A lower bound for the maximum topological entropy of $4k+2-$cycles of interval maps

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Outline

The type of an orbit

Let $f \in \mathcal{C}^0([0,1])$ and let P be a *finite totally invariant* set of f, (that is, $f(P) = P$ — so P is a periodic orbit or union of periodic orbits).

Intrinsic information about the map is encoded in the set P.

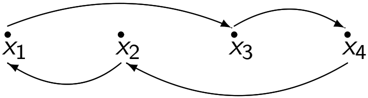
We can think of the set P as a permutation θ induced by $f|_P$ in a natural way: if $P = \{p_1 < p_2 < \cdots < p_n\}$ then we define $\theta_P \colon \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$:

$$\theta_P(i) = j \quad \Longleftrightarrow \quad f(p_i) = p_j.$$

The permutation θ_P is called the *type* of P.

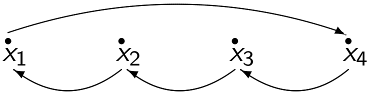
An example

The period 4 orbit $f(x_1) = x_3,\ f(x_2) = x_1,\ f(x_3) = x_4,\ f(x_4) = x_2$ has a different type than the period 4 orbit $g(x_1) = x_4,\ g(x_2) = x_1,\ g(x_3) = x_2,\ g(x_4) = x_3$:



f

$(1, 3, 4, 2)$



g

$(1, 4, 3, 2)$

Statement of the problem

For the sets P_n of permutations of length n and the sets C_n of cyclic permutations of length n ($n \in \mathbb{N}$), can we identify those elements which represent the periodic orbits and invariant sets which are the most complicated in terms of their dynamics?

We use the *topological entropy* to measure the complexity of these permutations.

Definition of the topological entropy of a permutation

The *topological entropy of a permutation* θ, which will be denoted by $h(\theta)$, is defined as follows:

$$h(\theta) := \inf\{h(f) \colon f \in \mathcal{C}^0([0,1]) \text{ has an invariant set of type } \theta\},$$

where $h(f)$ denotes the topological entropy of f.

The topological entropy of a map was first defined in [AKM]. It is a topological invariant which measures the dynamical complexity of f (for more information on the definition and basic properties of the topological entropy, see also [ALM]).

[AKM] R. L. Adler, A. G. Konheim, and M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. **114** (1965), 309–319.

[ALM] Lluís Alsedà, Jaume Llibre, and Michał Misiurewicz, *Combinatorial dynamics and entropy in dimension one*, second ed., Advanced Series in Nonlinear Dynamics, vol. 5, World Scientific Publishing Co. Inc., River Edge, NJ, 2000.

The minimisation problem I

The converse problem, that is to characterise the elements of C_n with the smallest possible entropy, has been solved by several authors and in several steps.

The formula for the smallest possible entropy in C_n was obtained in [BGMY] (see also [ALM]).

[BGMY] L. Block, J. Guckenheimer, M. Misiurewicz and L.-S. Young, *Periodic points and topological entropy of one dimensional maps*, Global theory of dynamical systems, 18–34, Lecture Notes in Math. **819**, Springer, Berlin, 1980.

[ALM] Lluís Alsedà, Jaume Llibre, and Michał Misiurewicz, *Combinatorial dynamics and entropy in dimension one*, second ed., Advanced Series in Nonlinear Dynamics, vol. 5, World Scientific Publishing Co. Inc., River Edge, NJ, 2000.

The minimisation problem II

The characterisation of the elements of C_n with minimum complexity has been given in [Stefan, Block] when n is odd or a power of two and in [ALS, BICop] (independently) in the case $n = 2^q p$ with $q > 0$ and $p \geq 3$. The last case was also obtained simultaneously and independently by L. Snoha and C. W. Ho. A unified description can be found in [ALM].

[Stefan] P. Štefan, *A theorem of Šarkovskii on the existence of periodic orbits of continuous endomorphisms of the real line*, Comm. Math. Phys. **54** (1977), 237–248.

[Block] L. Block, *Simple periodic orbits of mappings of the interval*, Trans. Amer. Math. Soc. **254** (1979), 391–398.

[ALS] L. Alsedà, J. Llibre and R. Serra, *Bifurcation for a circle map family associated with the Van der Pol equation*, Théorie de l'iteration et ses applications, 177–186, Editions du CNRS, Paris, 1982.

[BICop] L. S. Block and W. A. Coppel, *Stratification of continuous maps of an interval*, Trans. Amer. Math. Soc. **297** (1986), 587–604.

[ALM] Lluís Alsedà, Jaume Llibre, and Michał Misiurewicz, *Combinatorial dynamics and entropy in dimension one*, second ed., Advanced Series in Nonlinear Dynamics, vol. 5, World Scientific Publishing Co. Inc., River Edge, NJ, 2000.

Computation of the topological entropy of a permutation I

Typically, computing the entropy of a map is difficult.

However, the computation of the entropy of a permutation can be easily done by using the following algebraic tools:

The connect the dots map

If S is a finite, fully invariant set for f of type θ, then there is a unique map $f_\theta \colon [1, n] \longrightarrow [1, n]$ which satisfies

1. $f_\theta(i) = \theta(i), \quad$ for $i \in \{1, \dots, n\}$,
2. f_θ is affine on each interval $I_i = \{x \in \mathbb{R} : i \le x \le i + 1\}$ for each $i \in \{1, \dots, n-1\}$.

The map f_θ is known as the *connect-the-dots map* and clearly it has an invariant set of type θ.

Example: The connect the dots map of the permutation $\theta = (1, 3, 4, 2)$

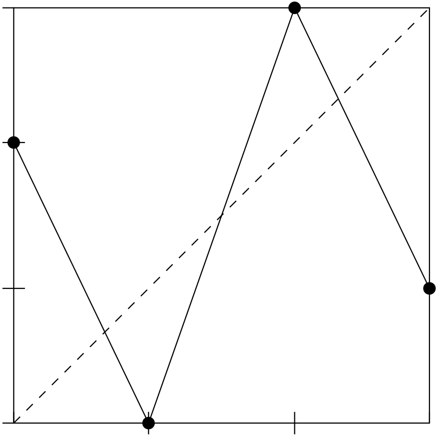


Figure: The connect the dots map $f_{(1,3,4,2)}$

Computation of the topological entropy of a permutation II

The Markov matrix

From the connect-the-dots map we can construct a matrix $M(\theta) = (m_{ij})$ with m_{ij} given by:

$$m_{ij} = \begin{cases} 1, & \text{if } f_\theta(I_i) \supset I_j \\ 0, & \text{otherwise} \end{cases}$$

for $i, j \in \{1, \ldots, n-1\}$.
It is well known that

$$h(\theta) = h(f_\theta) = \max\{\log(\rho(M(\theta))), 0\},$$

where $\rho(M(\theta))$ is the spectral radius of $M(\theta)$.

Example: The computation of the entropy of a permutation

We take $\theta = (1, 3, 4, 2)$.

According to the permutation data:

$$\begin{array}{ccc} 1 & \longmapsto & 3 \\ 2 & \longmapsto & 1 \\ 3 & \longmapsto & 4 \\ 4 & \longmapsto & 2 \end{array}$$

Thus, since $I_1 = [1, 2]$, $I_2 = [2, 3]$ and $I_3 = [3, 4]$, we obtain

$$M(\theta) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Finally,

$$h(\theta) = \log \rho(M(\theta)) = \log(1 + \sqrt{2})$$

History: The asymptotic bound

In [MiNi] it is obtained an asymptotic result which shows that the maximum entropy for n-cycles and n-permutations approaches $\log(2n/\pi)$ as $n \to \infty$. To prove this result, they constructed a "diamond shaped" family of *cyclic* permutations of period $n \equiv 1 \pmod 4$ which had the required asymptotic growth rate.

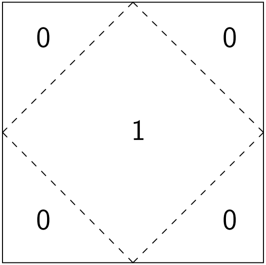


Figure: An approximate shape of the matrix for a cycle with maximal entropy.

[MiNi] Michał Misiurewicz and Zbigniew Nitecki, *Combinatorial patterns for maps of the interval*, Mem. Amer. Math. Soc. **94** (1991), no. 456, vi+112.

History: The odd case

Geller and Tolosa [GT] extended the Misiurewicz-Nitecki definition to a family of periodic orbits of period $n \equiv 3 \pmod 4$ and proved that this family in fact did have maximum entropy amongst all n-permutations such that $n \equiv \{1, 3\} \pmod 4$ (this family was later shown to be unique [GW]). Since the family described is a family of cyclic permutations, the question of which n-cycles and n-permutations have maximum topological entropy for n odd has been completely answered.

[GT] William Geller and Juán Tolosa, *Maximal entropy odd orbit types*, Trans. Amer. Math. Soc. **329** (1992), no. 1, 161–171.

[GW] William Geller and Benjamin Weiss, *Uniqueness of maximal entropy odd orbit types*, Proc. Amer. Math. Soc. **123** (1995), no. 6, 1917–1922.

History: The even case for permutations

For the case n even, the maximum entropy n-permutations are non-cyclic. All maximum entropy n-permutations for n even were described by King [K, K2] and independently by Geller and Zhang [GZ].

- **[K]** Deborah M. King, *Maximal entropy of permutations of even order*, Ergodic Theory Dynam. Systems **17** (1997), no. 6, 1409–1417.
- **[K2]** Deborah M. King, *Non-uniqueness of even order permutations with maximal entropy*, Ergodic Theory Dynam. Systems **20** (2000), no. 3, 801–807.
- **[GZ]** William Geller and Zhenhua Zhang, *Maximal entropy permutations of even size*, Proc. Amer. Math. Soc. **126** (1998), no. 12, 3709–3713.

History: The even case for cyclic permutations

Two families of maximum entropy $4k$-cycles have been described [KS] and have recently been shown to be the only two families with maximum entropy (up to a reversal of orientation) [KS2].

[KS] Deborah M. King and John B. Strantzen, *Maximum entropy of cycles of even period*, Mem. Amer. Math. Soc. **152** (2001), no. 723, viii+59.

[KS2] Deborah M. King and John B. Strantzen, *Cycles of period 4k which attain maximum topological entropy*, (2005), Preprint.

Aim

We want to classify the maximum entropy $4k+2$-cycles, thus solving the remaining case in the characterization of the maximum entropy cycles for the interval.

What we really do?

We formulate a conjecture on the shape of the maximum entropy $4k+2$-cycles supported by adhoc numerical simulations.

Thus this is indeed an excercise of numerical simulation on the maximum entropy cycles.

Strategy: Jungreis Theorem

The set C_n (of cyclic permutations of order n) is endowed with a partial order, usually called the *forcing relation* (see [J] or [MiNi] for details). It has been shown that topological entropy respects this partial order on C_n (see [MiNi]), so that if ϕ is smaller than θ in the forcing relation then $h(\phi) \leq h(\theta)$. As a consequence, any candidates for maximum entropy cycles must be forcing-maximal in C_n.

[J] Irwin Jungreis, *Some results on the Šarkovskiĭ partial ordering of permutations*, Trans. Amer. Math. Soc. **325** (1991), no. 1, 319–344.

[MiNi] Michał Misiurewicz and Zbigniew Nitecki, *Combinatorial patterns for maps of the interval*, Mem. Amer. Math. Soc. **94** (1991), no. 456, vi+112.

Jungreis Cycles (definition and consequences)

Definition (Maximodal)

An n-permutation θ will be called *maximodal* if every point $1, 2, \dots, n$ is either a local maximum or a local minimum for f_θ.

Definition (Jungreis Cycle)

An n-cycle θ will be called a *Jungreis cycle* if it is maximodal and f_θ satisfies one of the following conditions:

`zero exceptions` all maximum values are above all minimum values, or

`one exception` exactly one maximum value is less than some minimum value and exactly one minimum value is greater than some maximum value.

The sets of Jungreis n-cycles satisfying the first (respectively second) condition will be denoted by C_n^0 (respectively C_n^1).

The following result is an immediate consequence of Corollary 9.6 and Theorem 9.13 of [J] together with the previously stated fact that topological entropy respects the forcing relation:

Theorem

Each forcing maximal cycle in C_n is a Jungreis cycle. Consequently, each maximum entropy cycle in C_n is a Jungreis cycle.

Thus, to compute the maximum entropy n-cycles it is enough to explore the class of all Jungreis n-cycles and choose the ones with larger entropy.

[J] Irwin Jungreis, *Some results on the Šarkovskiĭ partial ordering of permutations*, Trans. Amer. Math. Soc. **325** (1991), no. 1, 319–344.

First attempt: “brute force”

It consists in generating all maximodal n-cycles in lexicographic order (by using the material described in [KS] Section 2.4: Permutations — 2.4.1 Lexicographic ordering) and discarding those which are not Jungreis. For the remaining ones the Markov matrix and its spectral radius is computed (by using the power method), thus allowing us to choose the maximal ones.

Using this aproach we have computed the topological entropy of all Jungreis cycles in C_n for $n \leq 17$ and determined the maximum entropy cycles.

In doing this we have obtained the unknown maximal entropy cycles for periods $6, 10$ and 14 and tested the feasibility of this naive approach.

[KS] Donald L. Kreher and Douglas Stinson, *Combinatorial Algorithms:Generation, Enumeration & Search*, CRC Press series on Discrete Mathematics and its applications, Boca Ratón, Florida, U.S.A., 1999.

The number of Jungreis cycles for low periods

To have an idea of the computational complexity of this task, see the following table:

n	Card(C_n^0)	Card(C_n^1)	Total
4	2	0	2
5	2	1	3
6	7	5	12
7	24	15	39
8	72	105	177
9	288	561	849
10	1452	3228	4680
11	8640	20548	29188
12	43320	145572	188892
13	259200	1084512	1343712
14	1814760	8486268	10301028
15	14515200	73104480	87619680
16	101606400	636109560	737715960
17	812851200	5937577920	6750429120

The results for low periods

Notation

In what follows each permutation $\theta \in P_n$ will also be written as a sequence $(\theta(1), \theta(2), \ldots, \theta(n))$.

n	Maximum entropy cycles	Entropy
4	(2,4,1,3), (3,1,4,2)	0.881373587 . . .
5	(2,4,1,5,3)	1.083936863 . . .
6	(3,6,2,5,1,4)	1.256056722 . . .
7	(4,6,2,7,1,5,3)	1.454520522 . . .
8	(4,6,1,8,2,7,3,5), (5,3,7,2,8,1,6,4)	1.609651344 . . .
9	(4,6,2,8,1,9,3,7,5)	1.721042556 . . .
10	(6,4,9,3,8,2,10,1,7,5)	1.815568127 . . .
11	(6,8,4,10,2,11,1,9,3,7,5)	1.929670502 . . .
12	(6,8,4,10,3,11,1,12,2,9,5,7), (7,5,9,2,12,1,11,3,10,4,8,6)	2.024121348 . . .
13	(6,8,4,10,2,12,1,13,3,11,5,9,7)	2.101379638 . . .
14	(7,9,4,10,1,14,2,12,3,13,5,11,6,8)	2.169240867 . . .
15	(8,10,6,12,4,14,2,15,1,13,3,11,5,9,7)	2.247430219 . . .
16	(8,10,6,12,3,15,1,16,2,14,4,13,5,11,7,9), (9,7,11,5,13,4,14,2,16,1,15,3,12,6,10,8)	2.315471390 . . .
17	(8,10,6,12,4,14,2,16,1,17,3,15,5,13,7,11,9)	2.374577194 . . .

Remark

All the above Jungreis cycles are (of course) maximodal and belong to C_n^0.

Conclusions drawn from the "brute force" method

Remark

- The results for periods 6, 10 and 14 in the above table are new.
- All cycles in the above table have zero exceptions (belong to C_n^0).

The "brute force" method does not work for periods 18 and larger. If we want more numerical information we have to devise clever methods.

The programs

The C++ code of the programs that we have used to perform the computations in the paper, together with a file with brief instructions describing how to compile and use them, are available from `http://www.mat.uab.cat/~alseda/research/`

Machinery to study C_n^0: the cross product

Definition of $\otimes$

$\alpha \otimes \beta := \alpha \oplus \delta(\beta)$, where

$$(a_1, a_2, \dots, a_p) \oplus (b_1, b_2, \dots, b_p) = (a_1, b_1, a_2, b_2, \dots, a_p, b_p)$$

and

$$\delta(a_1, a_2, \dots, a_p) = (n - a_p, n - a_{p-1}, \dots, n - a_2, n - a_1),$$

with $n := 2p + 1$.

Example

$$(7, 4, 1, 9, 2, 3, 5, 6, 8) \otimes (7, 4, 2, 3, 1, 9, 5, 6, 8) =$$
$$(7, 11, 4, 13, 1, 14, 9, 10, 2, 18, 3, 16, 5, 17, 6, 15, 8, 12)$$

Machinery to study C_n^0: the dot product

Definition of $\odot$

$\alpha \odot \beta := \sigma^-(\delta(\alpha)) \oplus \sigma^+(\beta)$, where

- $\sigma^+\big(a_1, a_2, \dots, a_p\big) := \big(\widehat{\sigma}^+(a_1), \widehat{\sigma}^+(a_2), \dots, \widehat{\sigma}^+(a_p)\big)$;

  with $\widehat{\sigma}^+(a) := \begin{cases} a+1 & \text{when } a < p \\ 1 & \text{when } a = p. \end{cases}$

- $\sigma^-\big(a_1, a_2, \dots, a_p\big) := \big(\widehat{\sigma}^-(a_1), \widehat{\sigma}^-(a_2), \dots, \widehat{\sigma}^-(a_p)\big)$;

  with $\widehat{\sigma}^-(a) := \begin{cases} a-1 & \text{when } a > p+1 \\ 2p & \text{when } a = p+1. \end{cases}$

Example

$$(7,4,1,9,2,3,5,6,8) \odot (7,4,2,3,1,9,5,6,8) =$$
$$\big(10,8,12,5,13,3,15,4,16,2,18,1,17,6,14,7,11,9\big)$$

Properties of the cross and dot products I

To study the $\otimes$ and $\odot$ products we need the following:

Definition

For a sequence $\alpha = (c_1, c_2, \ldots, c_n)$ with $1 \leq c_i \leq n$, the *dual* of α, denoted by $d(\alpha)$, is the sequence

$$(n+1-c_n, n+1-c_{n-1}, \ldots, n+1-c_1).$$

Observe that $d(d(\alpha)) = \alpha$.

Remark

It is well known that when $\theta \in P_n$, the entropies of θ and $d(\theta)$ are equal, since the corresponding connect-the-dots maps are topologically conjugate.

It is very easy to see that

- $d(\alpha \oplus \beta) = \delta(\beta) \oplus \delta(\alpha)$.

Then,

Proposition (Properties of the cross and dot products)

1. $d(\alpha \otimes \beta) = \beta \otimes \alpha$ *and* $d(\alpha \odot \beta) = \beta \odot \alpha$.
2. *If* $\theta \in C_n^{0,m}$ *then* $\theta = \theta_1 \otimes \theta_2$ *for some* $\theta_1, \theta_2 \in P_{n/2}$, *where* $C_n^{0,m}$ *denotes the subset of* C_n^0 *which contains all cycles for which* $f_\theta(1)$ *is a minimum.*
3. *If* $\theta \in C_n^{0,M}$ *then* $\theta = \theta_1 \odot \theta_2$ *for some* $\theta_1, \theta_2 \in P_{n/2}$, *where* $C_n^{0,M}$ *denotes the subset of* C_n^0 *which contains all cycles for which* $f_\theta(1)$ *is a maximum.*

Remark (to the properties of the cross and dot products)

For every $\alpha, \beta \in P_p$ it follows that $\alpha \otimes \beta$ and $\alpha \odot \beta$ are elements of P_{2p} which *are always maximodal and have zero exceptions*. Moreover, *$f_{\alpha\otimes\beta}(1)$ is a minimum whereas $f_{\alpha\odot\beta}(1)$ is a maximum*. Despite these facts, the converse of Properties 2 and 3 above do not hold since, in general, $\alpha \otimes \beta$ and $\alpha \odot \beta$ need not be cycles. Apart from this, if the cross (respectively dot) product of two elements of P_p is a cycle (that is, belongs to C_{2p}), then it clearly belongs to $C_{2p}^{0,m}$ $\left(\text{respectively } C_{2p}^{0,M}\right)$.

Examples

- $(3, 1, 2) \otimes (1, 2, 3) = (3, 4, 1, 5, 2, 6)$ is not a cycle because it contains the cycle $\{1, 3\}$, and
- $(2, 3, 1) \odot (1, 2, 3) = (5, 2, 6, 3, 4, 1)$, is not a cycle because has 2 as a fixed point.

Algorithmic strategy to generate $\mathbf{C_n^0}$

1. We create a list, $\mathcal{A}$, consisting of all elements of $P_{n/2}$ endowed with any order $\preceq$ (a natural candidate is the lexicographic order).
2. We compute all products $\alpha \otimes \beta$ and $\alpha \odot \beta$, for $\alpha, \beta \in \mathcal{A}$, and in each case, check whether the obtained permutation is a cycle. Note that since the entropies of a cycle and its dual are equal, in view of Property 1 above, we only have to consider those products $\alpha \otimes \beta$ and $\alpha \odot \beta$ for $\alpha, \beta \in \mathcal{A}$ such that $\alpha \preceq \beta$.
3. Choose the maximum entropy cycles among all of them

Drawbacks of the above algorithm

This algorithm is still inefficient, since we spend a lot of time performing products which do not produce cycles. For instance, there is a substantial proportion of permutations $\alpha \in P_{n/2}$ such that $\alpha \otimes \beta$ and $\alpha \odot \beta$ are not cycles for any β. To improve efficiency these α's should be discarded from $\mathcal{A}$.

Observe that neither condition can be derived from the other (for instance, $(2,3,1) \otimes \beta$ never gives a cycle, while $(2,3,1) \odot (3,2,1)$ is a cycle). Next we state some results which allow us to decide whether a permutation α can be deleted from $\mathcal{A}$.

Simplification of the list $\mathcal{A}$

Definition

For each $\alpha \in P_p$, we define $\phi_\alpha \colon O[1, 2p] \longrightarrow [1, p]$ by $\phi_\alpha(2i-1) = \alpha(i)$, for $1 \leq i \leq p$; where:

Notation

For $a, b \in \mathbb{N}$ with $a \leq b$, we denote

$$[a, b] := \{m \in \mathbb{N} : \ a \leq m \leq b\};$$

$$O[a, b] := \{m \in \mathbb{N} : m \text{ is odd and } a \leq m \leq b\}.$$

Proposition

Let $\alpha \in P_p$. If ϕ_α has a cycle which does not contain p then $\alpha \otimes \beta$ and $\alpha \odot \beta$ are not cycles for any $\beta \in P_p$.

Conclusions for $\mathbf{C_n^0}$

We implement the above algorithm by removing from the list $\mathcal{A}$ all permutations $\alpha \in P_{n/2}$ such that ϕ_α has a cycle not containing p. This has significantly shortened the length of $\mathcal{A}$, thus reducing the combinatorial complexity of the task.

The algorithm has been implemented in C++ and executed for period 18 in eight separate parallel jobs (four for each kind of product — cross and dot) on a cluster of Dual Xeon computers at 2.66GHz, with an execution time of about 6.5 hours.

This procedure has given

$$(10, 8, 12, 5, 13, 3, 15, 4, 16, 2, 18, 1, 17, 6, 14, 7, 11, 9) = (7, 4, 1, 9, 2, 3, 5, 6, 8) \odot (7, 4, 2, 3, 1, 9, 5, 6, 8)$$

as the maximum entropy cycle in C_{18}^0, with entropy $\log(11.33428901405\ldots)$.

Study of C_n^1: initial definitions.

Let A^- be the set of sequences $(a_1, a_2, \ldots, a_p)$ of length p such that $a_i > p$ for a unique $1 \leq i \leq p$.

Let also A^+ be the set of sequences $(a_1, a_2, \ldots, a_p)$ of length p such that $a_i \leq p$ for a unique $1 \leq i \leq p$.

As in the case C_n^0, we define $C_n^{1,m}$ to be the subset of C_n^1 which contains all cycles for which $f_\theta(1)$ is a minimum, and $C_n^{1,M}$ to be the subset of C_n^1 which contains all cycles for which $f_\theta(1)$ is a maximum.

Characterisation of $C_n^{1,m}$

Proposition

If $\theta \in C_n^{1,m}$ *then* $\theta = \theta_1 \otimes \theta_2$ *where for* $\ell = 1, 2$, $\theta_\ell = (\theta_\ell^1, \theta_\ell^2, \cdots, \theta_\ell^{n/2}) \in A^-$ *and satisfies the following properties:*

(1.m) $\theta_\ell^i \neq n$ *for* $1 \leq i \leq n/2$,

(2.m) *there is an* $1 \leq i \leq n/2$ *such that* $\theta_\ell^i = 1$,

(3.m) *if* $\theta_\ell^i = n - 1$ *for some* i *then* $i = 1$,

(4.m) $\theta_\ell^1 \neq 1$.

Characterisation of $C_n^{1,M}$

Proposition

If $\theta \in C_m^{1,M}$ *then* $\theta = \theta_1 \otimes \theta_2$ *where, for* $\ell = 1, 2$, $\theta_\ell = (\theta_\ell^1, \theta_\ell^2, \cdots, \theta_\ell^{n/2}) \in A^+$ *and satisfies the following properties:*

(1.M) $\theta_\ell^i \neq 1$ *for* $1 \leq i \leq n/2$,

(2.M) *there is an* $1 \leq i \leq n/2$ *such that* $\theta_\ell^i = n$,

(3.M) *if* $\theta_\ell^i \neq 2$ *for* $1 \leq i \leq p$.

Remark

The converse of the above two propositions does not hold. Indeed, the cross product of two elements from A^- (respectively A^+) satisfying (1–4.m) (respectively (1–3.M)) may not give an element of $C_n^{1,m}$ (respectively $C_n^{1,M}$) for two reasons:

1. such a product may fail to give a cycle
2. even when the product of two elements from $A^\pm$ gives a cycle, we cannot guarantee in advance that this cycle will be maximodal.

Apart from this, if the cross product of two elements from A^- (respectively A^+) verifying (1–4.m) (respectively (1–3.M)) gives a maximodal cycle then this cycle must belong to $C_n^{1,m}$ (respectively $C_n^{1,M}$).

The two examples mentioned in the above remark

We provide both examples with sequences from A^- satisfying (1–4.m) and $p = 9$.

Example (A product may not give a cycle:)

$$(10, 1, 3, 4, 5, 6, 7, 8, 9) \otimes (2, 5, 8, 1, 3, 9, 6, 12, 4) = \\ (10, 15, 1, 7, 3, 13, 4, 10, 5, 16, 6, 18, 7, 11, 8, 14, 9, 17)$$

(this 18-permutation is not a cycle because it contains the 2-cycle $\{7, 4\}$).

Example (A product may not be maximodal:)

$$(10, 1, 3, 4, 5, 6, 7, 8, 9) \otimes (17, 8, 7, 6, 5, 3, 2, 1, 4) = \\ (10, 15, 1, 18, 3, 17, 4, 16, 5, 14, 6, 13, 7, 12, 8, 11, 9, 2).$$

Algorithmic strategy to generate $\mathbf{C_n^1}$

1. We create two lists. To generate $C_n^{1,m}$ we need to create the list $\mathcal{Y}$ of all the elements of A^- satisfying the properties (1–4.m), endowed with any order $\preceq$ (a natural candidate is the lexicographic order).
2. We have to compute all the products $\alpha \otimes \beta$ for $\alpha, \beta \in \mathcal{Y}$ and, in each case, check whether the obtained permutation is a maximodal cycle. As before, since the entropies of a cycle and its dual are equal, it is enough to consider only the products $\alpha \otimes \beta$ for $\alpha, \beta \in \mathcal{Y}$ such that $\alpha \preceq \beta$.
3. Analogously, to generate $C_n^{1,M}$ we need to create the list $\mathcal{Z}$ of all the elements of A^+ satisfying the properties (1–3.M), endowed with an order $\preceq$.
4. We perform all products $\alpha \otimes \beta$ for $\alpha, \beta \in \mathcal{Z}$ such that $\alpha \preceq \beta$ and the product is a maximodal cycle.

Results

The algorithm has been implemented in C++ and executed for period 18 splitting the task into 16 subtasks (eight dealing with each list $\mathcal{Y}$ and $\mathcal{Z}$) that have been executed in parallel on a cluster of Dual Xeon computers at 2.66GHz, with an execution time of about three months (with a typical CPU usage higher than 45% for each job). This procedure has given

$$(9, 8, 12, 6, 13, 3, 15, 4, 16, 2, 18, 1, 17, 5, 14, 7, 11, 10) = \\ (9, 12, 13, 15, 16, 18, 17, 14, 11) \otimes (9, 12, 14, 18, 17, 15, 16, 13, 11)$$

and

$$(9, 8, 12, 6, 13, 2, 18, 1, 17, 3, 15, 4, 16, 5, 14, 7, 11, 10) = \\ (9, 12, 13, 18, 17, 15, 16, 14, 11) \otimes (9, 12, 14, 15, 16, 18, 17, 13, 11)$$

as the maximum entropy cycles in C_{18}^1, with entropy $\log(11.321231505957\ldots)$.

Conclusions for C_{18}

- The maximum entropy cycle for period 18 is the one in C_{18}^0 reported before.

Conclusions for C_{18}

- The maximum entropy cycle for period 18 is the one in C_{18}^0 reported before.
- It is clear that this method cannot be extended to periods larger than 18, as the execution time is prohibitive. To continue our investigation to higher periods it has been necessary to focus our attention on a restricted set of cycles; one which is most likely to include those of highest entropy.

Comparing the efficiency of the two strategies for period 18 (measured in number of permutations to deal with)

Brute force method: $18! = 6402373705728000$ permutations.
Products method (compare the first two numbers in the column "# of list" with $9! = 362880$ and the last two with $9^2 \cdot 9! = 29393280$ — these are upper bounds for these sizes):

list	# of list N	Product	# of product permutations $N \cdot (N+1)/2$
$\mathcal{A}$	201600	$\otimes$	20321380800
$\mathcal{A}$	201600	$\odot$	20321380800
$\mathcal{Y}$	9334080	$\otimes$	43562529390240
$\mathcal{Z}$	11128320	$\otimes$	61919758575360
Total number of permutations			105522930727200

Thus, $\frac{18!}{105522930727200} = 60.67\cdots$ *much* less effort when we use the products method.

Further restrictions for $n = 22$ and beyond

Finding the maximum entropy n-cycle (and thus checking the validity of the conjecture) by using the algorithm described above is not feasible in computational terms when $n > 18$. Instead, it is necessary to make further restrictions on the subclass of cycles to be explored. They are supported by the following two conjectures:

Conjecture

For each period n, the maximum entropy n-cycle belongs to C_n^0 (holds for all periods $n \leq 18$).

Conjecture

For $k \geq 2$ and $n = 4k + 2$, if ψ_n is the entropy-maximal n-permutation as defined in [K], the maximum entropy cycle ϕ_n will be such that $\phi_n(i) = \psi_n(n + 1 - i)$ for all $i \in \left[1, \frac{k}{2} + 1\right] \cup \left[\frac{7k}{2} + 2, n\right]$, k even, or $\phi_n(i) = \psi_n(i)$ for all $i \in [1, k - 1] \cup [3k + 4, n]$, k odd (see the next table).

(Too complicate) definition of the conjectured families

Next we introduce three families of $4k+2-$cycles with $k>1$ (one for k odd and two for k even). They are elements of C_n^0 and have been obtained by generalising the computational results which have been reported in the previous sections. The topological entropy of the cycles generated by these families is monotonically increasing as $k \to \infty$.

Recall:

Notation

For $a, b \in \mathbb{N}$ with $1 \leq a \leq b$, we denote

$$\begin{aligned} [a, b] &:= \{m \in \mathbb{N} : \ a \leq m \leq b\}; \\ O[a, b] &:= \{m \in \mathbb{N} : m \text{ is odd and } a \leq m \leq b\}; \\ E[a, b] &:= \{m \in \mathbb{N} : m \text{ is even and } a \leq m \leq b\}. \end{aligned}$$

Definition for $k = 2p$, p odd

$$\theta_n : j \to \begin{cases} 4p+1+j, & \text{if } j \in O[1,p] \\ 4p+2+j, & \text{if } j \in O[p+2,3p] \\ 4p-1+j, & \text{if } j \in O[3p+2,4p+3] \\ 12p+6-j, & \text{if } j \in O[4p+5,5p+2] \\ 12p+3-j, & \text{if } j \in O[5p+4,7p] \\ 12p+4-j, & \text{if } j \in O[7p+2,n-1] \\ 4p+2-j, & \text{if } j \in E[2,p+1] \\ 4p+3-j, & \text{if } j \in E[p+3,3p+1] \\ 4p+4-j, & \text{if } j \in E[3p+3,4p+2] \\ j-4p-3, & \text{if } j \in E[4p+4,5p+3] \\ j-4p-2, & \text{if } j \in E[5p+5,7p+1] \\ j-4p-1, & \text{if } j \in E[7p+3,n]. \end{cases}$$

Definition for $k = 2p$, p even

$$\theta_n : j \to \begin{cases} 4p+1+j, & \text{if } j \in O[1, p+1] \\ 4p+j, & \text{if } j \in O[p+3, 3p+1] \\ 4p-1+j, & \text{if } j \in O[3p+3, 4p+3] \\ 12p+6-j, & \text{if } j \in O[4p+5, 5p+3] \\ 12p+5-j, & \text{if } j \in O[5p+5, 7p+1] \\ 12p+4-j, & \text{if } j \in O[7p+3, n-1] \\ 4p+2-j, & \text{if } j \in E[2, p] \\ 4p+1-j, & \text{if } j \in E[p+2, 3p] \\ 4p+4-j, & \text{if } j \in E[3p+2, 4p+2] \\ j-4p-3, & \text{if } j \in E[4p+4, 5p+2] \\ j-4p, & \text{if } j \in E[5p+4, 7p] \\ j-4p-1, & \text{if } j \in E[7p+2, n]. \end{cases}$$

Definition for $k \geq 3$, odd

$$
\theta_n : j \to \begin{cases}
2k - j + 2, & \text{if } j \in O[1, k-2] \\
k + 1, & \text{if } j = k \\
2k - j, & \text{if } j \in O[k+2, 2k-1] \\
j - 2k + 1, & \text{if } j \in O[2k+1, 3k-2] \\
j - 2k, & \text{if } j \in O[3k, 3k+2] \\
j - 2k - 1, & \text{if } j \in O[3k+4, n-1] \\
2k + 1 + j, & \text{if } j \in E[2, k-1] \\
3k + 1, & \text{if } j = k + 1 \\
2k + 3 + j, & \text{if } j \in E[k+3, 2k-2] \\
6k + 2 - j, & \text{if } j \in E[2k, 3k-1] \\
6k + 5 - j, & \text{if } j \in E[3k+1, 3k+3] \\
6k + 4 - j, & \text{if } j \in E[3k+5, n].
\end{cases}
$$

The table: the conjectured cycles for “low” periods

Period n	Max permutation ψ_n	Entropy	Max cycle θ_n	Entropy
10	(5,7,3,9,1,10,2,8,4,6)	1.8427299...	(6,4,9,3,8,2,10,1,7,5)	1.8155681...
14	(7,9,5,11,3,13,1, 14,2,12,4,10,6,8)	2.1832659...	(7,9,4,10,1,14,2, 12,3,13,5,11,6,8)	2.1692408...
18	(9,11,7,13,5,15,3,17,1, 18,2,16,4,14,6,12,8,10)	2.4362460...	(10,8,12,5,13,3,15,4,16, 2,18,1,17,6,14,7,11,9)	2.4278325...
22	(11,13,9,15,7,17,5, 19,3,21,1,22,2,4, 18,6,16,8,14,10,12)	2.6377584...	(11,13,9,15,6,16,3, 21,1,22,2,20,4,18,5, 19,7,17,8,14,10,12)	2.6320413...
26	(13,15,11,17,9,19,7,21,5, 23,3,25,1,26,2,24,4,22, 6,20,8,18,10,16,12,14)	2.8052961...	(14,12,16,10,19,9,21,7, 23,5,22,4,24,2,26,1,25, 3,20,6,18,8,17,11,15,13)	2.8011896...
30	(15,17,13,19,11,21,9, 23,7,25,5,27,3,29,1, 30,2,28,4,26,6,24,8, 22,10,20,12,18,14,16)	2.9487002 ...	(15,17,13,19,11,21,8, 22,5,27,3,29,1,30,2, 28,4,26,6,24,7,25,9, 23,10,20,12,18,14,16)	2.9454988...
34	(17,19,15,21,13,23,11,25, 9,27,7,29,5,31,3,33,1, 34,2,32,4,30,6,28,8,26, 10,24,12,22,14,20,16,18)	3.0740659 ...	(18,16,20,14,22,11,23,9, 25,7,27,5,29,6,30,4,32, 2,34,1,33,3,31,8,28,10, 26,12,24,13,21,15,19,17)	3.0716352...

Properties of the conjectured families

1. For $n = 4k + 2$, k even, the map f_{θ_n} has a local maximum at $j = 1$ whilst for $n = 4k + 2$, k odd, the map f_{θ_n} has a local minimum at $j = 1$.
2. Each cycle θ_n is maximodal and f_{θ_n} has all maximum values greater than all minimum values (that is, $\theta_n \in C_n^0$).
3. For $n = 4k + 2$, k even, f_{θ_n} has a global minimum at $j = 2k + 4$ whilst for $n = 4k + 2$, k odd, f_{θ_n} has a global minimum at $j = 2k - 1$.
4. For $n = 4k + 2$, k even, f_{θ_n} has a global maximum at $j = 2k + 3$ whilst for $n = 4k + 2$, k odd, f_{θ_n} has a global maximum at $j = 2k$.

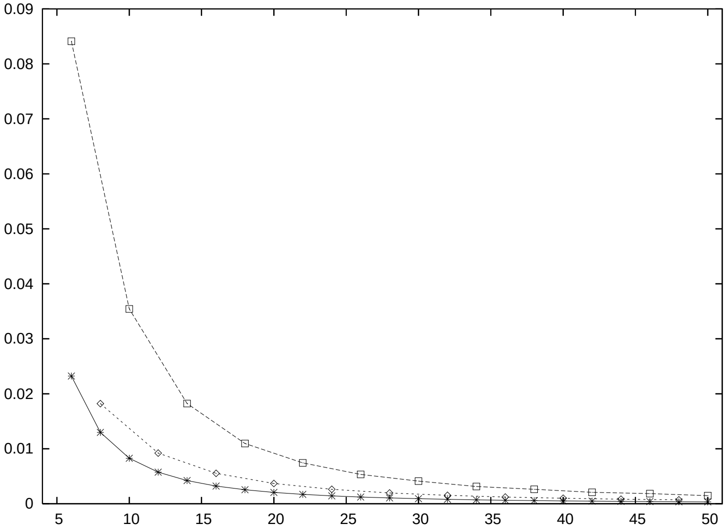
Remark

The entropy-maximal 6-cycle is not generated by the formulae given by the above definitions.

Asymptotic behaviour of the conjectured families: a first quality test

The figure shows the asymptotic behaviour of the entropies of the cycles in the conjectured families. The three curves above represent the difference between the Misiurewicz-Nitecki bound, $\log(2n/\pi)$, and the entropies of (i) the maximum entropy n-permutation, for $n \in E[6, 50]$ (lower curve), (ii) the maximum entropy $4k$-cycle, for $k \in [2, 12]$ (centre curve), (iii) the cycle θ_{4k+2}, for $k \in [1, 12]$ (upper curve).

The figure



Finding maximal elements for periods 22, 26 and 30 assuming that the conjectures hold

For $n \in \{22, 26, 30\}$ the above conjectures allow us to consider the classes $C_n^* \subset C_n^0$ of cycles ϕ such that $\phi(i) = \psi_n(i)$ when $n \in \{22, 30\}$ and $\phi(i) = \psi_n(n+1-i)$ when $n = 26$, for all $i \in \{1, 2, \dots, 2k-8, 2k+11, \dots, n-1, n\}$.

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The permutations θ_n of the conjectured families belong to C_n^*. Also, every $\phi \in C_n^*$ has $n-18$ positions fixed according to ψ_n. However, for periods $n = 4k+2 \geq 34$ this is no longer true (θ_{34} only agrees with ψ_{34} in 10 positions, thus leaving 24 free positions).

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Thus, the task of finding the highest entropy cycle in C_n^* with $n \in \{22, 26, 30\}$ has a computational complexity not larger than that of finding the highest entropy cycle in C_{18}^0, while this problem is unsolvable with the conjectures and techniques devised in this paper for periods $n = 4k+2 \geq 34$.

Algorithmic strategy to generate $\mathbf{C}_\mathbf{n}^*$, $\mathbf{n} \in \{22, 26, 30\}$

As already defined, the elements of C_n^* are the maximodal cycles of period $n = 4k + 2$ that have a certain pattern at the beginning and at the end of the cycle, which is determined by ψ_n. Moreover, the parity of k completely determines the structure of C_n^*.

Remark

We should bear in mind a crucial property that influences the whole strategy of this computation: *the permutations ψ_n are self-dual* (this can be checked directly from the definition of ψ_n, see also [K]).

Algorithm for the case k odd

All cycles $\theta \in C_n^*$ have $f_\theta(1)$ as a minimum. Hence, $\theta = \theta_1 \otimes \theta_2$ for some $\theta_1, \theta_2 \in P_{2k+1}$. Then,

1. We create a list $\mathcal{A}^* \subset P_{2k+1}$ such that each $\alpha \in \mathcal{A}^*$ satisfies $\alpha(i) = \psi_n(2i-1)$ for $i \in \{1, 2, \dots, k-4, k+6, \dots, 2k+1\}$ (see the next table) and ϕ_α has no cycle not containing $2k+1$ (that is, we discard those permutations α such that $\alpha \otimes \beta$ is not a cycle for any $\beta \in P_{2k+1}$). We also endow the list $\mathcal{A}^*$ with any order $\preceq$ (a natural candidate is the lexicographic order). It follows from the definition of the cross product and the fact that ψ_n is self-dual that for every $\alpha, \beta \in \mathcal{A}^*$, we have that $\alpha \otimes \beta \in C_n^*$.
2. To generate all elements from C_n^* we compute all products $\alpha \otimes \beta$ for $\alpha, \beta \in \mathcal{A}^*$ such that $\alpha \preceq \beta$, and in each case, we check whether the obtained permutation is a cycle.

Algorithm for the case k even

All cycles $\theta \in C_n^*$ have $f_\theta(1)$ as a maximum. Hence, $\theta = \theta_1 \odot \theta_2$ for some $\theta_1, \theta_2 \in P_{2k+1}$. Then,

1. We create an ordered list $\mathcal{A}^* \subset P_{2k+1}$ such that each $\alpha \in \mathcal{A}^*$ satisfies

$$\alpha(i) = \begin{cases} \psi_n(n+1-2i) - 1 & \text{if } \psi_n(n+1-2i) \neq 1 \\ p & \text{otherwise} \end{cases}$$

   for $i \in \{1, 2, \ldots, k-4, k+6, \ldots, 2k+1\}$ (see the next table) and ϕ_α has no cycle not containing $2k+1$. Then as above, for every $\alpha, \beta \in \mathcal{A}^*$, $\alpha \odot \beta \in C_n^*$.

2. To generate all elements from C_n^* we compute all products $\alpha \odot \beta$ for $\alpha, \beta \in \mathcal{A}^*$ such that $\alpha \preceq \beta$, and in each case, we check whether the obtained permutation is a cycle.

The structure table for the lists $\mathcal{A}^*$

The structure of the $\mathcal{A}^*$-lists and the type of products to consider to compute the maximum entropy cycle in C_n^* are the following.

n	Structure of the restricted $\mathcal{A}$-list	Product
22	(11; $\mathbf{P_9}$;10)	$\otimes$
26	(11,9; $\mathbf{P_9^{13}}$;10,12)	$\odot$
30	(15,13,11; $\mathbf{P_9}$;10,12,14)	$\otimes$

When we write $\mathbf{P_9^p}$ we mean that the list is generated by successively inserting in the corresponding place, each permutation $\alpha \in P_9$ and then replacing 9 by p (when we omit the superscript we mean that the last step —replacing 9 by p— is omitted). Any permutation $\alpha \in P_{n/2}$ such that ϕ_α has a cycle not containing $n/2$ is discarded from $\mathcal{A}^*$.

Conclusions

- We have found numerically that the maximum entropy cycle in C_n^* is θ_n for $n = 22,\ 26,\ 30.$ Moreover, the entropy of any other cycle in the class is strictly smaller than $h(\theta_n)$ (up to duality).

Conclusions

- We have found numerically that the maximum entropy cycle in C_n^* is θ_n for $n = 22,\ 26,\ 30$. Moreover, the entropy of any other cycle in the class is strictly smaller than $h(\theta_n)$ (up to duality).
- In the worst case, the conjectured families of cycles provide a good lower bound on the maximum topological entropy of cycles in C_{4k+2}. Indeed, the sequence of topological entropies of the cycles generated by each family is monotonically increasing as $k \to \infty$ and furthermore, if we combine the three sequences into a single sequence, the new sequence obtained is also monotonically increasing as $k \to \infty$.

A word of caution

A degree of caution should be taken. As remarked by a referee, in the search for entropy-maximising cycles of order n, first there was a distinction between n odd and $n = 2m$; then between m odd and $m = 2k$; and now, conjecturally, between k odd and $k = 2\ell$. In this situation we could think that we are facing an infinite cascade of such distinctions. However, since we have found no single cycle amongst those generated with entropy larger than θ_n, we believe that the above conjectures are reasonable.

The next step

Either prove that the cycles of the family θ_{4k+2} are, indeed, the maximal ones; or find a counterexample to this statement leading to a new (better) conjecture.

For the first objective we need new clever ideas to afford the difficulty of the problem.

For the second objective we need new clever ideas to treat higher periods more efficiently and in resonable time. Also, some more "CPU power" woud be helpful.