REVERSE CARLESON MEASURES IN HARDY SPACES

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ABSTRACT. We give a necessary and sufficient condition for a measure μ in the closed unit disk to be a reverse Carleson measure for Hardy spaces. This extends a previous result of Lefèvre, Li, Queffélec and Rodríguez-Piazza [LLQR]. We also provide a simple example showing that the analogue for the Paley-Wiener space does not hold. As it turns out the analogue never holds in any model space.

1. INTRODUCTION

For $1 \le p < \infty$ let H^p be the Hardy space on the unit disk $\mathbb{D}$ equipped with its usual norm

$$\|f\|_p = \left(\sup_{r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \, \frac{d\theta}{2\pi} \right)^{1/p} .$$

Denote by $M_+(\mathbb{D})$ the set of positive, finite Borel measures supported on $\mathbb{D}$, and let $\mu \in M_+(\mathbb{D})$. A well known theorem by Carleson (see [Gar, Chap.I Th. 5.6]) states that H^p embeds into $L^p(\mathbb{D}, \mu)$: there exists $C > 0$ such that

$$\|f\|_{L^p(\mathbb{D},\mu)} \le C\|f\|_p, \quad f \in H^p, \tag{1.1}$$

if and only if μ satisfies the Carleson condition: there exists $C > 0$ such that for all arcs I in $\partial\mathbb{D}$

$$\mu(S_I) \le C|I|, \tag{1.2}$$

where $S_I = \{z \in \overline{\mathbb{D}} : 1 - |I| \le |z| \le 1, z/|z| \in I\}$ is the usual Carleson window. This theorem has been extended to several other spaces, like Bergman, Fock, model spaces etc., and we refer the reader to the huge bibliography on this topic for further information.

Note that H^p contains a dense set of continuous functions for which the embedding (1.1) still makes sense when the measure has a part supported on the boundary. Then (1.2) implies that the restriction of the measure μ to the boundary has to be absolutely continuous with respect to Lebesgue measure and with bounded Radon-Nikodym derivative. It is thus possible to consider, more generally, positive, finite Borel measures supported on the closed unit disk: $M_+(\overline{\mathbb{D}})$.

Here, we are interested in reverse Carleson inequalities $\|f\|_p \le C\|f\|_{L^p(\overline{\mathbb{D}},\mu)}$, $f \in C(\overline{\mathbb{D}}) \cap H^p(\mathbb{D})$, $1 < p < \infty$. In [LLQR] Lefèvre et al. proved that when μ is already a Carleson measure

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Date: January 23, 2014.

2010 *Mathematics Subject Classification.* 30H10, 30J99.

Key words and phrases. Hardy space, Carleson measure, reverse Carleson measure, reproducing kernel, reproducing kernel thesis, model space.

First author supported by ANR FRAB: ANR-09-BLAN-0058-02 and CTP network "Analyse mathématique et applications". Other authors supported by the Generalitat de Catalunya (grants 2009 SGR 01303 and 00420) and the spanish Ministerio de Ciencia e Innovación (projects MTM2011-27932-C02-01 and MTM2011-0374).

these hold if and only it there exists $C > 0$ such that for all arcs $I \subset \partial\mathbb{D}$

$$\mu(S_I) \geq C|I|.$$

We will show that this result can be deduced from a well-known balayage argument which does not require the Carleson condition. It will be clear from this argument that we have a reproducing kernel thesis for the reverse embedding: if the embedding holds on the reproducing kernels, then it actually holds for every function.

It turns out that the interesting part of the measure has to be supported on the boundary, while the part supported in the disk can be dropped.

Finally, we provide a simple example showing that the analogous reproducing kernel thesis for the reverse embedding in the Paley-Wiener space does not hold. We will actually show that the reproducing kernel thesis for the reverse embedding never holds in any model space. In a previous version of this work, the construction valid in the Paley-Wiener space was generalized to the situation of so-called one-component inner functions. We are very grateful to Anton Baranov who suggested the shorter proof presented below and which gives the result in general model spaces.

We shall use the following standard notation: $f \lesssim g$ means that there is a constant C independent of the relevant variables such that $f \leq Cg$, and $f \simeq g$ means that $f \lesssim g$ and $g \lesssim f$.

2. Main result

For $1 < p < \infty$ and $\lambda \in \mathbb{D}$ consider the reproducing kernel in H^p

$$k_\lambda(z) = \frac{1}{1 - \overline{\lambda} z}, \quad z \in \mathbb{D},$$

and its normalized companion

$$K_\lambda := \frac{k_\lambda}{\|k_\lambda\|_p} \ .$$

A standard computation shows that $\|k_\lambda\|_p \simeq (1 - |\lambda|)^{-1/p'}$, where $1/p + 1/p' = 1$.

Our main result reads as follows.

Theorem 2.1. *Let $1 < p < \infty$ and let $\mu \in M_+(\overline{\mathbb{D}})$. Then the following assertions are equivalent:*

(1) *There exists $C_1 > 0$ such that for every function $f \in H^p \cap C(\overline{\mathbb{D}})$,*

$$\int_{\overline{\mathbb{D}}} |f|^p d\mu \geq C_1 \|f\|_p^p \ ,$$

(2) *There exists $C_2 > 0$ such that for every $\lambda \in \mathbb{D}$,*

$$\int_{\overline{\mathbb{D}}} |K_\lambda|^p d\mu \geq C_2 \ ,$$

(3) *There exists $C_3 > 0$ such that for every arc $I \subset \partial\mathbb{D}$,*

$$\mu(S_I) \geq C_3|I| \ .$$

(4) *There exists $C_4 > 0$ such that the Radon-Nikodym derivative of $\mu|_{\partial\mathbb{D}}$ with respect to the length measure is bounded below by C_4.*

The key implication of the above result if of course $(2) \Longrightarrow (3)$ which is based on a balayage argument.

Observe that in this theorem we do not require absolute continuity of the restriction $\mu|_{\partial\mathbb{D}}$. Still, if we want to extend (1) to the entire H^p-space, then, in order that $\int_{\overline{\mathbb{D}}} |f|^p d\mu$ makes sense for every function in H^p, we need to impose absolute continuity on $\mu|_{\partial\mathbb{D}}$. Note that the integral $\int_{\overline{\mathbb{D}}} |f|^p d\mu$ can be infinite for certain $f \in H^p$ when the Radon-Nikodym derivative of $\mu|_{\partial\mathbb{D}}$ is not bounded.

Proof. (1) $\Rightarrow$ (2) is clear.

(3) $\Rightarrow$ (4). Take $h > 0$ so that $|I|/h$ is a large integer N and consider the modified Carleson window

$$S_{I,h} = \{z \in \overline{\mathbb{D}} : 1 - h \le |z| \le 1,\ z/|z| \in I\}\ .$$

Split I into N subarcs I_k such that $|I_k| = h$ (and hence $S_{I_k,h} = S_{I_k}$). Then

$$\mu(S_{I,h}) = \mu(\bigcup_{k=1}^{N} S_{I_k,h}) = \sum_{k=1}^{N} \mu(S_{I_k,h}) \ge C_3 \sum_{k=1}^{N} |I_k| = C_3|I|.$$

Now, for every open set O in $\overline{\mathbb{D}}$ for which $I \subset O$ there exists $h > 0$ such that $S_{I,h} \subset O$. Since $\mu \in M_+(\mathbb{D}^-)$ is outer regular (see [Ru, Theorem 2.18]) we thus have

$$\mu(I) = \inf_{I \subset O \text{ open in } \overline{\mathbb{D}}} \mu(O) \ge \inf_{h>0} \mu(S_{I,h}) \ge C_3|I|.$$

We deduce that the Lebesgue measure on $\partial\mathbb{D}$, denoted by m, is absolutely continuous with respect to the restriction of μ to $\partial\mathbb{D}$ and that the corresponding Radon-Nikodym derivative of μ is bounded below by C_3. In particular one can choose $C_4 = C_3$.

$(4) \Rightarrow (1)$ Clearly, for all $f \in H^p$,

$$\int_{\overline{\mathbb{D}}} |f|^p d\mu \ge \int_{\partial\mathbb{D}} |f|^p d\mu \ge C_4 \int_{\partial\mathbb{D}} |f|^p dm = C_4 \|f\|_p^p$$

(in particular, one can choose $C_1 = C_4$).

(2) $\Rightarrow$ (3). Observe that when $p = 2$, then $|K_\lambda(z)|^2$ is nothing but the Poisson kernel, for which the arguments below are very transparent. Let us however do the argument for general p. By hypothesis, integrating over $S_{I,h}$ with respect to area measure dA on $\overline{\mathbb{D}}$ we get

$$C_2|I| \times h \le \int_{S_{I,h}} \int_{\overline{\mathbb{D}}} |K_\lambda|^p d\mu\, dA(\lambda) \simeq \int_{\overline{\mathbb{D}}} \int_{S_{I,h}} \frac{(1 - |\lambda|^2)^{p/p'}}{|1 - \overline{\lambda} z|^p} dA(\lambda) d\mu(z).$$

Set

$$\varphi_h(z) = \frac{1}{h} \int_{S_{I,h}} \frac{(1 - |\lambda|^2)^{p/p'}}{|1 - \overline{\lambda} z|^p} dA(\lambda) = \frac{1}{h} \int_{S_{I,h}} \frac{(1 - |\lambda|^2)^{p-1}}{|1 - \overline{\lambda} z|^p} dA(\lambda),$$

so that the previous estimate becomes

$$\int_{\overline{\mathbb{D}}} \varphi_h(z) d\mu(z) \gtrsim |I|\ . \tag{2.1}$$

We claim that

$$\lim_{h \to 0} \varphi_h(z) \begin{cases} \simeq 1 & \text{if } z \in \overline{I}, \\ = 0 & \text{otherwise.} \end{cases}$$

Indeed, if $z\notin \overline{I}$, then there are $\delta, h_0>0$ such that for every $0<h<h_0$ and for every $\lambda\in S_{I,h}$, we have $|1-\overline{\lambda}z|\geq\delta>0$, and the result follows from the estimate

$$0\leq\varphi_h(z)=\frac{1}{h}\int_{S_{I,h}}\frac{(1-|\lambda|^2)^{p-1}}{|1-\overline{\lambda}z|^p}dA(\lambda)\leq\frac{1}{\delta^p}\frac{|I|\times h}{h}\times(2h)^{p-1}\lesssim h^{p-1}.$$

Suppose now that $z=e^{i\theta_0}\in\overline{I}$. Let $h\leq|I|$, then setting $\lambda=(1-t)e^{i\theta}$ for $\lambda\in S_{I,h}$ we have

$$\begin{aligned}\varphi_h(z) &= \frac{1}{h}\int_{S_{I,h}}\frac{(1-|\lambda|^2)^{p-1}}{|1-\overline{\lambda}z|^p}dA(\lambda)\geq\frac{1}{h}\int_{e^{i\theta}\in I}\int_0^h\frac{t^{p-1}}{|e^{i\theta_0}-(1-t)e^{i\theta}|^p}(1-t)dtd\theta\\ &\gtrsim \frac{1}{h}\int_0^h\int_{|\theta-\theta_0|\leq t, e^{i\theta}\in I}\frac{t^{p-1}}{|\theta-\theta_0|^p+t^p}d\theta dt\\ &\geq \frac{1}{h}\int_0^h\int_{|\theta-\theta_0|\leq t, e^{i\theta}\in I}\frac{t^{p-1}}{2t^p}d\theta dt.\end{aligned}$$

Since $0\leq t\leq h\leq|I|$ and $z=e^{it}\in\overline{I}$, the set $\{e^{i\theta}:|\theta-\theta_0|\leq t, e^{i\theta}\in I\}$ contains an interval of length at least $t/2$, we get

$$\varphi_h(z)\ \gtrsim\ \frac{1}{h}\int_0^h\frac{t}{2}\times\frac{t^{p-1}}{2t^p}dt\simeq 1.$$

On the other hand, integrating in polar coordinates, we get

$$\begin{aligned}\varphi_h(z) &= \frac{1}{h}\int_{S_{I,h}}\frac{(1-|\lambda|^2)^{p-1}}{|1-\overline{\lambda}z|^p}dA(\lambda)=\frac{1}{h}\int_{1-h}^1(1-r^2)^{p-1}\int_I\frac{1}{|1-re^{i(\theta-\theta_0)}|^p}d\theta rdr\\ &\lesssim \frac{1}{h}\int_0^h t^{p-1}\frac{1}{t^{p/p'}}dt\simeq 1.\end{aligned}$$

Hence φ_h converges pointwise to a function comparable to $\chi_{\overline{I}}$, and φ_h is uniformly bounded in h. Now, from (2.1) and by dominated convergence we finally deduce that

$$\mu(\overline{I})=\int_{\mathbb{D}^-}\chi_{\overline{I}}d\mu\simeq\int_{\overline{\mathbb{D}}}\lim_{h\to0}\varphi_h(z)d\mu(z)=\lim_{h\to0}\int_{\overline{\mathbb{D}}}\varphi_h(z)d\mu(z)\gtrsim|I|\,.$$

■

Remark. The following example shows that the reproducing kernel thesis fails for the reverse Carleson inequalities in the Paley-Wiener space PW_π, the space of Fourier transforms of square integrable functions on $[-\pi,\pi]$. In Section 2 we will show how it can be adapted to any model space.

Consider the sequence $S=\{x_n\}_{n\in\mathbb{Z}\setminus\{0\}}$, where

$$x_n=\begin{cases}n+1/8 & \text{if } n \text{ is even}\\ n-1/8 & \text{if } n \text{ is odd.}\end{cases}$$

By the Kadets-Ingham theorem (see e.g. [Nik, Theorem D4.1.2]) S would be a minimal sampling sequence if we added the point 0. Since S is not sampling the discrete measure $\mu:=\sum_{n\neq0}\delta_{x_n}$ does not satisfy the reverse inequality $\|f\|_{L^2(\mathbb{R})}\lesssim\|f\|_{L^2(\mu)}$, $f\in PW_\pi$.

Let us see that, on the other hand, the μ-norm of the normalized reproducing kernels

$$K_\lambda(z) = c_\lambda \operatorname{sinc}(\pi(z-\lambda)) = c_\lambda \frac{\sin(\pi(z-\lambda))}{\pi(z-\lambda)}, \qquad c_\lambda^2 \simeq (1+|\operatorname{Im}\lambda|)e^{-2\pi|\operatorname{Im}\lambda|},$$

is uniformly bounded from below. If λ is such that $|\operatorname{Im}\lambda| > 1$ then $|\sin(\pi(x_n-\lambda))| \simeq e^{\pi|\operatorname{Im}\lambda|}$, and hence

$$\int_{\mathbb{C}} |K_\lambda(x)|^2 d\mu(x) = \sum_{n\neq 0} c_\lambda^2 \left|\frac{\sin(\pi(x_n-\lambda))}{\pi(x_n-\lambda)}\right|^2 \simeq \sum_{n\neq 0} \frac{|\operatorname{Im}\lambda|}{|x_n-\lambda|^2} \simeq 1.$$

It is thus enough to consider points $\lambda \in \mathbb{C}$ with $|\operatorname{Im}\ \lambda| \leq 1$. Let x_{n_0} be the point of S closest to λ; then there is $\delta > 0$, independent of λ, such that

$$\int_{\mathbb{C}} |K_\lambda(x)|^2 d\mu(x) = \sum_{n\neq 0} |K_\lambda(x_n)|^2 \geq \left|\frac{\sin(\pi(x_{n_0}-\lambda))}{\pi(x_{n_0}-\lambda)}\right|^2 \geq \delta\ .$$

It is interesting to point out that μ is a Carleson measure for PW_π, since S is in a strip and separated.

3. Failure in general model spaces

The previous construction can be generalized to certain model spaces in the disk. The model space associated to an inner function Θ is $K_\Theta = H^2 \ominus \Theta H^2$, and the reproducing kernel corresponding to $\lambda \in \mathbb{D}$ is given by

$$k_\lambda^\Theta(z) = \frac{1-\overline{\Theta(\lambda)}\Theta(z)}{1-\bar{\lambda}z}, \quad z \in \mathbb{D}.$$

If Θ is a finite Blaschke product of degree strictly bigger than one, picking for instance $\mu = \delta_0$ we immediatly get the reverse inequality on reproducing kernels (see (3.1)). Clearly μ is Carleson, and since the degree of Θ is not one, we can combine two linearly independent functions of K_Θ vanishing at 0.

In the general case we need to construct a measure supported on $\mathbb{T}$.

Theorem 3.1. *Let Θ be an inner function which is not a finite Blaschke product. Then there exists a measure μ on $\mathbb{T}$ such that $K_\Theta \subset L^2(\mu)$, the measure μ satisfies the reverse estimate on reproducing kernels k_λ^Θ,*

$$\|k_\lambda^\Theta\|_{L^2(\mu)} \geq C\|k_\lambda^\Theta\|_2, \qquad \lambda \in \mathbb{D}, \tag{3.1}$$

but the reverse Carleson embedding for the space K_Θ does not hold.

Proof. Let us first assume that Θ vanishes at $z_0 = 0$, and let $\Theta = z\Theta_0$. Denote by μ the Clark measure for Θ_0, that is μ is defined by

$$\operatorname{Re}\frac{1+\Theta_0(z)}{1-\Theta_0(z)} = \int_{\mathbb{T}} \frac{1-|z|^2}{|\tau-z|^2} d\mu(\tau).$$

Clark introduced these measures in [Cl]. Observe first that $K_\Theta = \mathbb{C} \oplus zK_{\Theta_0}$ which implies that $K_\Theta \subset L^2(\mu)$. Continuous functions are dense in K_Θ (see [Al3] or [CMR, p.187]) and functions in K_Θ are μ-measurable (see [P]). In particular, if we had reverse embedding on continuous functions then we would have it on the whole space K_Θ. It is thus sufficient to find a non-zero function $f \in K_\Theta$ with zero $L^2(\mu)$-norm. To this end, pick $f(z) = 1 - \Theta_0(z)$ which belongs to K_Θ. Clearly $f = 0$ μ-a.e, so that $\int |f|^2 d\mu = 0$.

Let us show that (3.1) is satisfied. Any reproducing kernel k_λ^Θ has representation as the (orthogonal) sum

$$k_\lambda^\Theta(z) = k_\lambda^{\Theta_0}(z) + \overline{\Theta_0(\lambda)}\Theta_0(z),$$

and

$$\|k_\lambda^\Theta\|_2^2 = \|k_\lambda^{\Theta_0}\|_2^2 + |\Theta_0(\lambda)|^2.$$

In particular,

$$\|k_\lambda^{\Theta_0}\|_{L^2(\mu)} \le \|k_\lambda^\Theta\|_{L^2(\mu)} + \big(\mu(\mathbb{T})\big)^{1/2}.$$

Also, since μ is a Clark measure for K_{Θ_0}, we have

$$\|k_\lambda^{\Theta_0}\|_{L^2(\mu)} = \|k_\lambda^{\Theta_0}\|_2.$$

Thus, we clearly have

$$\|k_\lambda^\Theta\|_{L^2(\mu)} \ge \|k_\lambda^{\Theta_0}\|_{L^2(\mu)} - \big(\mu(\mathbb{T})\big)^{1/2} \ge \|k_\lambda^\Theta\|_2 - 1 - \big(\mu(\mathbb{T})\big)^{1/2} \ge \frac{1}{2}\|k_\lambda^\Theta\|_2$$

for λ such that $\|k_\lambda^\Theta\|_2 \ge 2(1 + \big(\mu(\mathbb{T})\big)^{1/2})$.

Assume that there exists a sequence λ_n such that $\|k_{\lambda_n}^\Theta\|_2 \le 2(1 + \big(\mu(\mathbb{T})\big)^{1/2})$ and (3.1) does not hold for λ_n with any positive C. Since the norms of the kernels are supposed bounded on λ_n, this implies that

$$\|k_{\lambda_n}^\Theta\|_{L^2(\mu)} \to 0.$$

Passing if necessary to a subsequence we may assume that $\lambda_n \to \lambda_0$, and it follows from the fact that the norms $\|k_{\lambda_n}^\Theta\|_2$ are uniformly bounded that even in the case when $\lambda_0 \in \mathbb{T}$ we still have that the kernel $k_{\lambda_0}^\Theta$ is correctly defined and belongs to K_Θ. Then, by the Fatou lemma, we have that $\|k_{\lambda_0}^\Theta\|_{L^2(\mu)} = 0$. Hence, $1 - \overline{\Theta(\lambda_0)}\Theta(z) = 0$ μ-a.e. But $\Theta_0(z) = 1$ μ-a.e. Thus, $1 - \overline{\Theta(\lambda_0)}z = 0$ μ-a.e. z, which is impossible if the support of μ contains at least two points (but it does, since Θ_0 is not a single Blaschke factor).

Let us now discuss the situation when Θ does not vanish at 0. For $a = \Theta(0)$, the Frostman shift

$$\Theta_a(z) := \frac{\Theta(z) - a}{1 - \overline{a}\Theta(z)}$$

has a zero at 0. By the above discussions, there is a measure μ_a such that $K_{\Theta_a} \subset L^2(\mu_a)$, the reverse estimate holds on the kernels:

$$\|k_\lambda^{\Theta_a}\|_{L^2(\mu_a)} \ge C\|k_\lambda^{\Theta_a}\|_2, \quad \lambda \in \mathbb{D}, \tag{3.2}$$

and there is a non zero function $f_0 \in K_{\Theta_a}$ with $\|f_0\|_{L^2(\mu_a)} = 0$. Recall that the Crofoot transform $U_\Theta^a : K_\Theta \longrightarrow K_{\Theta_a}$, defined by

$$(U_\Theta^a f)(z) := \frac{\sqrt{1 - |a|^2}}{1 - \overline{a}\Theta(z)} f(z), \quad z \in \mathbb{D},$$

is isometric onto K_{Θ_a}. An easy computation gives

$$k_\lambda^{\Theta_a} = U_\Theta^a(c_{a,\lambda} k_\lambda^\Theta), \quad \lambda \in \mathbb{D},$$

where
$$c_{a,\lambda} = \frac{\sqrt{1-|a|^2}}{1-\overline{a}\Theta(\lambda)}.$$
Setting now
$$d\mu(z) = \frac{1-|a|^2}{|1-\overline{a}\Theta(z)|^2} d\mu_a(z)$$
(which is well defined since $|a| < 1$), we have for every $f \in K_\Theta$,
$$\|f\|_{L^2(\mu)} = \|U_\Theta^a f\|_{L^2(\mu_a)}.$$
Using (3.2) and the isometry property of the Crofoot transform, we get
$$C\|k_\lambda^\Theta\|_2 = C\|U_\Theta^a k_\lambda^\Theta\|_2 \leq \|U_\Theta^a k_\lambda^\Theta\|_{L^2(\mu_a)} = \|k_\lambda^\Theta\|_{L^2(\mu)},$$
and so μ satisfies (3.1). We also have the Carleson measure condition for this measure: for every $f \in K_\Theta$
$$\|f\|_2 = \|U_\Theta^a f\|_2 \gtrsim \|U_\Theta^a f\|_{L^2(\mu_a)} = \|f\|_{L^2(\mu)}.$$
Finally, since there is $0 \neq f_0 \in K_{\Theta_a}$ with $\|f_0\|_{L^2(\mu_a)} = 0$, take the unique $0 \neq g_0 \in K_\Theta$ with $U_\Theta^a g_0 = f_0$, then
$$\|g_0\|_{L^2(\mu)} = \|U_\Theta^a g_0\|_{L^2(\mu_a)} = \|f_0\|_{L^2(\mu_a)} = 0.$$
■

Note that the above proof actually works for finite Blaschke product with degree at least 3.

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