a given measure μ is absolutely continuous with respect to the Hausdorff measure associated to a certain measure function ϕ. In this respect, note that if μ satisfies a global estimate of the form

$$\mu(Q) \leqslant c\phi(r(Q))$$

for any cube $Q \subset Q_0$, and some positive constant c, then $\mu \ll \mathcal{H}_\phi$, directly from the definition. Standard measure-theoretical arguments show that a local control is actually sufficient to get the same conclusion.

PROPOSITION 1.1. *With the notation above,*
(a) *if*

$$\limsup_{n\to\infty} \frac{\mu(Q_n(x))}{\phi(2^{-n})} < \infty \quad \text{for } \mu\text{-a.e. } x,$$

then $\mu \ll \mathcal{H}_\phi$;
(b) *if*

$$\limsup_{n\to\infty} \frac{\mu(Q_n(x))}{\phi(2^{-n})} = \infty \quad \text{for } \mu\text{-a.e. } x,$$

then μ *is singular with respect to* $\mathcal{H}_\phi$.

Proof. (a) Choose E, with $\mu(E) > 0$. By localization, it is enough to assume that $\mu(Q_n(x)) \leqslant M\phi(2^{-n})$ for all $x \in E$, each $n \in \mathbb{N}$ and some $M > 0$. Suppose that $E \subset \bigcup_k Q_k$, where the Q_k are cubes. Fix such a Q_k and let $n \in \mathbb{N}$ be such that $2^{-n} \leqslant r(Q_k) < 2^{-(n-1)}$. Then Q_k is covered by, at most, 2^N cubes $Q_k^1, \ldots, Q_k^{2^N} \in \mathcal{F}_n$ and $\{Q_k^j : j = 1, 2, \ldots, 2^N, k = 1, 2, \ldots\}$ is a covering of E by dyadic cubes. We can also assume that $Q_k^j \cap E \neq \emptyset$ for all k and j, so $\phi(r(Q_k^j)) \geqslant M^{-1}\mu(Q_k^j)$. Then,

$$\sum_{k=1}^{\infty} \phi(r(Q_k)) \geqslant 2^{-N} \sum_{k=1}^{\infty} \sum_{j=1}^{2^N} \phi(r(Q_k^j)) \geqslant M^{-1} 2^{-N} \mu(E).$$

This shows that $M_\phi(E) > 0$.
(b) Let A be the set of points x of the unit square Q_0 for which

$$\limsup_{n\to\infty} \frac{\mu(Q_n(x))}{\phi(2^{-n})} = \infty.$$

So, $\mu(A) = \mu(Q_0)$. Given $M > 0$, let A_M be the set of points $x \in Q_0$ such that

$$\limsup_{n\to\infty} \frac{\mu(Q_n(x))}{\phi(2^{-n})} > M.$$

Hence, if $x \in A_M$, there exists n such that $\mu(Q_n(x)) > M\phi(2^{-n})$. Denote $Q(x) = Q_n(x)$. Then

$$A_M \subset \bigcup_{x \in A_M} Q(x) = \bigcup_j Q_j,$$

where $\{Q_j\}$ is a collection of pairwise disjoint dyadic cubes. Then

$$\mathcal{H}_\phi(A_M) \leqslant \sum \phi(r(Q_j)) \leqslant M^{-1} \sum_j \mu(Q_j) \leqslant M^{-1}\mu(Q_0).$$

Therefore $\mathcal{H}_\phi(A) = 0$. □

Remark. It is well known that Hausdorff measures allow one to compare sets of Lebesgue measure zero. If $M_\phi(E) > 0$, then the smaller $\phi(t)$ is when t approaches 0, the bigger E is, in this sense. In this paper we are specially concerned with measures which 'live' on sets of Hausdorff dimension smaller than N, so this means that we will mainly deal with measure functions such that $\lim_{t\to 0}(\phi(t)/t^N) = \infty$.

2. *The logarithmic transform*

Suppose we are given a probability measure μ in Q_0. For each $Q \in \mathcal{F}_n$, define

$$Z_n|_Q = \mu(Q)/|Q|.$$

Then (Z_n) is an $\widetilde{\mathcal{F}}_n$-measurable sequence of functions which measures the density of μ, with respect to Lebesgue measure, for any scale. The following proposition clarifies the structure of the logarithmic transform $\log Z_n$.

Proposition 2.1. *If μ and Z_n are as above then there are a martingale (S_n, μ) in Q_0 and a non-negative, non-decreasing, predictable sequence P_n such that*

$$\log Z_n = S_n + P_n \quad \mu\text{-a.e.}$$

In particular, $\mu(Q_n) = \exp\{S_n - H_n\}$ for any dyadic cube Q_n of the generation n. In fact, except for additive constants, one has

$$S_n(x) = \sum_{k=1}^{n} \sum_{i=1}^{2^N} \frac{\mu(Q_k^i)}{\mu(Q_{k-1})} \log\left(\frac{\mu(Q_k)}{\mu(Q_k^i)}\right)$$

and

$$\begin{aligned} P_n(x) &= \sum_{k=1}^{n} \sum_{i=1}^{2^N} \frac{\mu(Q_k^i)}{\mu(Q_{k-1})} \log\left(2^N \frac{\mu(Q_k^i)}{\mu(Q_{k-1})}\right) \\ &= n\log 2^N - \sum_{k=1}^{n} \sum_{i=1}^{2^N} \frac{\mu(Q_k^i)}{\mu(Q_{k-1})} \log\left(\frac{\mu(Q_{k-1})}{\mu(Q_k^i)}\right) \\ &= nN\log 2 - H_n(x), \end{aligned}$$

where, for each k, Q_k is the dyadic cube of the generation k that contains x and $\{Q_k^i\}_{i=1}^{2^N}$ are its 2^N dyadic brothers.

Proof. Fix $Q_{k-1} \in \mathcal{F}_{k-1}$, with $\mu(Q_{k-1}) > 0$ and let $\{Q_k^i\} \subset \mathcal{F}_k$ be its dyadic sons. Suppose that $\log Z_k = S_k + P_k$, with (S_k, μ) martingale and (P_k) predictable. Let S_k^i, Z_k^i, P_k (respectively S_{k-1}, Z_{k-1}, P_{k-1}) be the restrictions of S_k, Z_k, P_k to Q_k^i (respectively Q_{k-1}). Predictability and the martingale condition imply then that

$$0 = \sum_{i=1}^{2^N} \mu(Q_k^i)\left(\log \frac{Z_k^i}{Z_{k-1}} - (P_k - P_{k-1})\right)$$
$$= \sum_{i=1}^{2^N} \mu(Q_k^i) \log \left(2^N \frac{\mu(Q_k^i)}{\mu(Q_{k-1})} \right) - (P_k - P_{k-1})\mu(Q_{k-1}).$$

So,

$$P_k - P_{k-1} = \sum_{i=1}^{2^N} \frac{\mu(Q_k^i)}{\mu(Q_{k-1})} \log \left(2^N \frac{\mu(Q_k^i)}{\mu(Q_{k-1})} \right)$$

and the proposition follows with the choice of P_n in the statement. Positivity of P_n is a consequence of the following elementary fact: if $\lambda_i \geqslant 0$ and $\sum_{i=1}^m \lambda_i = 1$, then $\sum_{i=1}^m \lambda_i \log \lambda_i^{-1} \leqslant \log m$. □

Remark 2.1. Since we will only be interested in properties of μ which hold μ-a.e., the relevant values of x for us are those for which $\mu(Q_k(x)) > 0$ for all k. In this case, note that all the Q_k^i such that $\mu(Q_k^i) = 0$ are not relevant in the definition of S_n and P_n in the proposition above. Also, in the rest of the paper, all the inequalities must be understood μ-a.e.

Remark 2.2. If $N = 1$, then

$$S_n(x) = \sum_{k=1}^{n} \frac{\mu(I_k')}{\mu(I_{k-1})} \log \left(\frac{\mu(I_k)}{\mu(I_k')} \right),$$

and

$$P_n(x) = n \log 2 - \sum_{k=1}^{n} \left[\frac{\mu(I_k)}{\mu(I_{k-1})} \log \left(\frac{\mu(I_{k-1})}{\mu(I_k)} \right) + \frac{\mu(I_k')}{\mu(I_{k-1})} \log \left(\frac{\mu(I_{k-1})}{\mu(I_k')} \right) \right]$$

where I_k is the dyadic interval of $\mathcal{F}_k$ which contains x and I_k' is its dyadic brother.

Remark 2.3. The quadratic characteristic of the martingale (S_n, μ) is given by

$$\langle S \rangle_n(x) = \sum_{k=1}^{n} \sum_{j=1}^{2^N} \frac{\mu(Q_k^j)}{\mu(Q_{k-1})} \left[\sum_{i=1}^{2^N} \frac{\mu(Q_k^i)}{\mu(Q_{k-1})} \log \left(\frac{\mu(Q_k^j)}{\mu(Q_k^i)} \right) \right]^2$$

where, as before, $\{Q_k\}$ is the 'dyadic tower' of cubes containing x and $Q_{k-1} = \bigcup_{i=1}^{2^N} Q_k^i$ is the dyadic decomposition of Q_{k-1}. An elementary computation shows that

$$\langle S \rangle_n(x) = \sum_{k=1}^{n} \left[\sum_{i=1}^{2^N} \frac{\mu(Q_k^i)}{\mu(Q_{k-1})} \log^2 \left(\frac{\mu(Q_{k-1})}{\mu(Q_k^i)} \right) - \left(\sum_{i=1}^{2^N} \frac{\mu(Q_k^i)}{\mu(Q_{k-1})} \log \left(\frac{\mu(Q_{k-1})}{\mu(Q_k^i)} \right) \right)^2 \right].$$

Remark 2.4. Observe that if μ is Lebesgue measure in Q_0, then $S_n = P_n = 0$. Actually, $P_n(x)$ measures how far μ is from Lebesgue measure according to the doubling behaviour of μ on cubes containing x. Furthermore, P_n can be understood in terms of means. Fix k and let $\lambda_k^i = \mu(Q_k^i)/\mu(Q_{k-1})$. Then the arithmetic mean of the λ_i is 2^{-N} and

$$P_n = \sum_{k=1}^{n} \log\left(\frac{\prod_{i=1}^{2^N} (\lambda_k^i)^{\lambda_k^i}}{2^{-N}}\right).$$

3. *On the upper bound of the LIL for the class of martingales arising in* § 2

Definition. We say that a dyadic martingale (S_n, μ) in $Q_0 \subset \mathbb{R}^N$ *satisfies condition* $\mathcal{E}$ *with constant* $c > 0$ if

$$(\exp(tS_n - ct^2\langle S\rangle_n), \mu)$$

is a supermartingale for any $t > 0$.

From § 1, we remember that, for a dyadic martingale $S_n = \sum_{k=1}^n X_k$ with respect to a probability measure μ in Q_0, then

$$\langle S\rangle_n(x) = \sum_{k=1}^{n} \sum_{j=1}^{2^N} \lambda_k^j (X_k^j)^2$$

where $\lambda_k^j = \mu(Q_k^j)/\mu(Q_{k-1})$, $X_k^j = S_k - S_{k-1}|_{Q_k^j}$, $\{Q_k\}$ is the dyadic tower of cubes containing x and $\{Q_k^j\}$ are the dyadic sons of Q_{k-1}.

Proposition 3.1. *Let (S_n, μ) be a dyadic martingale in Q_0. Then (S_n, μ) satisfies condition $\mathcal{E}$ with constant $c > 0$ if and only if, for each $k \in \mathbb{N}$ and each $Q_{k-1} \in \mathcal{F}_{k-1}$, the following inequality holds:*

$$\sum_{j=1}^{2^N} \lambda_k^j \exp\{tX_k^j\} \leqslant \exp\left\{ct^2 \sum_{j=1}^{2^N} \lambda_k^j (X_k^j)^2\right\}, \quad \text{for } t > 0,$$

where λ_k^j and X_k^j are as above for $j = 1, \ldots, 2^N$.

Proof. The inequality stated in the proposition is equivalent to

$$\int_{Q_{k-1}} \exp(tX_k - ct^2(\langle S\rangle_k - \langle S\rangle_{k-1}))\, d\mu \leqslant \mu(Q_{k-1})$$

which is in fact equivalent to

$$E[\exp(tS_k - ct^2\langle S\rangle_k)/\mathcal{F}_{k-1}] \leqslant \exp(tS_{k-1} - ct^2\langle S\rangle_{k-1})$$

which is just condition $\mathcal{E}$ with constant c. □

The first part of the following theorem is the key point of the section.

Theorem 3.2. (a) *There is $c = c(N) > 0$ such that, if μ is any probability measure in $Q_0 \subset \mathbb{R}^N$, and (S_n, μ) is the dyadic martingale associated to μ as in* § 2, *then (S_n, μ) satisfies condition $\mathcal{E}$ with constant c.*

(b) *If* (S_n, μ) *is a dyadic martingale in* $Q_0 \subset \mathbb{R}^N$ *and* μ *is dyadic doubling then* S_n *satisfies condition* $\mathcal{E}$ *with a constant* c *depending only on* N *and the doubling constant of* μ*.*

Theorem 3.2 is a consequence of the following two lemmas, which will be proved at the end of the section.

Lemma 3.3. *Let* $m \in \mathbb{N}$, $m \geqslant 2$. *Then there exists a constant* $c = c(m) > 0$ *such that for any* $t \geqslant 0$ *one has*

$$\sum_{j=1}^{m} \lambda_j \exp(tX_j) \leqslant \exp\left(ct^2 \sum_{j=1}^{m} \lambda_j X_j^2\right),$$

where $0 \leqslant \lambda_j \leqslant 1$ *for* $j = 1, \ldots, m$, $\sum \lambda_j = 1$, $H = \sum \lambda_j \log \lambda_j^{-1}$ *and*

$$X_j = \log \lambda_j + H = \sum_i \lambda_i \log(\lambda_j/\lambda_i).$$

Lemma 3.4. *Let* $m \in \mathbb{N}$ *and* $0 < \delta \leqslant 1/m$. *Suppose that* $\lambda_i \geqslant \delta$ *for* $1 \leqslant i \leqslant m$, *and* $\sum_{i=1}^{m} \lambda_i = 1$. *If* $(x_i)_{i=1}^{m}$ *are real numbers such that* $\sum_{i=1}^{m} \lambda_i x_i = 0$ *then, for any* $t \geqslant 0$ *and any* $c \geqslant 3/(4\delta)$, *we have*

$$\sum_{i=1}^{m} \lambda_i \exp(tx_i) \leqslant \exp\left(ct^2 \sum_{i=1}^{m} \lambda_i x_i^2\right).$$

Remarks. 1. The use of continuous exponential inequalities has been a usual tool to prove the upper bound of the LIL for general martingales [**22**, **23**]. In the dyadic case, what we have called 'condition $\mathcal{E}$' is a reformulation of the 'exponential transformation', extensively used by Makarov in [**17**]. We will show that the discrete exponential inequality given by Lemma 3.3 is all that we need to get an upper bound of the LIL for the special class of martingales described in § 2. On the other hand, we believe that part (b) of Theorem 3.2 is probably well known and has previously appeared in the literature in more or less explicit ways [**1**, **6**].

2. An easy observation that will be useful in the proof of the LIL is the fact that condition $\mathcal{E}$ is preserved by stopping-times. More precisely, if (S_n, μ) is a dyadic martingale in Q_0 that satisfies condition $\mathcal{E}$ with constant $c > 0$ and τ is any stopping-time in Q_0, then the stopped martingale

$$S_n^\tau(x) = S_{\min\{\tau(x), n\}}(x)$$

also satisfies condition $\mathcal{E}$ with the same constant c. This is obvious from the definition of stopping-time.

3. The proofs of Lemmas 3.3 and 3.4 give explicit values of the constant c which are not necessarily the best ones.

Now we will see that condition $\mathcal{E}$ is all that we need to get an upper bound of the LIL. The following lemma is the key for such reduction.

Given a dyadic martingale (S_n, μ), put

$$S_n^* = \max\{S_1, \ldots, S_n\}, \quad S_\infty^* = \lim_n S_n^*, \quad \langle S \rangle_\infty = \lim_n \langle S \rangle_n.$$

Lemma 3.5. *Let (S_n,μ) be a dyadic martingale in $Q_0 \subset \mathbb{R}^N$ satisfying condition $\mathcal{E}$ with constant $c>0$. Suppose that $S_0=0$. Then, for any $M>0$, $N>0$,*

$$\mu\{x \in Q_0 : \exists n \in \mathbb{N},\ S_n^*(x) > M,\ \langle S\rangle_n(x) \leqslant N\} \leqslant \exp\left\{-\frac{M^2}{4cN}\right\}.$$

Proof. Given $M>0$, let τ be the stopping-time defined by

$$\tau(x)=k \iff S_1(x)\leqslant M,\ \ldots,\ S_{k-1}(x)\leqslant M,\ S_k(x)>M$$

and let (S_n^τ) be the stopped martingale. Let

$$A=\{x : \exists n \in \mathbb{N},\ S_n^*(x)>M,\ \langle S\rangle_n \leqslant N\}.$$

From Remark 2, (S_n^τ) satisfies condition $\mathcal{E}$ with constant c. From the supermartingale assumption,

$$1=\int_{Q_0}\exp(tS_0^\tau - ct^2\langle S^\tau\rangle_0)\,d\mu \geqslant \int_{Q_0}\exp(tS_n^\tau - ct^2\langle S^\tau\rangle_n)\,d\mu \geqslant \int_A \exp(tS_n^\tau - ct^2\langle S^\tau\rangle_n)\,d\mu$$

for any $t>0$ and each $n\in\mathbb{N}$.

Since $\langle S^\tau\rangle_n \leqslant \langle S\rangle_n$, it follows from the definition of τ that if $x\in A$, then $S_n^\tau(x)\geqslant M$ and $\langle S^\tau\rangle_n(x)\leqslant N$ eventually. Therefore, by Fatou's lemma,

$$\exp(tM-ct^2N)\mu(A) \leqslant \varliminf_n \int_A \exp(tS_n^\tau - ct^2\langle S^\tau\rangle_n)\,d\mu \leqslant 1$$

and the lemma follows with the choice $t=M/2cN$. □

Corollary 3.6. *If (S_n,μ) is as in Lemma 3.5, then*

$$\mu\{x\in Q_0 : S_\infty^*(x)>M, \langle S\rangle_\infty(x)\leqslant N\} \leqslant \exp\left\{-\frac{M^2}{4cN}\right\}.$$

Now, the proof of the LIL (upper bound) is standard [**17**, **1**]. We include it for completeness.

Theorem 3.7 (LIL, upper bound). *Let (S_n,μ) be a dyadic martingale in Q_0 satisfying condition $\mathcal{E}$ with constant $c>0$. Then*

$$\varlimsup_{n\to\infty}\frac{S_n}{\sqrt{4c\langle S\rangle_n \log\log\langle S\rangle_n}} \leqslant 1$$

almost everywhere on the set $\{\langle S\rangle_\infty=\infty\}$.

Proof. Fix $R>1$, and define, for $k\in\mathbb{N}$, the following sets:

$$A=\{\langle S\rangle_\infty=\infty,\ S_n > R\sqrt{4c\langle S\rangle_n\log\log\langle S\rangle_n} \text{ for infinitely many } n\},$$
$$A_k=\big\{x\in Q_0 : \exists n\in\mathbb{N} : R^k\leqslant \langle S\rangle_n(x) < R^{k+1},\ S_n(x) > R\sqrt{4c\langle S\rangle_n(x)\log\log\langle S\rangle_n(x)}\big\}.$$

It is easy to check that

$$A \subset \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

and, since

$$A_k \subset \{x : \exists n : S_n^*(x) > R\sqrt{4cR^k \log\log R^k},\ \langle S\rangle_n(x) \leqslant R^{k+1}\},$$

it follows by Lemma 3.5 that

$$\mu(A_k) \leqslant \exp\left\{ -\frac{R^2 4cR^k \log\log R^k}{4cR^{k+1}} \right\} = \frac{1}{k^R} \exp\{-R\log\log R\}.$$

Therefore,

$$\mu(A) \leqslant e^{-R\log\log R} \sum_{k=n}^{\infty} \frac{1}{k^R}$$

for any $n \in \mathbb{N}$ and, since $R > 1$ it follows that $\mu(A) = 0$. This finishes the proof. $\square$

Corollary 3.8. *There exist a constant $C_1 = C_1(N) > 0$ such that, if (S_n, μ) is the martingale associated to μ as in § 2, then*

$$\varlimsup_{n\to\infty} \frac{S_n(x)}{\sqrt{\langle S\rangle_n(x) \log\log\langle S\rangle_n(x)}} \leqslant C_1$$

for μ-a.e. $x \in \{x \in Q_0 : \langle S\rangle_\infty(x) = \infty\}$. If μ is a dyadic doubling measure, there exists a constant $C_2 > 0$ such that

$$\varlimsup_{n\to\infty} \frac{S_n(x)}{\sqrt{\langle S\rangle_n(x) \log\log\langle S\rangle_n(x)}} \geqslant C_2 \tag{3.1}$$

μ-a.e. at $\{x \in Q_0 : \langle S\rangle_\infty(x) = \infty\}$.

The upper bound follows from the previous result, while the lower bound follows from well-known results [**22**], because the assumption on the measure μ implies that the martingale (S_n, μ) has bounded increments.

Proof of Lemma 3.3. Let $c = c(m)$ be a constant to be determined later. We have to show that the function

$$f(t) = \log\left(\sum_{j=1}^{m} \lambda_j \exp(t\log\lambda_j)\right) - ct^2 \sum_{j=1}^{m} \lambda_j X_j^2 + tH$$

is negative if $t \geqslant 0$. It turns out that $f(0) = f'(0) = 0$ and so it is sufficient to show that $f''(t) \leqslant 0$ if $t \geqslant 0$. Computing $f''(t)$, one is led to prove that for any $t \geqslant 0$,

$$\sum_{i<j} \lambda_i\lambda_j(\log\lambda_i - \log\lambda_j)^2 k_{ij}(t) \leqslant 2c \sum_{j=1}^{m} \lambda_j X_j^2 \tag{3.2}$$

where

$$k_{ij}(t)=\frac{\exp(t\log\lambda_i+t\log\lambda_j)}{(\sum_{j=1}^m\lambda_j\exp(t\log\lambda_j))^2}.$$

Observe that $k_{ij}(0)=1$ and an easy calculation shows that (3.2) holds with $c=\frac{1}{2}$ when $t=0$. Since $\sum_{j=1}^m\lambda_j=1$, we have $0\leqslant\lambda_j\leqslant 1$,

$$\lambda_{j_0}=\max\{\lambda_j : j=1,\dots,m\}\geqslant 1/m$$

and

$$k_{ij}(t)\leqslant\frac{\exp(t\log\lambda_i+t\log\lambda_j)}{\lambda_{j_0}^2\exp(2t\log\lambda_{j_0})}\leqslant m^2.$$

Hence (3.2) holds with $c=\frac{1}{2}m^2$. □

Proof of Lemma 3.4. As above, we will see that if $c\geqslant 3/(4\delta)$ then the function

$$f(t)=\log\left(\sum_{i=1}^m\lambda_i\exp(tx_i)\right)-ct^2\sum_{i=1}^m\lambda_ix_i^2$$

is negative for $t\geqslant 0$. Since $f(0)=f'(0)=0$, it is enough to see that $f''(t)\leqslant 0$ if $t\geqslant 0$. Computation shows that the sign of f'' is given by

$$\sum_{i,j}\lambda_i\lambda_j\exp(t(x_i+x_j))\Big(x_i^2-x_ix_j-2c\sum_k\lambda_kx_k^2\Big).$$

Now observe that

$$x_i^2-x_ix_j-2c\sum_k\lambda_kx_k^2\leqslant x_i^2-x_ix_j-2c(\lambda_ix_i^2+\lambda_jx_j^2)\leqslant 0$$

as soon as $c\geqslant 3/(4\delta)$. □

4. *Characterization of the singularity and absolute continuity of μ in terms of the sequence (P_n)*

In this section we will see that singularity and absolute continuity of the measure μ can be expressed in terms of the boundedness of the sequence (P_n). Note that, since P_n is non-decreasing, there always exists $P_\infty(x)=\lim_{n\to\infty}P_n(x)$. Notation is as in § 2.

The following proposition relates $\langle S\rangle_n$ and P_n.

Proposition 4.1. *With the notation of § 2, we have, for every $n\in\mathbb{N}$:*

(a) $\langle S\rangle_n\leqslant 2P_n$;

(b) *if the measure μ is dyadic doubling, then*

$$cP_n\leqslant\langle S\rangle_n\leqslant 2P_n$$

where $c=c(\mu)>0$ only depends on the doubling constant of μ.

The inequalities must be understood μ-a.e.

Using Proposition 2.1 and Remark 2.3, we see that Proposition 4.1 is a direct consequence of the following elementary lemma.

LEMMA 4.2. *Let $m \in \mathbb{N}$, and $(\lambda_i)_{i=1}^m$ be such that $\lambda_i \geqslant 0$ for $i = 1, \ldots, m$, $\sum_{i=1}^m \lambda_i = 1$. Then:*

(a)
$$\sum_{i=1}^{m} \lambda_i \log^2 \frac{1}{\lambda_i} - \left(\sum_{i=1}^{m} \lambda_i \log \frac{1}{\lambda_i} \right)^2 \leqslant 2 \sum_{i=1}^{m} \lambda_i \log(m\lambda_i);$$

(b) *if, furthermore, $M^{-1} \leqslant \lambda_i / \lambda_j \leqslant M$ for some $M > 0$ and any i and j, then there exists $c > 0$, only depending on M, such that*
$$c \sum_{i=1}^{m} \lambda_i \log(m\lambda_i) \leqslant \sum_{i=1}^{m} \lambda_i \log^2 \frac{1}{\lambda_i} - \left(\sum_{i=1}^{m} \lambda_i \log \frac{1}{\lambda_i} \right)^2.$$

Proof. We only sketch the proof of (a). The result is obvious if $m = 1$. Fix m and suppose, by induction, that the result is true for smaller m. Then, it is enough to show that the maximum of
$$f(\lambda_1, \ldots, \lambda_m) = \sum_{i=1}^{m} \lambda_i \log^2 \frac{1}{\lambda_i} - \left(\sum_{i=1}^{m} \lambda_i \log \frac{1}{\lambda_i} \right)^2 - 2 \sum_{i=1}^{m} \lambda_i \log \lambda_i$$
on the set $A = \{(\lambda_1, \ldots, \lambda_m) : \lambda_i > 0, \sum_{i=1}^m \lambda_i = 1\}$ is $2 \log m$.

From Lagrange multipliers it is easily seen that if $\lambda_1, \ldots, \lambda_m$ is the only critical point of f at A, then $\log^2 \lambda_i - 2H \log \lambda_i$ must be independent of i (here $H = \sum_{k=1}^m \lambda_k \log {\lambda_k}^{-1}$). Since each $\log \lambda_i$ is negative, this implies that $\lambda_1 = \ldots = \lambda_m = 1/m$. The result now follows from the facts that $f(1/m, \ldots, 1/m) = 2 \log m$ and $(1/m, \ldots, 1/m)$ cannot be a minimum. $\square$

THEOREM 4.3. *Let μ be a probability measure in Q_0. Then*

(a) *μ is singular (with respect to Lebesgue measure) if and only if $P_\infty(x) = \infty$ at μ-a.e. $x \in Q_0$;*

(b) *μ is absolutely continuous if and only if $P_\infty(x) < \infty$ at μ-a.e. $x \in Q_0$.*

Proof. We only prove (a), the proof of (b) being similar. Suppose first that μ is singular. If $\mu\{P_\infty < \infty\} > 0$, then, by Proposition 4.1 and the theorem of Doob stated in § 1.4, it follows that $\mu\{\sup_n |S_n| < \infty\} > 0$. Since $\log Z_n = S_n + P_n$ and $P_n \geqslant 0$, we can choose $E \subset Q_0$, with $\mu(E) > 0$, and $c > 0$ such that
$$c^{-1} \leqslant \frac{\mu(Q_n(x))}{|Q_n(x)|} \leqslant c$$
for all $n \in \mathbb{N}$ and each $x \in E$, which is impossible if μ is singular. Thus, this contradiction shows that $\mu\{P_\infty < \infty\} = 0$. For the other implication assume $P_\infty = \infty$ μ-a.e. From a theorem of Levy [**23**, p. 157; **20**, p. 519], we have $|S_n| = o(\langle S \rangle_n)$ μ-a.e. on $\{\langle S \rangle_\infty = \infty\}$. Since $\langle S \rangle_n \leqslant 2P_n$, it follows that $|S_n| = o(P_n)$ μ-a.e. Therefore $\log Z_n(x) \to +\infty$ for μ-a.e. $x \in Q_0$ and
$$\frac{\mu(Q_n(x))}{|Q_n(x)|} \to +\infty \quad \text{as } n \to \infty$$
for μ-a.e. $x \in Q_0$. This implies that μ is singular. $\square$

Corollary 4.4. *If μ is a singular probability measure in Q_0, then there is $c > 0$ such that*

$$\varlimsup_{n\to\infty} \frac{\log(\mu(Q_n(x))/|Q_n(x)|) - P_n(x)}{\sqrt{P_n(x)\log\log P_n(x)}} \leqslant c$$

for μ-a.e. $x \in Q_0$.

Proof. Recall that $S_n = \log(\mu(Q_n(X))/|Q_n(x)|) - P_n(x)$ is a μ-martingale. By Theorem 4.3, $\mu\{P_\infty < \infty\} = 0$. Hence, the result is obvious on the set $\{\sup_n |S_n| < \infty\}$. On its complement, we use the upper bound of the LIL for the martingale S_n and the estimate in Proposition 4.1. □

Theorem 4.3 should be compared to Corollary 1.2 in [**9**], which says that a dyadic doubling positive measure μ is singular if and only if

$$\sum_{k=1}^{\infty}\sum_{i=1}^{2^N}\left(\log\left(2^N\frac{\mu(Q_k^i)}{\mu(Q_{k-1})}\right)\right)^2 = \infty$$

a.e.(dx) $x \in Q_0$. Here Q_{k-1} is the dyadic cube of generation $k-1$ which contains x and $\{Q_k^i\}$, for $i = 1, \dots, 2^N$, are its 2^N dyadic sons.

5. *Control of S_n by H_n*

5.1. *Control of $\langle S\rangle_n$ by H_n*

From § 2, for a given probability measure in Q_0, we have the representation

$$\mu(Q_n(x)) = \exp\{S_n(x) - H_n(x)\}$$

where (S_n, μ) is a dyadic martingale,

$$H_n(x) = \sum_{k=1}^{n} h_k(x), \quad h_k(x) = \sum_{i=1}^{2^N} \lambda_k^i(x)\log\frac{1}{\lambda_k^i(x)}$$

and, from now on, $\lambda_k^i(x) = \mu(Q_k^i)/\mu(Q_{k-1}(x))$, $\{Q_k^i\}_{i=1}^{2^n}$ being the dyadic sons of $Q_{k-1}(x)$. Since, by the upper bound of the law of the iterated logarithm, S_n can be controlled by $\langle S\rangle_n$, the purpose of this section is to determine which conditions on the doubling behaviour of μ imply that $S_n(x) = o(H_n(x))$ for μ-a.e. x. We remember that, if (S_n) is as above, then

$$\begin{aligned}\langle S\rangle_n &= \sum_{k=1}^{n}\sum_{j=1}^{2^N}\lambda_k^j\left(\sum_{i=1}^{2^N}\lambda_k^i\log\frac{\lambda_k^j}{\lambda_k^i}\right)^2 \\ &= \sum_{k=1}^{n}\left[\sum_{i=1}^{2^N}\lambda_k^i\log^2\frac{1}{\lambda_k^i} - \left(\sum_{i=1}^{2^N}\lambda_k^i\log\frac{1}{\lambda_k^i}\right)^2\right].\end{aligned}$$

We start with some technical lemmas.

Lemma 5.1. *Let $m \in \mathbb{N}$ with $m \geqslant 2$. Then there exists $c = c(m) > 0$ such that if $(\lambda_i)_{i=1}^m$ are non-negative and $\sum_{i=1}^m \lambda_i = 1$, then*

$$0 \leqslant \sum_{i=1}^m \lambda_i \log^2 \frac{1}{\lambda_i} - \left(\sum_{i=1}^m \lambda_i \log \frac{1}{\lambda_i} \right)^2$$
$$\leqslant c \left(\sum_{i=1}^m \lambda_i \log \frac{1}{\lambda_i} \right) \log \left(\frac{\log m}{\sum_{i=1}^m \lambda_i \log \lambda_i^{-1}} \right).$$

Proof. We will distinguish two cases.

1. There is j, with $1 \leqslant j \leqslant m$, such that $1/100m \leqslant \lambda_j \leqslant \frac{3}{4}$. Since $\sum_{i=1}^m \lambda_i \log^2 \lambda_i^{-1} \leqslant \log^2 m$ and $\sum_{i=1}^m \lambda_i \log \lambda_i^{-1} \geqslant \lambda_j \log \lambda_j^{-1} \geqslant c(m)$ then the conclusion follows from Lemma 4.2(a).

2. There is j with $\lambda_j \geqslant \frac{3}{4}$ and $\lambda_i \leqslant 1/100m$ if $i \neq j$. Assume that $\lambda_m \geqslant \frac{3}{4}$, and $\lambda_i \leqslant 1/100m$ for $i = 1, \ldots, m-1$. Let $\varepsilon = \max\{\lambda_1, \ldots, \lambda_{m-1}\} \leqslant 1/100m$. Since $\sum_{i=1}^{m-1} \lambda_i + \lambda_m = 1 \leqslant (m-1)\varepsilon + \lambda_m$, it follows that $\lambda_m \geqslant 1 - (m-1)\varepsilon$. Therefore,

$$\begin{aligned} \sum_{i=1}^m \lambda_i \log^2 \lambda_i^{-1} &\leqslant (m-1)\varepsilon \log^2 \varepsilon^{-1} + [1-(m-1)\varepsilon] \log^2 \frac{1}{1-(m-1)\varepsilon} \\ &\leqslant (m-1)\varepsilon \log^2 \varepsilon^{-1} + \left(\frac{1}{1-(m-1)\varepsilon} - 1 \right)^2 \\ &\leqslant 2(m-1)\varepsilon \log^2 \varepsilon^{-1} \end{aligned} \tag{5.1}$$

where the fact that $x \log x^{-1}$ is increasing in $[0, e^{-1}] \supset [0, \varepsilon]$ and decreasing in $[1-\varepsilon(m-1), 1]$ and the estimate $\log(1+t) \leqslant t$ $(t > 0)$ have been used. On the other hand, an easy computation shows that

$$\frac{\varepsilon \log \varepsilon^{-1}}{\log m} \leqslant \frac{\sum_{i=1}^m \lambda_i \log \lambda_i^{-1}}{\log m} \leqslant \tfrac{1}{3} \leqslant \frac{1}{e}$$

so

$$\frac{\sum_{i=1}^m \lambda_i \log \lambda_i^{-1}}{\log m} \log \left(\frac{\log m}{\sum_{i=1}^m \lambda_i \log \lambda_i^{-1}} \right) \geqslant \frac{\varepsilon \log \varepsilon^{-1}}{\log m} \log \left(\frac{\log m}{\varepsilon \log \varepsilon^{-1}} \right)$$

and we finally get

$$\left(\sum_{i=1}^m \lambda_i \log \lambda_i^{-1} \right) \log \left(\frac{\log m}{\sum_{i=1}^m \lambda_i \log \lambda_i^{-1}} \right) \geqslant \tfrac{1}{2} \varepsilon \log^2 \varepsilon. \tag{5.2}$$

The result follows now from (5.1) and (5.2). □

The following lemma shows how to control the quadratic characteristic in terms of the total entropy.

Lemma 5.2. *Let $0 \leqslant h_k \leqslant 1$, $k = 1, 2, \ldots, n$, and $H_n = \sum_{k=1}^n h_k$. Then,*

$$\sum_{k=1}^n h_k \log \frac{1}{h_k} \leqslant H_n \log \left(\frac{n}{H_n} \right).$$

Proof. The result follows from Jensen's inequality for the convex function $\log x^{-1}$. □

Observe that the opposite inequality in Lemma 5.2 cannot be true: put $h_1 = \ldots = h_m = 1$ with $h_k = 0$ if $n \geqslant k > m$.

Now we are ready to control $\langle S \rangle_n$ by H_n. Corollary 5.3 is a direct consequence of Lemmas 5.1 and 5.2.

Corollary 5.3. *If, for each $k \in \mathbb{N}$,*

$$h_k(x) = \sum_{i=1}^{2^N} \lambda_k^i(x) \log \frac{1}{\lambda_k^i(x)}$$

is the entropy at step k and $H_n(x) = \sum_{k=1}^{n} h_k(x)$, then

$$\langle S \rangle_n(x) \leqslant C H_n(x) \log \left(\frac{nN \log 2}{H_n(x)} \right)$$

where C is some positive constant depending only on N.

Now Corollary 5.3 and the LIL (Corollary 3.8) give the following reformulation of the LIL in terms of the total entropy.

Corollary 5.4. *Let S_n, $\langle S_n \rangle_n$ and H_n be as at the beginning of this section and $T_n(x) = nN(\log 2) H_n(x)^{-1}$. Then*

$$\limsup_{n \to \infty} \frac{S_n}{(H_n(\log T_n) \log\log(H_n \log T_n))^{1/2}} \leqslant C < \infty$$

μ-a.e. on the set $\{\langle S \rangle_\infty = \infty\}$, where $C = C(N) > 0$. In particular,

$$\limsup_{n \to \infty} \frac{S_n}{(H_n \log(2T_n) \log\log(H_n \log(2T_n)))^{1/2}} \leqslant C < \infty$$

μ-a.e. on the set $\{\lim_n H_n = \infty\}$.

Observe that $2T_n(x) \geqslant 2$ for all n. This is technically convenient for the subsequent applications.

Lemma 5.5. *Let h_k, with $0 \leqslant h_k \leqslant 1$, and H_n be as in Lemma 5.2 and $T_n = nN(\log 2) H_n^{-1}$. Suppose that*

$$\lim_{n \to \infty} \frac{H_n}{(\log n)(\log\log\log n)} = \infty.$$

Then, $H_n \log(2T_n) \log\log(H_n \log(2T_n)) = o(H_n^2)$ as $n \to \infty$.

Proof. If n is large enough, using the notation $\log^{(3)} n = \log\log\log n$, we have

$$\begin{aligned}
&\frac{H_n \log(2T_n)\log\log(H_n\log(2T_n))}{H_n^2}\\
&\qquad\leqslant \log\left(\frac{2nN\log 2}{(\log n)(\log^{(3)} n)}\right)\\
&\qquad\quad\times\left(\frac{\log\log H_n}{H_n}+\frac{\log^{(3)}(2N(\log 2)n(\log n)^{-1}(\log^{(3)} n)^{-1})}{H_n}\right)\\
&\qquad\leqslant C(\log n)\left(\varepsilon\frac{\log\log((\log n)(\log^{(3)} n))}{(\log n)(\log^{(3)} n)}+\varepsilon\frac{1}{\log n}\right)\leqslant 2C\varepsilon. \qquad \square
\end{aligned}$$

The following result says, in a precise way, that if H_n is big enough then the expression in $\langle S\rangle_n$ appearing in the LIL can be controlled by H_n.

THEOREM 5.6. *Let S_n, $\langle S\rangle_n$, H_n be as at the beginning of the section. Let $x \in Q_0$ be such that*

$$\frac{H_n(x)}{(\log n)(\log\log\log n)} \to \infty \quad \text{as } n \to \infty.$$

Then $\sqrt{\langle S\rangle_n(x)\log^+\log^+\langle S\rangle_n(x)} = o(H_n(x))$ as $n \to \infty$. In particular, $S_n(x) = o(H_n(x))$ for μ-almost every point

$$x \in \left\{x \in Q_0 : \frac{H_n(x)}{(\log n)(\log\log\log n)} \to \infty \text{ as } n \to \infty\right\}.$$

Proof. Just combine Corollary 5.3, Corollary 5.4, Lemma 5.5 and Doob's theorem in § 1.4. $\square$

5.2. *Uniform conditions*

We introduce the following quantities:

$$\delta_k = \min\left\{1-\frac{\mu(Q_k^i)}{\mu(Q_{k-1})} : Q_{k-1}\in\mathcal{F}_{k-1},\ i=1,\ldots,2^N\right\}$$

for each $k \in \mathbb{N}$, where, as usual, $\{Q_k^i\}_{i=1}^{2^N}$ are the dyadic sons of Q_{k-1}. Note that δ_k informs about the worst doubling behaviour of μ at the generation k. Observe also that, from the additivity of μ, it follows that $0 \leqslant \delta_k \leqslant 1-2^{-N}$ for each k. These extreme values correspond to the extreme doubling behaviour of μ: δ_k near $1-2^{-N}$ means that μ behaves like Lebesgue measure and δ_k near 0 means that μ doubles very badly inside some Q_{k-1}.

The following technical lemma says how small the entropy in the function of this uniform quantity can be.

LEMMA 5.7. (a) *Let* $m \in \mathbb{N}$, $m \geqslant 2$, *and* $0 \leqslant \delta \leqslant 1 - m^{-1}$. *Then*

$$\min\left\{\sum_{i=1}^{m} \lambda_i \log \lambda_i^{-1} : \sum_{i=1}^{m} \lambda_i = 1, 0 \leqslant \lambda_i \leqslant 1-\delta\right\}$$
$$= p(1-\delta)\log\frac{1}{1-\delta} + (1-p(1-\delta))\log\frac{1}{1-p(1-\delta)}$$

where $p = [(1-\delta)^{-1}]$ *is the integer part of* $(1-\delta)^{-1}$.

(b)

$$\max\left\{\sum_{i=1}^{m} \lambda_i \log \lambda_i^{-1} : \sum_{i=1}^{m} \lambda_i = 1, 0 \leqslant \lambda_i \leqslant \max_i \lambda_i = 1-\delta\right\}$$
$$= (1-\delta)\log\frac{1}{1-\delta} + \delta\log\frac{m-1}{\delta}.$$

Proof. (a) We claim that some λ_i of the extremal configuration must be equal to $1-\delta$. If not, the extremal configuration would be of the form $\lambda_1, \ldots, \lambda_q, 0, \ldots, 0$, where $1 \leqslant q \leqslant m$ and $0 < \lambda_i < 1-\delta$ $(i = 1, \ldots, q)$. But this would imply that the minimum

$$\min\left\{\sum_{i=1}^{q} \lambda_i \log \lambda_i^{-1} : \sum_{i=1}^{q} \lambda_i = 1,\ 0 < \lambda_i < 1-\delta\right\}$$

is attained at the set $\{(\lambda_1, \ldots, \lambda_q) : \sum_{i=1}^{q} \lambda_i = 1,\ 0 < \lambda_i < 1-\delta\}$ which is impossible since, by Lagrange, the only critical point at this set is $\lambda_1 = \ldots = \lambda_q = 1/q$ and it is a local maximum.

Therefore, we can assume that $\lambda_1 = 1-\delta$. Now we argue by induction on m. We distinguish two cases.

1: $0 \leqslant \delta < \frac{1}{2}$. Then $p = [(1-\delta)^{-1}] = 1$ and, since $\sum_{i=2}^{m} \lambda_i = \delta$, it follows that

$$\sum_{i=2}^{m} \frac{\lambda_i}{\delta}\log\left(\frac{\delta}{\lambda_i}\right) \geqslant 0, \qquad \text{so} \quad \sum_{i=2}^{m} \lambda_i \log\frac{1}{\lambda_i} \geqslant \delta\log\frac{1}{\delta}.$$

Therefore,

$$\sum_{i=1}^{m} \lambda_i \log\frac{1}{\lambda_i} \geqslant \delta\log\frac{1}{\delta} + (1-\delta)\log\frac{1}{1-\delta}.$$

2: $\frac{1}{2} \leqslant \delta \leqslant 1 - m^{-1}$. Here, put $\mu_i = \lambda_i/\delta$ for $i = 2, \ldots, m$. Then

$$\sum_{i=2}^{m} \mu_i = 1 \quad \text{and} \quad 0 \leqslant \mu_i \leqslant \frac{1-\delta}{\delta} = 1-\delta'$$

where $\delta' = (2\delta - 1)/\delta$. Note that

$$0 \leqslant \delta' \leqslant 1 - \frac{1}{m-1} \quad \text{and} \quad \frac{1}{1-\delta'} = \frac{1}{1-\delta} - 1,$$

so $[(1-\delta')^{-1}] = p-1$. By the induction hypothesis,

$$\begin{aligned}\sum_{i=2}^{m} \mu_i \log \frac{1}{\mu_i} = \sum_{i=2}^{m} \frac{\lambda_i}{\delta} \log\left(\frac{\delta}{\lambda_i}\right) &\geqslant (p-1)(1-\delta') \log \frac{1}{1-\delta'} \\ &\quad + (1-(p-1)(1-\delta')) \log \frac{1}{1-(p-1)(1-\delta')} \\ &= (p-1)\frac{1-\delta}{\delta} \log \frac{\delta}{1-\delta} + \frac{1-p(1-\delta)}{\delta} \log \frac{\delta}{1-p(1-\delta)}.\end{aligned}$$

So

$$\begin{aligned}\sum_{i=1}^{m} \lambda_i \log \lambda_i^{-1} \geqslant (1-\delta)\log\frac{1}{1-\delta} + \delta \log \frac{1}{\delta} + (p-1)(1-\delta)\log\frac{\delta}{1-\delta} \\ + (1-p(1-\delta))\log\frac{\delta}{1-p(1-\delta)} \\ = p(1-\delta)\log\frac{1}{1-\delta} + (1-p(1-\delta))\log\frac{1}{1-p(1-\delta)}.\end{aligned}$$

(b) Assume that $\lambda_i = \max_i \lambda_i = 1-\delta$. Then $\sum_{i=2}^{m} \lambda_i = \delta$ and, under this restriction, the maximum of $\sum_{i=2}^{m} \lambda_i \log 1/\lambda_i$ is attained when $\lambda_2 = \ldots = \lambda_m = \delta/m-1$. Thus

$$\begin{aligned}\sum_{i=1}^{m} \lambda_i \log \lambda_i^{-1} &\leqslant (1-\delta)\log\frac{1}{1-\delta} + \frac{(m-1)\delta}{m-1}\log\frac{m-1}{\delta} \\ &= (1-\delta)\log\frac{1}{1-\delta} + \delta\log\frac{m-1}{\delta},\end{aligned}$$

and the lemma is proved. □

REMARK 5.1. In the situation of Lemma 5.5 note that, if $0 \leqslant \delta < \frac{1}{2}$ or $m=2$, then

$$p(1-\delta)\log\frac{1}{1-\delta} + (1-p(1-\delta))\log\frac{1}{1-p(1-\delta)} = \delta\log\frac{1}{\delta} + (1-\delta)\log\frac{1}{1-\delta}.$$

REMARK 5.2. If p, δ and m are as in Lemma 5.5, then

$$p(1-\delta)\log\frac{1}{1-\delta} + (1-p(1-\delta))\log\frac{1}{1-p(1-\delta)} \asymp \delta\log\frac{1}{\delta}$$

with comparison constants that depend only on m.

The upper bound of the law of the iterated logarithm, Lemma 5.7 and Remark 5.2 lead to the following formulations of Theorem 5.6 in terms of the quantities δ_k.

COROLLARY 5.8. *If*

$$\frac{\sum_{k=1}^{n} \delta_k \log \delta_k^{-1}}{(\log n)(\log\log\log n)} \to +\infty,$$

then $\sqrt{\langle S\rangle_n(x)\log^+\log^+\langle S\rangle_n(x)} = o(H_n(x))$ *as* $n\to\infty$ *for* μ*-a.e* $x\in Q_0$. *In particular,* $S_n(x)=o(H_n(x))$ *as* $n\to\infty$, *for* μ*-almost every* $x\in Q_0$.

Corollary 5.9. *If*

$$\delta_k\Big/\left(\frac{\log\log\log k}{k\log k}\right)\to\infty\quad\text{as } k\to\infty,$$

then

$$\sqrt{\langle S\rangle_n(x)\log^+\log^+\langle S\rangle_n(x)} = o(H_n(x))\quad\text{as } n\to\infty$$

and $S_n(x)=o(H_n(x))$ *as* $n\to\infty$, *for* μ*-almost every* $x\in Q_0$.

Proof. Just observe that the hypothesis yields

$$\lim_{n\to\infty}\frac{\sum_{k=1}^n \delta_k\log\delta_k^{-1}}{(\log n)(\log\log\log n)}=\infty$$

and apply Corollary 5.8. □

6. *Estimates of the support of μ in terms of its doubling behaviour*

In this section we apply the results in § 5 to determine measure functions ϕ for which $\mu\ll\mathcal{H}_\phi$, under restrictions on the doubling behaviour of μ. Let S_n, $\langle S\rangle_n$, $H_n=\sum_{k=1}^n h_k$ and $h_k(x)=\sum_{i=1}^{2^N}\lambda_k^i\log(\lambda_k^i)^{-1}$ be as in § 5.

The first result is an immediate consequence of Corollary 5.4.

Theorem 6.1. *Let* $(A_n)_{n=1}^\infty$ *be a sequence of numbers such that*

$$0\leqslant A_n-A_{n-1}\leqslant N\log 2\quad \textit{for all } n,$$

and such that

$$\lim_{n\to\infty}\frac{A_n}{(\log n)(\log\log\log n)}=\infty.$$

Suppose that $H_n(x)\geqslant A_n$ *for* $n\geqslant n_0(x)$, μ*-almost every* $x\in Q_0$. *Let* $T_n=nN(\log 2)A_n^{-1}$. *Then,* $\mu\ll\mathcal{H}_\phi$, *where* ϕ *is any measure function such that*

$$\phi(2^{-n})=\exp\{C(A_n(\log 2T_n)\log\log(A_n\log(2T_n)))^{1/2}-A_n\}$$

for some $C=C(N)>0$. *In particular, for any* $\varepsilon>0$, $\mu\ll\mathcal{H}_{\phi_\varepsilon}$ *where* $\phi(2^{-n})=\exp\{-(1-\varepsilon)A_n\}$.

Proof. Corollary 5.4 gives

$$\mu(Q_n(x))\leqslant\exp\{C(H_n(x)\log(2T_n(x))\log\log(H_n(x)\log(2T_n(x))))^{1/2}-H_n(x)\}$$

eventually for μ-a.e. x. Now, for $n\geqslant n_0(x)$, $T_n(x)\leqslant T_n$ and hence

$$\begin{aligned}H_n(x)-C(H_n(x)\log(2T_n(x))\log\log(H_n(x)\log(2T_n(x))))^{1/2}\\ \geqslant\sqrt{H_n(x)}\big(\sqrt{H_n(x)}-C\sqrt{\log 2T_n\log\log(H_n(x)\log(2T_n))}\big).\end{aligned}$$

Since $H_n(x)\geqslant A_n\geqslant(\log n)(\log^{(3)}n)$, it is sufficient to observe that the function $\sqrt{x}-C\sqrt{(\log 2T_n)\log\log(x\log 2T_n)}$ is increasing in x if $x\geqslant A_n$. □

REMARK. It is easily checked that if ϕ_1 and ϕ_2 are two measure functions such that $\phi_1(2^{-n}) = \phi_2(2^{-n}) = \exp\{-B_n\}$, where $0 \leqslant B_n - B_{n-1} \leqslant C$ for all n then $\phi_1(t) \asymp \phi_2(t)$ for all $t \in [0,1]$, and, therefore $M_{\phi_1}(E) > 0$ if and only if $M_{\phi_2}(E) > 0$. Thus, there is no ambiguity in the statements of Theorem 6.1 and the results below.

COROLLARY 6.2. (a) *Let $0 < \beta \leqslant N$ and assume $H_n(x) \geqslant \beta n \log 2$ μ-almost every $x \in Q_0$. Then $\mu \ll \mathcal{H}_{\phi_\beta}$, where*

$$\phi_\beta(t) = t^\beta \exp\{C(\log t^{-1} \log\log\log t^{-1})^{1/2}\},$$

for some $C = C(N) > 0$.

(b) *Assume that μ is a dyadic doubling measure which is singular with respect to Lebesgue measure. Let $0 < \beta < N$ and assume that $H_n(x) \leqslant \beta n \log 2$ μ-almost every $x \in Q_0$. Then μ is singular with respect to $\mathcal{H}_{\phi_\beta}$, where*

$$\phi_\beta(t) = t^\beta \exp\{C_1(\log t^{-1} \log\log\log t^{-1})^{1/2}\}$$

for some $C_1 = C_1(N) > 0$.

Proof. Part (a) follows from Theorem 6.1.

To prove (b) recall that $\mu(Q_n(x)) = \exp(S_n(x) - H_n(x))$ where (S_n, μ) is a martingale. Since μ is dyadic doubling, the lower bound in the law of the iterated logarithm holds, that is,

$$\limsup_{n\to\infty} \frac{S_n(x)}{\sqrt{\langle S\rangle_n(x) \log\log\langle S\rangle_n(x)}} \geqslant C$$

at μ-a.e. $x \in \{x \in Q_0 : \langle S\rangle_\infty(x) = \infty\}$. Since μ is a dyadic doubling measure, Proposition 4.11(b) tells us that $\langle S\rangle_n(x)$ is comparable to

$$P_n(x) = nN \log 2 - H_n(x).$$

Also, since μ is singular with respect to Lebesgue measure, Theorem 4.3 tells that $\langle S\rangle_\infty(x) = \infty$ μ-a.e. $x \in Q_0$. Therefore, the lower bound in the law of the iterated logarithm reads

$$\limsup_{n\to\infty} \frac{S_n(x)}{\sqrt{(nN\log 2 - H_n(x)) \log\log(nN\log 2 - H_n(x))}} \geqslant C \quad \mu\text{-a.e. } x \in Q_0.$$

Hence, for μ-a.e. $x \in Q_0$, there exist a sequence of natural numbers n_k tending to infinity for which

$$\mu(Q_{n_k}(x)) \\ \geqslant \exp\big(C\sqrt{(n_k N\log 2 - H_{n_k}(x)) \log\log(n_k N\log 2 - H_{n_k}(x))} - H_{n_k}(x)\big).$$

Now, we deduce that

$$\begin{aligned}\mu(Q_{n_k}(x)) &\geqslant \exp\big(C(\beta,N)C\sqrt{n_k \log\log n_k} - \beta n_k \log 2\big) \\ &= 2^{-\beta n_k} \exp\big(C_2\sqrt{n_k \log\log n_k}\big) \quad \mu\text{-a.e. } x \in Q_0.\end{aligned}$$

Applying part (b) of Proposition 1.1, we deduce that μ is singular with respect to

$\mathcal{H}_{\phi_\beta}$ where

$$\phi_\beta(t) = t^\beta \exp\{C_1(\log t^{-1} \log\log\log t^{-1})^{1/2}\}. \qquad \square$$

Let (δ_k) be as in § 5.2.

COROLLARY 6.3. (a) *Suppose that*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n \delta_k \log \delta_k^{-1}}{(\log n)(\log\log\log n)} = \infty.$$

Then, for each $\varepsilon > 0$, we have $\mu \ll \mathcal{H}_{\phi_\varepsilon}$ where ϕ_ε is any measure function such that

$$\phi_\varepsilon(2^{-n}) = \exp\Bigg\{-(1-\varepsilon)\sum_{k=1}^n \Bigg(p_k(1-\delta_k)\log\frac{1}{1-\delta_k} + (1-p_k(1-\delta_k))\log\frac{1}{1-p_k(1-\delta_k)}\Bigg)\Bigg\}$$

where $p_k = [(1-\delta_k)^{-1}]$. In particular, there exists $c = c(N) > 0$ such that $\mu \ll \mathcal{H}_\phi$ where ϕ is any measure function such that

$$\phi(2^{-n}) = \exp\Bigg\{-c\sum_{k=1}^n \delta_k \log\frac{1}{\delta_k}\Bigg\}.$$

(b) *Let μ be a singular dyadic doubling measure in Q_0 which is singular with respect to Lebesgue measure. Then μ is singular with respect to $\mathcal{H}_\phi$ where ϕ is any measure function such that*

$$\phi(2^{-n}) = \exp\Bigg(-\sum_{b=1}^n (1-\delta_k)\log\frac{1}{1-\delta_k} + \delta_k \log\frac{2^N-1}{\delta_k}\Bigg).$$

Proof. Statement (a) follows from Theorem 6.1, Lemma 5.7 and Remark 5.2 in § 5.2. To prove (b) we argue as in the proof of Corollary 6.2(b). We know that for μ-a.e. $x \in Q_0$, there exists a sequence n_k of natural numbers, $n_k \to \infty$, for which

$$\mu(Q_{n_k}(x)) \geqslant \exp\big(C\sqrt{(Nn_k\log 2 - H_{n_k}(x))\log\log(Nn_k\log 2 - H_{n_k}(x))} - H_{n_k}(x)\big).$$

Since $\delta_k(x) \leqslant \delta_k$ and $(1-x)\log(1-x)^{-1} + x\log((2^N-1)/x)$ is increasing in x if $0 \leqslant x \leqslant 1-2^{-N}$, applying Lemma 5.7, we deduce that

$$H_n(x) \leqslant \sum_{k=1}^n (1-\delta_k)\log\frac{1}{1-\delta_k} + \delta_k\log\frac{2^N-1}{\delta_k}.$$

Hence,

$$\mu(Q_{n_k}(x)) \geqslant \phi(2^{-n_k})\exp\big(C\sqrt{P_{n_k}(x)\log\log P_{n_k}(x)}\big).$$

Since μ is singular with respect to Lebesgue measure, $P_{n_k}(x) \to \infty$ μ-a.e. $x \in Q_0$. So, part (b) of Proposition 1.1 shows that μ is singular with respect to $\mathcal{H}_\phi$. $\square$

The hypothesis in Theorem 6.1 and Corollary 6.3 may seem unnatural, but some restrictions on the growth of H_n and δ_k are necessary to avoid measures with point masses which obviously cannot be absolutely continuous with respect to any Hausdorff measure. Actually it turns out that if $\delta_k = k^{-1}(\log k)^{-1}(\log\log k)^{\varepsilon}$, with $\varepsilon > 0$, it could happen that μ has point masses, while if $\delta_k k(\log k)/\log\log\log k \to \infty$, Corollary 6.3 can be applied. Our method does not give enough information to fill in this narrow gap.

In the case that the sequence (δ_k) is bounded below by some positive constant (that is, μ dyadic doubling) we recover Heurteaux's result (recall Remark 5.2 of § 5.2; see also § 8).

Corollary 6.4. (a) *Suppose that* $\delta_k \geqslant \delta > 0$ *for all* $k \in \mathbb{N}$, *and let* $p = [(1-\delta)^{-1}]$ *and*

$$\beta = \frac{p(1-\delta)\log(1-\delta)^{-1} + (1-p(1-\delta))\log(1-p(1-\delta))^{-1}}{\log 2}.$$

Then $\mu \ll \mathcal{H}_{\phi_\beta}$, *where* ϕ_β *is as in Corollary* 6.2(a). *In particular, if* $N = 1$,

$$\beta = \frac{\delta\log\delta^{-1} + (1-\delta)\log(1-\delta)^{-1}}{\log 2}.$$

(b) *Let* μ *be a dyadic doubling singular measure. Assume* $\delta_k(x) \leqslant \delta < 1 - 2^{-N}$ *at* μ*-a.e.* $x \in Q_0$, *for* $k = 1, 2, \ldots$. *Then* μ *is singular with respect to* $\mathcal{H}_{\phi_\gamma}$ *where* ϕ_γ *is as in Corollary* 6.2(b), *with*

$$\gamma = (\log 2)^{-1}\left((1-\delta)\log\frac{1}{1-\delta} + \delta\log\left(\frac{2^N-1}{\delta}\right)\right).$$

Proof. (a) A direct application of Lemma 5.7 gives

$$\begin{aligned} h_k(x) &= \sum_{i=1}^{2^N} \lambda_k^i(x)\log\frac{1}{\lambda_k^i(x)} \\ &\geqslant p(1-\delta)\log\frac{1}{1-\delta} + (1-p(1-\delta))\log\frac{1}{1-p(1-\delta)}. \end{aligned}$$

The result follows now from Corollary 6.2.

(b) As in the proof of Corollary 6.2(b), we know that for μ-a.e. $x \in Q_0$, there exists a sequence of natural numbers n_k, tending to infinity, for which

$$\begin{aligned} &\mu(Q_{n_k}(x)) \\ &\quad \geqslant \exp\big(C\sqrt{(Nn_k\log 2 - H_{n_k}(x))\log\log(Nn_k\log 2 - H_{n_k}(x))} - H_{n_k}(x)\big). \end{aligned}$$

Now, since $\delta_k(x) \leqslant \delta < 1 - 2^{-N}$, μ-a.e. $x \in Q_0$, we deduce that for μ-a.e. $x \in Q_0$, the dyadic square $Q_k(x)$ contains a dyadic cube of the next generation Q_{k+1}^i, with $\mu(Q_{k+1}^i) \geqslant (1-\delta_k(x))\mu(Q_k) \geqslant (1-\delta)\mu(Q_k)$. Then, applying Lemma 5.7(b) we deduce that

$$H_n(x) \leqslant n\left((1-\delta)\log\frac{1}{(1-\delta)} + \delta\log\frac{2^N-1}{\delta}\right)$$

and deduce that

$$\mu(Q_{n_k}(x)) \geqslant 2^{-n_k\beta} \exp\bigl(C(\delta,N)C\sqrt{n_k \log\log n_k}\bigr)$$

for μ-a.e. $x \in Q_0$ and for infinitely many n_k. An application of part (b) of Proposition 1.1 finishes the proof. $\square$

As announced in the introduction, we can obtain the same results under the assumption that the doubling behaviour of μ is good for many scales, that is, suppose that $\delta_k \geqslant \delta > 0$ for each $k \in A$, where $A \subset \mathbb{N}$. Let $p = \bigl[(1-\delta)^{-1}\bigr]$. Then, by Lemma 5.5,

$$h_k = \sum_{i=1}^{2^N} \lambda_k^i \log \frac{1}{\lambda_k^i} \geqslant p(1-\delta)\log\frac{1}{1-\delta} + (1-p(1-\delta))\log\frac{1}{1-p(1-\delta)}$$

if $k \in A$, so

$$H_n \geqslant \biggl(p(1-\delta)\log\frac{1}{1-\delta} + (1-p(1-\delta))\log\frac{1}{1-p(1-\delta)}\biggr)\operatorname{card}(A\cap[1,n]).$$

The following two corollaries are now a consequence of Corollary 6.3.

Corollary 6.5. (a) *If $\delta_k \geqslant \delta > 0$ for all $k \in A \subset \mathbb{N}$ and*

$$\lim_{n\to\infty} \frac{\operatorname{card}(A\cap[1,n])}{n} = 1$$

then $\mu \ll \mathcal{H}_{\beta'}$ for any

$$\beta' < \frac{p(1-\delta)\log(1-\delta)^{-1} + (1-p(1-\delta))\log(1-p(1-\delta))^{-1}}{\log 2}.$$

In particular, if $N = 1$ then $\mu \ll \mathcal{H}_{\beta'}$ for any

$$\beta' < \frac{\delta\log\delta^{-1} + (1-\delta)\log(1-\delta)^{-1}}{\log 2}.$$

(b) *Let μ be a dyadic doubling measure in Q_0 which is singular with respect to Lebesgue measure. Assume $\delta_k \leqslant \delta < 1 - 2^{-N}$ for all $k \in A \subset \mathbb{N}$ and*

$$\lim_{n\to\infty} \frac{\operatorname{card}(A\cap[1,n])}{n} = 1.$$

Then μ is singular with respect to $\mathcal{H}_{\beta'}$ for any

$$\beta' > \frac{(1-\delta)\log(1-\delta)^{-1} + \delta\log((2^N-1)/\delta)}{\log 2}.$$

In particular, if $N = 1$, then μ is singular with respect to $\mathcal{H}_{\beta'}$ for any

$$\beta' > \frac{(1-\delta)\log(1-\delta)^{-1} + \delta\log\delta^{-1}}{\log 2}.$$

Corollary 6.6. *If $\inf_{k\in A}\delta_k > 0$ and $\underline{\lim}_{n\to\infty}(A\cap[1,n])/n > 0$, then there exists $\beta > 0$ such that $\mu \ll \mathcal{H}_\beta$.*

7. *Continuous formulation for measures in* $\mathbb{R}^N$

For each cube $Q \subset \mathbb{R}^N$, denote by $\{Q_i\}_{i=1}^{2^N}$ its natural decomposition into 2^N disjoint cubes of half size. Now, given a locally finite positive measure μ in $\mathbb{R}^N$, define, for $0 \leqslant t \leqslant \frac{1}{2}$,

$$\delta(t) = \inf\left\{1 - \mu(Q_i)/\mu(Q)\right\}$$

where the infimum is taken over all cubes in $\mathbb{R}^N$ of sidelength $r(Q) \geqslant 2t$, and where $\{Q_i\}_{i=1}^{2^N}$ is as above. The function $\delta(t)$ is non-decreasing and, when μ is restricted to the unit cube Q_0, it is easy to check that, with the notation of § 6,

$$\delta(2^{-k}) \leqslant \delta_k.$$

The following elementary proposition allows one to apply the results in § 6.

Proposition 7.1. *There is a positive constant c, depending only on N, such that*

$$\sum_{k=1}^{n} \delta_k \log \frac{1}{\delta_k} \geqslant c \int_{2^{-n}}^{1/2} \delta(t) \log \frac{1}{\delta(t)} \frac{dt}{t}.$$

Corollary 6.3 admits the following integral formulation.

Theorem 7.2. *There exists a positive constant c, depending only on N, such that if*

$$\lim_{t\to 0} \frac{\int_t^{1/2} \delta(s) \log \delta^{-1}(s) s^{-1}\, ds}{(\log\log t^{-1})(\log\log\log\log t^{-1})} = \infty,$$

then $\mu \ll \mathcal{H}_\phi$, where

$$\phi(t) = \exp\left\{ -c \int_t^{1/2} \delta(s) \log \frac{1}{\delta(s)} \frac{ds}{s} \right\}.$$

8. *Examples*

8.1. *An extremal class of dyadic doubling measures*

The construction of extremal measures in Heurteaux's result given in the introduction can be easily generalized to higher dimensions. This is very classical. Fix non-negative $\lambda_1, \dots, \lambda_{2^N}$ such that $\sum_{i=1}^{2^N} \lambda_i = 1$. Suppose that $Q_{k-1} \in \mathcal{F}_{k-1}$ and that $\mu(Q_{k-1})$ has been defined. If $\{Q_k^i\}_{i=1}^{2^N}$ is the dyadic decomposition of Q_{k-1}, define

$$\mu(Q_k^i) = \lambda_i \mu(Q_{k-1}).$$

Independently of the flexibility of these assignments for the different scales, it is clear that this also defines a Borel probability measure in Q_0. Then

$$H_n \geqslant n \sum_{i=1}^{2^N} \lambda_i \log \lambda_i^{-1}$$

and this implies, according to results in § 6,

$$\mu \ll \mathcal{H}_\beta \quad \text{for any } \beta < \biggl(\sum_{i=1}^{2^N} \lambda_i \log \lambda_i^{-1}\biggr)(\log 2)^{-1},$$

and μ is singular with respect to $\mathcal{H}_\beta$ for any

$$\beta > \biggl(\sum_{i=1}^{2^N} \lambda_i \log \lambda_i^{-1}\biggr)(\log 2)^{-1}.$$

8.2. *Some particular applications of the results in* § 6

When specialized to particular choices of the sequence (δ_k) in § 6, we can write down explicitly the measure functions ϕ for which $\mu \ll \mathcal{H}_\phi$. We collect some examples in Table 1.

TABLE 1.

δ_k	$\phi(t) \quad (\mu \ll \mathcal{H}_\phi)$
$\dfrac{\log\log k}{k \log k}$	$\exp\{-C(\log\log t^{-1})(\log\log\log t^{-1})\}$
k^{-1}	$\exp\{-C(\log\log t^{-1})^2\}$
$k^{-\alpha} \quad (0<\alpha<1)$	$\exp\{-C(\log t^{-1})^{1-\alpha}(\log\log t^{-1})\}$
$\dfrac{1}{\log k}$	$\exp\Bigl\{-C(\log t^{-1})\dfrac{\log\log\log t^{-1}}{\log\log t^{-1}}\Bigr\}$

8.3. *On the sharpness of the results in* § 6

We can modify the construction given in the introduction to allow different values of λ for different scales. Precisely, we consider a sequence $(\lambda_k)_{k=1}^\infty$ such that $0 \leqslant \lambda_k \leqslant \frac{1}{2}$ for each $k \in \mathbb{N}$, and, given $\mu(I_{k-1})$, define

$$\mu(I_k^+) = \lambda_k \mu(I_{k-1}), \qquad \mu(I_k^-) = (1-\lambda_k)\mu(I_{k-1}).$$

Since our final measure can have atoms in this general situation, it is worth pointing out that all dyadic intervals considered are always half-open (of the form $[\ ,\ )$). Then, since $\prod_{k=n}^\infty \lambda_k = 0$ for any $n \in \mathbb{N}$, it can be shown that the requisites to extend μ are satisfied [**20**], so this procedure also defines a Borel probability measure in $[0,1]$. As in § 8.1, we have now

$$\mu(I_n) = \prod_{k=1}^n \lambda_k^{\varepsilon_k} \prod_{k=1}^n (1-\lambda_k)^{1-\varepsilon_k}$$

where $\varepsilon_k = 1$ if the predecessor of I_n of the generation k is at right position and 0 if it is at left position. Observe that if $\sum_{k=1}^\infty \lambda_k < \infty$ then μ will have atoms at the dyadic numbers $\{m/2^n : n \in \mathbb{N}, m = 0,1,\dots,2^n-1\}$ in fact,

$$\mu[m/2^n, m/2^n + 1/2^k) = c(m,n) \prod_{j=n}^k (1-\lambda_j)$$

if $k \geqslant n$, which shows that μ has an atom at $\{m/2^n\}$. It can be shown that μ is actually concentrated at the dyadic numbers. Observe that a measure with atoms can never be absolutely continuous with respect to any $\mathcal{H}_\phi$, for any measure function ϕ.

8.4. *Symmetric measures*

A positive measure in Q_0 is called *dyadic-symmetric* if

$$\varepsilon_k = \inf\left\{\frac{\mu(Q_k^i)}{\mu(Q_{k-1}(x))} : x \in Q_0, i = 1, 2, \dots, 2^N\right\} \to \frac{1}{2^N}$$

as $k \to \infty$. With the notation of § 5.2, this is easily shown to be equivalent to $\lim_{k\to\infty} \delta_k = 1 - 2^{-N}$. Actually, observe that by additivity, $1 - 2^{-N} - \delta_k(x)$ is comparable to $2^{-N} - \varepsilon_k(x)$. Hence, dyadic symmetric measures are those dyadic doubling measures with a nice doubling behaviour for small scales. Examples of such measures can be given using the construction of § § 8.1 and 8.3. Concretely, given sequences $(\lambda_k^i)_{k=1}^\infty$ $(i = 1, \dots, 2^N)$ where $0 \leqslant \lambda_k^i$, such that $\sum_{i=1}^{2^N} \lambda_k^i = 1$ for each k, define inductively

$$\mu(Q_k^i) = \lambda_k^i \mu(Q_{k-1})$$

for $i = 1, \dots, 2^N$. If $\inf_i \lambda_k^i \to 2^{-N}$ as $k \to \infty$, the resulting measure is dyadic symmetric.

Now, let μ be a dyadic symmetric measure, so that

$$\lambda_k^i(x) = \frac{\mu(Q_k^i)}{\mu(Q_{k-1}(x))} \geqslant \varepsilon_k \to \frac{1}{2^N} \quad \text{as } k \to \infty, \text{ for } i = 1, \dots, 2^N.$$

Without loss of generality, we can assume that $2^{-(N+1)} \leqslant \varepsilon_k \leqslant 2^{-N}$ for any $k = 1, 2, \dots$. We will use the following elementary lemma which can be proved by induction.

Lemma 8.1. *Let $m \in \mathbb{N}$, $m \geqslant 2$ and $0 < \varepsilon \leqslant m^{-1}$. Then*

$$\min\left\{\sum_{i=1}^m \lambda_i \log\frac{1}{\lambda_i} : \lambda_i \geqslant \varepsilon, \sum_{i=1}^m \lambda_i = 1\right\}$$
$$= (m-1)\varepsilon \log\frac{1}{\varepsilon} + (1 - (m-1)\varepsilon) \log\frac{1}{1 - (m-1)\varepsilon}.$$

Hence, if μ is dyadic symmetric then we have

$$\sum_{i=1}^{2^N} \lambda_k^i(x) \log\frac{1}{\lambda_k^i(x)} \geqslant (2^N - 1)\varepsilon_k \log\frac{1}{\varepsilon_k}$$
$$+ (1 - (2^N - 1)\varepsilon_k) \log\frac{1}{1 - (2^N - 1)\varepsilon_k}$$
$$\to N \log 2 \quad \text{as } k \to \infty.$$

So

$$H_n \geqslant \sum_{k=1}^n \left((2^N - 1)\varepsilon_k \log\frac{1}{\varepsilon_k} + (1 - (2^N - 1)\varepsilon_k) \log\frac{1}{1 - (2^N - 1)\varepsilon_k}\right)$$

and, since $\varepsilon_k \to 2^{-N}$ as $k \to \infty$, the results in § 6 would imply that $\mu \ll \mathcal{H}_\beta$ for any β with $0 < \beta < N$. In fact, our method gives a sharper result. For that, we need a more careful estimate of $\mu(Q_n)$. We know that

$$\mu(Q_n) = \exp\{S_n - H_n\} = 2^{-nN} \exp\{S_n + Nn\log 2 - H_n\}$$

and, by Lemma 8.1,

$$\begin{aligned} P_n = Nn\log 2 - H_n &= Nn\log 2 - \sum_{k=1}^{n}\sum_{i=1}^{2^N} \lambda_k^i \log\frac{1}{\lambda_k^i} \\ &\leqslant \sum_{k=1}^{n}\left[N\log 2 - \left((2^N-1)\varepsilon_k \log\frac{1}{\varepsilon_k} + (1-(2^N-1)\varepsilon_k)\log\frac{1}{1-(2^N-1)\varepsilon_k}\right)\right]. \end{aligned}$$

An elementary computation shows that

$$\begin{aligned} 0 &\leqslant N\log 2 - \left((2^N-1)x\log\frac{1}{x} + (1-(2^N-1)x)\log\frac{1}{1-(2^N-1)x}\right) \\ &\leqslant C_N(2^{-N}-x)^2 \end{aligned}$$

if $2^{-(N+1)} \leqslant x \leqslant 2^{-N}$, where C_N only depends on N. Therefore

$$P_n = Nn\log 2 - H_n \leqslant C_N \sum_{k=1}^{n}\left(\frac{1}{2^N} - \varepsilon_k\right)^2.$$

At this point we should mention that we can assume that

$$\sum_{k=1}^{\infty}\left(\frac{1}{2^N} - \varepsilon_k\right)^2 = \infty.$$

Indeed, if this series converges then a well-known result of Carleson [**5**] implies that μ is absolutely continuous with respect to Lebesgue measure.

Now, since $\langle S\rangle_n \leqslant 2P_n = 2(Nn\log 2 - H_n)$, we have, for μ-almost every $x \in Q_0$, either $\sup_n |S_n(x)| < \infty$ or $S_n(x) = o\big(\sqrt{P_n(x)\log\log P_n(x)}\big)$ by the results in § 1 and the LIL. Therefore,

$$\mu(Q_n(x)) \leqslant 2^{-nN}\exp\left\{C(x) + C_N\sum_{k=1}^{n}\left(\frac{1}{2^N} - \varepsilon_k\right)^2\right\}.$$

So, we have the following.

THEOREM 8.2. (a) *If μ is a dyadic symmetric measure and (ε_k) are as above, then $\mu \ll \mathcal{H}_\phi$ where ϕ is any measure function such that*

$$\phi(2^{-n}) = 2^{-nN}\exp\left\{c\sum_{k=1}^{n}\left(\frac{1}{2^N} - \varepsilon_k\right)^2\right\}$$

for some $c = c(N) > 0$.

(b) *Let μ be a dyadic symmetric measure which is singular with respect to Lebesgue measure. Then μ is singular with respect to $\mathcal{H}_\phi$ where ϕ is any measure* *function such that*

$$\phi(2^{-n}) = 2^{-nN} \exp\left(c \sum_{k=1}^{n} \left(\frac{1}{2^N} - \varepsilon_k \right)^2 \right)$$

for some $c = c(N)$.

Proof. As in the proof of Corollary 6.2(b) we see that for μ-a.e. $x \in Q_0$, there exists a sequence n_k of natural numbers, $n_k \to \infty$, such that

$$\frac{\mu(Q_{n_k}(x))}{\ell(Q_{n_k}(x))^N} \geqslant \exp(C(P_{n_k}(x) \log\log P_{n_k}(x))^{1/2} + P_{n_k}(x)).$$

Applying Lemma 5.7(b), we deduce that

$$H_n(x) \leqslant \sum_{k=1}^{n} \left((1-\delta_k) \log \frac{1}{1-\delta_k} + \delta_k \log \frac{2^N - 1}{\delta_k} \right)$$

and $(1-x)\log(1-x)^{-1} + x\log((2^N-1)/x)$ is comparable to $(1-2^{-N}-x)^2$ for x close to $1-2^{-N}$. So, we deduce that

$$\frac{\mu(Q_{n_k}(x))}{\ell(Q_{n_k}(x))^N} \geqslant \exp\left(C(n) \sum_{k=1}^{n} (1-2^{-N}-\delta_k)^2 \right)$$

which finishes the proof. $\square$

When specializing these results to particular examples we have, for instance, those shown in Table 2.

TABLE 2.

ε_k	$\phi(t) \quad (\mu \ll \mathcal{H}_\phi)$
$\frac{1}{2^N} - \frac{1}{\sqrt{k}}$	$t^N \left(\log t^{-1}\right)^c$
$\frac{1}{2^N} - \frac{1}{k^{\alpha/2}} \quad (0 < \alpha < 1)$	$t^N \exp\{c\left(\log t^{-1}\right)^{1-\alpha}\}$
$\frac{1}{2^N} - \frac{1}{\sqrt{k \log k}}$	$t^N \exp\{c\left(\log\log t^{-1}\right)^c\}$

9. *Application to a certain class of degenerate elliptic equations*

Now we are ready to apply the previous results to the two situations described in the introduction. Suppose that $h : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and let μ_h be the Lebesgue–Stieltjes measure associated to h. Then if $M(t)$ is as in the introduction, it follows for instance that if $M(t) \leqslant M$, then $\delta_k \geqslant (M+1)^{-1}$ for all k. This observation and Corollary 6.4 prove part (a) in Corollary 9.1. Part (b) is derived from either Theorem 6.1 or Theorem 7.2.

COROLLARY 9.1. (a) *If* $M(t) \leqslant M < \infty$, *that is, if* h *is* M*-quasi-symmetric, then* μ_h *is absolutely continuous with respect to* $\mathcal{H}_{\phi_M}$, *where*

$$\phi_M(t) = t^{\beta(M)} \exp\{C(\log t^{-1} \log\log\log t^{-1})^{1/2}\}.$$

Here

$$\beta(M) = \frac{1}{M+1}\log_2(M+1) + \frac{M}{M+1}\log_2\left(\frac{M+1}{M}\right)$$

and C is some fixed absolute constant.

(b) *Assume that*

$$\lim_{t\to 0}\frac{\int_t^1 \frac{\log M(s)}{M(s)}\frac{ds}{s}}{(\log\log t^{-1})(\log\log\log\log t^{-1})} = \infty.$$

Then μ is absolutely continuous with respect to $\mathcal{H}_\phi$, where

$$\phi(t) = \exp\left\{ -C\int_t^1 \frac{\log M(s)}{M(s)}\frac{ds}{s}\right\}$$

and C is some fixed absolute constant.

The rest of the section is devoted to describing how these kinds of results inform us about the size of harmonic measure for certain degenerate elliptic equations.

To start with, consider a positive locally finite Borel measure μ in $\mathbb{R}$ such that $\mu(-\infty,+\infty)=\infty$, μ has no atoms and $\mu(I)>0$ for any interval I of the real line. Let $h:\mathbb{R}\to\mathbb{R}$ be the associated homeomorphism, that is $h(x)=\mu[0,x]$ if $x\geqslant 0$ and $h(x)=-\mu[x,0]$ for $x<0$. Then $h:\mathbb{R}\to\mathbb{R}$ is a continuous, strictly increasing homeomorphism of the real line. Define

$$u(x,y) = \tfrac{1}{2}\left(\frac{1}{y}\int_x^{x+y} h(t)\,dt + \frac{1}{y}\int_{x-y}^{x} h(t)\,dt\right),$$
$$v(x,y) = \tfrac{1}{2}\left(\frac{1}{y}\int_x^{x+y} h(t)\,dt - \frac{1}{y}\int_{x-y}^{x} h(t)\,dt\right),$$

for $x\in\mathbb{R}$, $y>0$. Then $\Phi:\mathbb{R}^2_+\to\mathbb{R}^2_+$, given by $\Phi(x,y)=(u(x,y),v(x,y))$, is called the Beurling–Ahlfors extension of h [**3**] (or simply, the Beurling–Ahlfors extension of μ).

Computing the derivatives of Φ, as in [**3**], it follows that

$$\Phi' = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} \alpha+\beta & \alpha'-\beta' \\ \alpha-\beta & \alpha'+\beta' \end{pmatrix}$$

where

$$\alpha = \tfrac{1}{2}\frac{h(x+y)-h(x)}{y}, \qquad \beta = \tfrac{1}{2}\frac{h(x)-h(x-y)}{y},$$
$$\alpha' = \tfrac{1}{2}\frac{\int_x^{x+y}[h(x+y)-h(t)]\,dt}{y^2}, \quad \beta' = \tfrac{1}{2}\frac{\int_{x-y}^{x}[h(t)-h(x-y)]\,dt}{y^2}.$$

Note that all α, β, α', β' are strictly positive. Then, $\det\Phi' = 2(\alpha\beta'+\beta\alpha')>0$ and it follows, for elementary topological reasons, that Φ is a C^1 homeomorphism from the upper half-plane into itself such that $\Phi(x,0)=h(x)$, for $x\in\mathbb{R}$.

The connection with elliptic operators is well known [**4**, **12**]. A simple computation with the chain rule and the change of variables formula shows that, if u is harmonic in $\mathbb{R}^2_+$, then $v=u\circ\Phi$ is a solution of the following equation in divergence form [**4**]:

$$Lv = \operatorname{div}(A(x,y)\nabla v) = 0$$

where $A(x,y) = \det \Phi'(x,y)(\Phi'(x,y))^{-1}((\Phi'(x,y))^t)^{-1}$.

An explicit computation of $A(x,y)$ gives

$$A(x,y) = \frac{1}{(\alpha\beta' + \alpha'\beta)(x,y)} \begin{pmatrix} \alpha'^2 + \beta'^2 & \alpha\alpha' - \beta\beta' \\ \alpha\alpha' - \beta\beta' & \alpha^2 + \beta^2 \end{pmatrix}_{(x,y)}.$$

Therefore,

$$(K(x,y))^{-1}\|(x,y)\|^2 \leqslant \langle A(x,y),(x,y)\rangle \leqslant K(x,y)\|(x,y)\|^2$$

where

$$K(x,y) = \left(\frac{\alpha^2 + \beta^2 + \alpha'^2 + \beta'^2}{\alpha\beta' + \alpha'\beta}\right)(x,y) \geqslant 2.$$

In a similar way to what we did in § 7, put

$$\delta(y) = \inf\left\{\frac{\mu(J)}{\mu(I)} : |I| \geqslant y\right\}, \quad M(y) = (\delta(y))^{-1} \tag{9.1}$$

where J is any of the two half-intervals in which I is divided. Note that, in the case of the line, $\delta(\,\cdot\,)$ is the same uniform function as that introduced in § 7, up to a change of scale: y instead of $2y$. Then $\delta(y) \leqslant \frac{1}{2}$, so $M(y) \geqslant 2$. Furthermore, $\delta(y)$ is increasing and $M(y)$ is decreasing for $y > 0$. The following lemma is elementary.

Lemma 9.2. *If α, β, α', β' are as above, then*

$$1 \leqslant \frac{\alpha}{\alpha'}, \frac{\beta}{\beta'} \leqslant 2M(y), \qquad (M(y)-1)^{-1} \leqslant \frac{\alpha}{\beta} \leqslant M(y) - 1.$$

In particular $K(x,y) \leqslant 4M^2(y)$.

Proof. That $\alpha' \leqslant \alpha$ and $\beta' \leqslant \beta$ is trivial from the definitions. On the other hand,

$$\begin{aligned} \frac{\alpha}{\alpha'} &= y\frac{h(x+y) - h(x)}{\int_x^{x+y}[h(x+y) - h(t)]\,dt} \leqslant y\frac{h(x+y) - h(x)}{\int_x^{x+y/2}[h(x+y) - h(t)]\,dt} \\ &\leqslant 2\frac{h(x+y) - h(x)}{h(x+y) - h(x+y/2)} = 2\frac{\mu[x, x+y]}{\mu[x+y/2, x+y]} \leqslant 2M(y). \end{aligned}$$

Also,

$$\frac{\alpha}{\beta} = \frac{h(x+y) - h(x)}{h(x) - h(x-y)} = \frac{\mu[x, x+y]}{\mu[x-y, x]} \leqslant M(2y) - 1 \leqslant M(y) - 1.$$

The other inequalities are analogous. Therefore,

$$\begin{aligned} K(x,y) = \frac{\alpha^2 + \beta^2 + \alpha'^2 + \beta'^2}{\alpha\beta' + \alpha'\beta} &\leqslant \frac{\alpha}{\alpha'}\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\frac{\beta}{\beta'} + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} \\ &\leqslant 4M(y)(M(y)-1) + 2(M(y)-1) \leqslant 4M^2(y). \end{aligned}$$

□

Now, if L is the elliptic operator obtained by pulling back the Laplacian in $\mathbb{R}^2_+$, as described above, $z_0 \in \mathbb{R}^2_+$ is fixed, and $w_0 = \Phi(z_0)$, then, for $E \subset \mathbb{R}$,

$$\omega_L(E, z_0) = \omega_\Delta(h(E), w_0) \asymp |h(E)| = \mu(E)$$

where $\omega_L(\,\cdot\,, z_0)$ (respectively $\omega_\Delta(\,\cdot\,, w_0)$) denotes harmonic measure in $\mathbb{R}^2_+$, from z_0 (respectively w_0) for the operator L (respectively the Laplacian). Here the notation $A \asymp B$ means that there exists a constant $C > 0$ which may depend on z_0 and h, but not on the set E, such that $C^{-1}A \leqslant B \leqslant CA$. We can summarize these preliminaries in the following proposition.

Proposition 9.3. *Let μ be a Borel, locally finite positive, infinite, measure in $\mathbb{R}$ without atoms and satisfying $\mu(I) > 0$ for any interval $I \subset \mathbb{R}$. Let Φ be the Beurling–Ahlfors extension of μ, and $L = \operatorname{div}(A(x,y)\nabla(\,\cdot\,))$, the pull-back of the Laplacian by Φ. Then, the ellipticity of A is related to the doubling behaviour of μ as follows:*

$$(4M^2(y))^{-1}\|(x,y)\|^2 \leqslant \langle A(x,y),(x,y)\rangle \leqslant 4M^2(y)\|(x,y)\|^2$$

where $M(y)$ is defined in (9.1). *Furthermore, $\omega_L(\,\cdot\,) \asymp \mu$ where ω_L is L-harmonic measure from some fixed point in $\mathbb{R}^2_+$. In particular, $\omega_L \ll \mathcal{H}_\phi$ if and only if $\mu \ll \mathcal{H}_\phi$.*

Now we are ready to apply Theorem 7.2 to this situation.

Theorem 9.4. *Let μ, L, ω_L, $M(y)$ be as in Proposition* 9.2. *Suppose that*

$$\lim_{t\to 0} \frac{\int_t^1 \frac{\log M(s)}{M(s)}\frac{ds}{s}}{(\log\log t^{-1})(\log\log\log\log t^{-1})} = \infty.$$

Then, there is some absolute constant $C > 0$ such that $\omega_L \ll \mathcal{H}_\phi$, where

$$\phi(t) = \exp\left\{-C\int_t^1 \frac{\log M(s)}{M(s)}\frac{ds}{s}\right\}.$$

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