Volume entropy for minimal presentations of surface groups in all ranks

Lluís Alsedà

in collaboration with David Juher, Jérôme Los and Francesc Mañosas

Departament de Matemàtiques
Universitat Autònoma de Barcelona

http://www.mat.uab.cat/∼alseda

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DEPARTAMENT DE MATEMÀTIQUES

Outline

Introduction

We compute the volume entropy of a special class of presentations (including the classical ones) called *minimal geometric presentations* for all surface groups of rank $n > 2$:

Theorem

For $n > 2$, let Γ be a surface group of rank n with a minimal geometric presentation P. Then, the volume entropy of Γ with respect to the presentation P is $\log(\lambda_n)$ where λ_n is the unique real root larger than one of the polynomial

$$Q_n(x) := x^n - 2(n-1)\sum_{j=1}^{n-1} x^j + 1.$$

Moreover, for $n \geq 4$, λ_n satisfies:

$$2n - 1 - \frac{1}{(2n-1)^{n-2}} < \lambda_n < 2n - 1.$$

Note that the volume entropy for all surface groups is encoded by a single, explicit polynomial whose degree is precisely the rank of the group, what seems a bit mysterious.

Key ideas

1. We use a dynamical system construction following an idea due to Bowen and Series [BS] and extended to all geometric presentations in [Los]. This dynamical system approach allows to compute the volume entropy of any geometric presentation P as the topological entropy of an explicit Bowen-Series-Like map. This map is defined on the circle (the infinity of the hyperbolic group) and it is a piecewise homeomorphism (non-necessarily continuous).

[BS] Rufus Bowen and Caroline Series.
Markov maps associated with Fuchsian groups.
Inst. Hautes Études Sci. Publ. Math., (50):153–170, 1979.

[Los] Jérôme Los.
Volume entropy for surface groups via Bowen-Series-like maps.
J. Topol., 7(1):120–154, 2014.

Key ideas

2. We consider a special class of minimal presentations with strong symmetry properties. For these presentations the Markov Matrix of the Bowen-Series-Like map (which is a non-negative inter matrix) has a special structure called *block circulant* in the case with an even number of generators (orientable or not) case and *disoriented block circulant* in the non-orientable case with an odd number of generators. As we will see the spectral radius of these matrices has very nice properties that allow the computation of it despite of the fact that the size of the matrix grows quadratically in n.
3. We use some standard Dynamical Systems tools to compute the topological entropy (including the so called *Rome Method* plus some more reductions (besides the one given the circulant property) to obtain the result just by computing the determinant of a functional 2×2 matrix.

History and related results

The dynamical system approach discussed above allows to compute the volume entropy of any geometric presentation P from an explicit Bowen-Series-Like map: $\Phi_P : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$.

First we developed an algorithm to compute the entropy of such maps, for the classical presentations of orientable surfaces, via the well known kneading invariant technique of Milnor and Thurston [MT]. The polynomial $Q_n(x)$ appears that way in the computation for all orientable surfaces of genus $g \leq 43$. Thanks to this conjecture about the polynomial it was possible to get the clues to prove the theorem by a the (different) Markov matrix method without the need of a computer.

[MT] John Milnor and William Thurston.
On iterated maps of the interval.
In *Dynamical systems (College Park, MD, 1986–87)*, volume 1342 of *Lecture Notes in Math.*, pages 465–563. Springer, Berlin, 1988.

History and related results

The notion of *hyperbolic group* was introduced by [Gr1], where the *growth function* plays a central role (as for other classes of groups). The growth function depends on the generating set X or on the presentation $P = \langle X/R \rangle$ of the group G. It is defined as the map $\mathbb{N} \mapsto \mathbb{N}$ such that

$$n \mapsto f_{G,P}(n) = \text{Card}\{g \in G : \text{length}_X(g) \leq n\}.$$

From the growth function $f_{G,P}$ several asymptotic functions are defined such as the *volume entropy* or the *growth series* also called the *Poincaré series*.

[Gr1] M. Gromov.
Hyperbolic groups.
In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.

History and related results

The computational issues appeared also at about the same period. An idea due to J. Cannon [Can84] allows an inductive way to describe geodesics in the Cayley graph $\mathrm{Cay}^1(G, P)$ via the notion of *cone types*. This notion has been intensively used later on by Epstein, Cannon, Levy, Holt, Patterson, Thurston [ECLHPT] with the introduction of a very large class of groups, called *automatic*, that contains the hyperbolic groups of Gromov. The computation of the growth function or the growth series becomes possible in principle from a geodesic automatic structure, when it exists. This is the case for hyperbolic groups.

[Can84] James W. Cannon.
The combinatorial structure of cocompact discrete hyperbolic groups.
Geom. Dedicata, 16(2):123–148, 1984.

[ECLHPT] David B. A. Epstein, James W. Cannon, Derek F. Holt, Silvio V. F. Levy, Michael S. Paterson, and William P. Thurston.
Word processing in groups.
Jones and Bartlett Publishers, Boston, MA, 1992.

History and related results

In practice, finding an explicit geodesic automatic structure from the presentation is not so simple. For free groups with the free presentation all the computations are easy and, for instance, the volume entropy is simply $\log(2n-1)$, for the free group of rank n (see for instance [DIH]).

[DIH] Pierre de la Harpe.
Topics in geometric group theory.
Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000.

History and related results: The case of surface groups

For the classical presentations of surface groups, the growth series appeared in a paper by Cannon and Wagreich [CW] without the explicit computation, leading to those series that were earlier obtained in a non published manuscript of Cannon [Can80]. For hyperbolic groups, the existence of a geodesic automatic structure for each presentation implies that the growth series is a rational function (see [ECLHPT, Can84]). In this case the volume entropy (sometimes called the critical exponent) is related to the largest pole of the growth series, i.e. the largest root of the denominator of the growth series (see for instance [Cal]). The result of Cannon and Wagreich for the classical presentations of surface groups states that the denominator of the growth series is precisely the polynomial Q_n.

[CW] J. W. Cannon and Ph. Wagreich.
Growth functions of surface groups.
Math. Ann., 293(2):239–257, 1992.

[Cal] D. Calegari.
The ergodic theory of hyperbolic groups.
Contemp. Math., 597:15–52, 2013.

[Can80] J. W. Cannon.
The growth of the closed surface groups and the compact hyperbolic coxeter groups.
1980.

History and related results

For any surface S, the classical presentation of the corresponding surface group $\Gamma = \pi_1(S)$ is geometric. These classical presentations are given by the minimal number of generators n and one relation of length $2n$. For orientable surfaces, n is even and equals $2g$, where g is the genus of the surface. In this case, the classical relation is a product of g commutators. In the non-orientable case, there is no restriction on the parity of n and the relation is given by the product of the squares of all generators (see for instance [Sti]).

[Sti] John Stillwell.
Classical topology and combinatorial group theory, volume 72 of *Graduate Texts in Mathematics*.
Springer-Verlag, New York, second edition, 1993.

History and related results: The rank 2 cases

The rank 2 cases (torus and Klein bottle) are, as usual, special: they are not hyperbolic, the growth function is quadratic and thus the volume entropy is 0.

For $n > 2$ all minimal geometric presentations are proved to have the minimal volume entropy, among geometric presentations.

Basic Definitions

Geometric presentations

Let $P = \langle X/R \rangle = \left\langle x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1} / R_1, \dots, R_k \right\rangle$ be a presentation of a group Γ.

The *Cayley graph*, $\mathrm{Cay}^1(\Gamma, P)$, of a group is a labelled directed graph constructed as follows:

- Each element g of Γ is assigned a vertex: the vertex set $V(\mathrm{Cay}^1(\Gamma, P))$ of $\mathrm{Cay}^1(\Gamma, P)$ is identified with Γ.
- For any $g \in Gamma$, $x \in X$, the vertices corresponding to the elements g and gx are joined by a directed edge labelled with x.

Thus the edge set $E(\mathrm{Cay}^1(\Gamma, P))$ consists of pairs of the form (g, gx), with $x \in X$ providing the label. In geometric group theory, the set X is usually assumed to be finite, symmetric (i.e. $X = X^{-1}$) and not containing the identity element of the group. In this case, the unlabelled Cayley graph is an ordinary graph: its edges are not oriented and it does not contain loops (single-element cycles).

Basic Definitions
Geometric presentations

The Cayley graph $\mathrm{Cay}^1(\Gamma, P)$ is a metric space and let B_m be the ball of radius m centred at the identity. We denote the cardinality of any finite set A by $|A|$. The *volume entropy* of Γ with respect to the presentation P is denoted by $h_{\mathrm{vol}}(\Gamma, P)$ and defined as:

$$\lim_{m\to\infty} \frac{1}{m} \log|B_m|.$$

A presentation P of a surface group $\Gamma = \pi_1(S)$ is called *geometric* if the Cayley 2-complex $\mathrm{Cay}^2(\Gamma, P)$ is a plane. In particular the Cayley graph $\mathrm{Cay}^1(\Gamma, P)$ is a planar graph.

A geometric presentation P is called *minimal* if the number of generators is minimal.

Basic Definitions

Geometric presentations

For a group of an orientable surface of genus g it is well known that the minimal number of generators is $2g$ (see [Sti] for instance) and, in this case, there is a presentation with a single relation of length $4g$. The standard classical presentation in this case is the following:

$$\left\langle x_1^{\pm 1}, y_1^{\pm 1}, x_2^{\pm 1}, y_2^{\pm 1}, \dots, x_g^{\pm 1}, y_g^{\pm 1} / \prod_{i=1}^{g} [x_i, y_i] \right\rangle,$$

where $[x_i, y_i] = x_i \cdot y_i \cdot x_i^{-1} \cdot y_i^{-1}$ is a commutator.

Basic Definitions

Geometric presentations

For a rank n group of a non-orientable surface there is also a classical presentation with a single relation of length $2n$:

$$\left\langle x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1} / \prod_{i=1}^{n} x_i^2 \right\rangle .$$

It is easy to check that such classical presentations are geometric.

Basic Definitions

Geometric presentations

Geometric presentations satisfy very simple combinatorial properties:

Lemma (Floyd and Plotnick [FP])

If $P = \langle x_1^{\pm 1}, \ldots, x_n^{\pm 1} / R_1, \ldots, R_k \rangle$ is a geometric presentation of a surface group Γ then P satisfies the following properties:

1. *The set $\{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}$ admits a cyclic ordering that is preserved by the Γ-action.*
2. *Each generator appears exactly twice (with plus or minus exponent) in the set $R = \{R_1, \ldots, R_k\}$ of relations.*
3. *Each pair of adjacent generators, according to the cyclic ordering (a), appears exactly once in R and defines uniquely a relation $R_i \in R$.*

Basic Definitions

Geometric presentations

The following statement is the main ingredient to compute the volume entropy of a geometric presentation. The statement also contains the main result about minimal geometric presentations. In what follows $\mathbb{S}^1$ will denote a (topological) circle. Recall that any surface group Γ is Gromov-hyperbolic [Gr1] and its boundary is: $\partial\Gamma \simeq \mathbb{S}^1$.

Let us introduce the notion of a Markov partition. Let W be a finite set of $\mathbb{S}^1$. An interval of $\mathbb{S}^1$ will be called *W-basic* if it is the closure of a connected component of $\mathbb{S}^1 \setminus W$. Observe that two different W-basic intervals have pairwise disjoint interiors. Let $\phi \colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ and let $W \subset \mathbb{S}^1$ be finite. We say that W is a *Markov partition of ϕ* if W is $\phi-$invariant (i.e., $\phi(W) \subset W$) and the image by ϕ of every basic interval is a union of basic intervals.

Basic Definitions

Geometric presentations

Theorem (Los [Los])

Let P be a geometric presentation of a surface group Γ. Then there exists a map $\Phi_P \colon \partial\Gamma = \mathbb{S}^1 \longrightarrow \partial\Gamma = \mathbb{S}^1$ with the following properties:

1. *The map Φ_P is Markov, i.e. it admits a finite Markov partition.*
2. *The topological entropy of Φ_P, $h_{top}(\Phi_P)$, is equal to the volume entropy $h_{vol}(\Gamma, P)$.*

In addition, the volume entropy is minimal, among geometric presentations, for all minimal geometric presentations.

Basic Definitions: Bowen-Series-Like map

Bigons

A *bigon* in $\text{Cay}^1(\Gamma, P)$ is a pair of distinct geodesics $\{\gamma_1, \gamma_2\}$ connecting two vertices $\{v, v'\} \in \text{Cay}^1(\Gamma, P)$. We denote by $B_v(x, y)$ the set of bigons $\{\gamma_1, \gamma_2\}$ whose initial vertex is v and so that the geodesic γ_1 starts at v by the edge labelled x and γ_2 starts at v by the edge labelled y, with $x \neq y$. By the Γ-action we can fix the initial vertex v to be the identity and we denote $B_{\text{id}}(x, y)$ by $B(x, y)$.

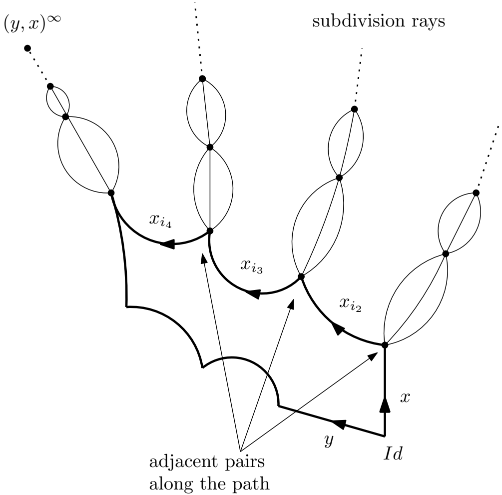
For geometric presentations of surface groups the set of bigons is particularly simple.

Lemma

If $P = \langle X / R \rangle$ is a geometric presentation of a surface group Γ then the set of bigons $B(x, y)$ is non empty if and only if (x, y) is an adjacent pair of generators, according to the cyclic ordering of Floyd–Plotnick Lemma. In addition, if (x, y) is an adjacent pair of generators there is a unique bigon $\beta(x, y) \in B(x, y)$ of finite minimal length, called minimal bigon.

Basic Definitions: Bowen-Series-Like map

Bigons



Basic Definitions: Bowen-Series-Like map

Bigon-Rays and $\partial\Gamma$

There is a canonical way to define a point on the boundary $\partial\Gamma = \mathbb{S}^1$.

By definition of $\partial\Gamma$, a point $\xi \in \partial\Gamma$ is the limit of geodesic rays, for instance starting at the identity, modulo the equivalence relation among rays that two rays are equivalent if they stay at a uniform bounded distance from each others (c.f. [Gr1]).

We construct a unique infinite sequence of adjacent pairs, bigons and vertices from any adjacent pair by the following process:

Basic Definitions: Bowen-Series-Like map

Bigon-Rays and $\partial\Gamma$: Notation

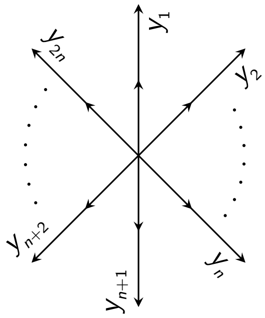
In what follows we denote the n generators (and their inverses) by $y_1, y_2, \ldots, y_{2n}$ in such a way that $y_{[i\pm1]_{2n}}$ $\left([k]_l := k \pmod{l} \right)$ are the elements adjacent to y_i with respect to the cyclic ordering from the above lemma. We denote an adjacent pair by $(y_i, y_{[i+1]_{2n}})$ where, by convention, the edges denoted y_i and $y_{[i+1]_{2n}}$ are adjacent and oriented from the vertex. We also adopt the convention that y_i is on the *left* of $y_{[i+1]_{2n}}$. This convention defines an orientation of the plane $\mathrm{Cay}^2(\Gamma, P)$.

The parity of the number of adjacent pairs at each vertex implies that $(y_i, y_{[i+1]_{2n}})$ defines an *opposite* pair, with respect to the cyclic ordering, defined by:

$$(y_i, y_{[i+1]_{2n}})^{\mathrm{opp}} := (y_{[i+n]_{2n}}, y_{[i+n+1]_{2n}}).$$

Basic Definitions

Geometric presentations: The labelling of the generators and a cyclic ordering



Basic Definitions: Bowen-Series-Like map

Bigon-Rays and $\partial\Gamma$

- Each adjacent pair, at the identity, defines a unique minimal bigon $\beta(y_i, y_{i+1})$ by the above lemma. The bigon $\beta(y_i, y_{i+1})$ is a pair of geodesics $\{\gamma_l, \gamma_r\}$, where the indices l, r stand for left and right, with respect to an orientation of the plane $\mathrm{Cay}^2(\Gamma, P)$. The geodesics $\{\gamma_l, \gamma_r\}$ connect the identity to a vertex $v_1 = v_1[\beta(y_i, y_{i+1})]$ (see the figure below).
- The two geodesics $\{\gamma_l, \gamma_r\}$ end at v_1 by two generators that are adjacent by the above lemma. Therefore the bigon $\beta(y_i, y_{i+1})$ defines a unique adjacent pair at v_1, called a *top pair* of $\beta(y_i, y_{i+1})$, which is denoted: $\mathrm{topp}[\beta(y_i, y_{i+1})]$, based at $v_1 = v_1[\beta(y_i, y_{i+1})]$ and is uniquely defined by (y_i, y_{i+1}).

Basic Definitions: Bowen-Series-Like map

Bigon-Rays and $\partial\Gamma$

- The pair $\text{topp}[\beta(y_i, y_{i+1})]$ defines an opposite pair at v_1, denoted by:

$$\big(\text{topp}[\beta(y_i, y_{i+1})]\big)^{\text{opp}} := (y_i, y_{i+1})^{(1)}.$$

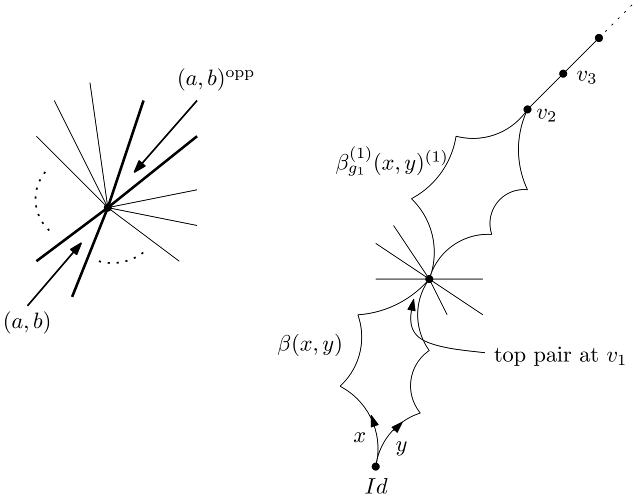
- We consider then the unique minimal bigon, at v_1, defined by the pair $(y_i, y_{i+1})^{(1)}$ by the above lemma:

$$\beta^{(1)}(y_i, y_{i+1}) := \beta_{v_1}[(y_i, y_{i+1})^{(1)}].$$

- The bigon $\beta^{(1)}(y_i, y_{i+1})$ defines a new top pair $\text{topp}[\beta^{(1)}(y_i, y_{i+1})]$, at the vertex v_2.

Basic Definitions: Bowen-Series-Like map

Bigon-Rays and $\partial\Gamma$: Opposite pairs and bigon rays



Basic Definitions: Bowen-Series-Like map

Bigon-Rays and $\partial\Gamma$

The above steps define, by induction, a unique infinite sequence of vertices and bigons:

$$\text{id}, v_1, v_2, \cdots$$

$$\beta(y_i, y_{i+1}), \beta^{(1)}(y_i, y_{i+1}), \beta^{(2)}(y_i, y_{i+1}) \cdots \ .$$

We denote the infinite concatenation of all these paths as:

$$\beta^{\infty}(y_i, y_{i+1}) := \lim_{k\to\infty} \beta(y_i, y_{i+1})\beta^{(1)}(y_i, y_{i+1}) \cdots \beta^{(k)}(y_i, y_{i+1}).$$

Lemma (Los [Los])

With the above notation the following statements hold.

1. *Each path in the collection: $\beta^{(0)}(y_i, y_{i+1})\beta^{(1)}(y_i, y_{i+1}) \cdots \beta^{(k)}(y_i, y_{i+1})$ is a geodesic segment, for all $k \in \mathbb{N}$.*
2. *Two geodesic segments in (a) stay at a uniform distance from each other for any $k \in \mathbb{N}$.*

In consequence, the infinite concatenation $\beta^{\infty}(y_i, y_{i+1})$ defines infinitely many geodesic rays with a unique limit point in $\partial\Gamma$. It will be denoted by $(y_i, y_{i+1})^{\infty}$.

Basic Definitions: Bowen-Series-Like map

Cylinders, definition of the BSL map

We define the *cylinder* of length one as the subset of the boundary:

$$\mathcal{C}_x := \{\xi \in \partial\Gamma : \text{there is a geodesic ray } \{\xi\} \text{ starting at id by } x \in X\}.$$

Lemma

Let $P = \langle X/R \rangle$ be a geometric presentation of Γ. The boundary $\partial\Gamma = \mathbb{S}^1$ is covered by the cylinder sets $\mathcal{C}_x$, $x \in X$ and:

1. *Two cylinders have non-empty intersection: $\mathcal{C}_x \bigcap \mathcal{C}_y \neq \emptyset$ if and only if (x, y) is an adjacent pair of generators.*
2. *Each cylinder $\mathcal{C}_x$, $x \in X$ is a non trivial connected interval of $\partial\Gamma$.*

Observe that the point $(y_i, y_{i+1})^\infty$ of the above lemma belongs, by definition, to the intersection $\mathcal{C}_{y_i} \bigcap \mathcal{C}_{y_{i+1}}$.

Basic Definitions: Bowen-Series-Like map

Cylinders, definition of the BSL map

We denote by I_{y_i} the interval[1] $[(y_{i-1}, y_i)^\infty, (y_i, y_{i+1})^\infty]$. Clearly I_{y_i} is a subset of $\mathcal{C}_{y_i}$ for every $y_i \in X$.

If $P = \langle X/R \rangle$ is a geometric presentation of a hyperbolic surface group Γ, we define the Bowen-Series-Like map $\Phi_P \colon \partial\Gamma \longrightarrow \partial\Gamma$ by

$$\Phi_P(\xi) = x^{-1}(\xi) \quad \text{if } \xi \in I_x,$$

where $x^{-1}(\xi)$ is the action, by homeomorphism, on $\partial\Gamma$ by the element x^{-1}.

The map Φ_P satisfies the following elementary properties:

(i) It depends explicitly on the presentation P.

(ii) Since $I_x \subset \mathcal{C}_x$, each $\xi \in I_x$ has a writing, as a limit of a ray, as $\{\xi\} = x \cdot \omega$. The image under Φ_P is given by:

$$\{\Phi_P(\xi)\} = \{x^{-1}(x \cdot \omega)\} = \{\omega\}.$$

That is, the map Φ_P is a shift map, on this particular writing as a ray.

[1] We consider the points in the circle ordered *clockwise*.

Basic Definitions: Markov partition for minimal geometric presentations

As it has been said, the map Φ_P admits a Markov partition.

We will define a particular presentation, which will be called *symmetric* which makes the map Φ_P and the Markov partition specially simple for this presentation.

The first step is to define subdivision points in each interval I_x, $x \in X$. Let us recall that the extreme points $(y, x)^\infty$ and $(x, z)^\infty$ of the intervals I_x are limit points of bigon rays $\beta^\infty(y, x)$ and $\beta^\infty(x, z)$.

Let us focus on $(y, x)^\infty$.

Basic Definitions: Markov partition for minimal geometric presentations

Let $\beta_v^\infty(y,x)$ be the bigon ray starting at the vertex $v \in \mathrm{Cay}^1(\Gamma, P)$. Observe that with this definition we can write:

$$\beta^\infty(y,x) = \beta(y,x) \cdot \beta_{v_1}^\infty[(y,x)^{(1)}].$$

The particular property of a minimal geometric presentation that is useful here is that there is only one relation R of even length $2n$, when Γ is a surface group of rank n. In this case, any bigon $\beta(y,x)$ has the form $\{\gamma_l, \gamma_r\}$ with $\gamma_l \cdot (\gamma_r)^{-1}$ being one of the words representing the relation R, up to cyclic permutation and inversion. This word starts with the letter y and terminates with the letter x^{-1}. So, we can write the two paths$\{\gamma_l, \gamma_r\}$ as:

$$\{y \cdot x'_{i_2} \cdots x'_{i_n}, x \cdot x_{i_2} \cdots x_{i_n}\}.$$

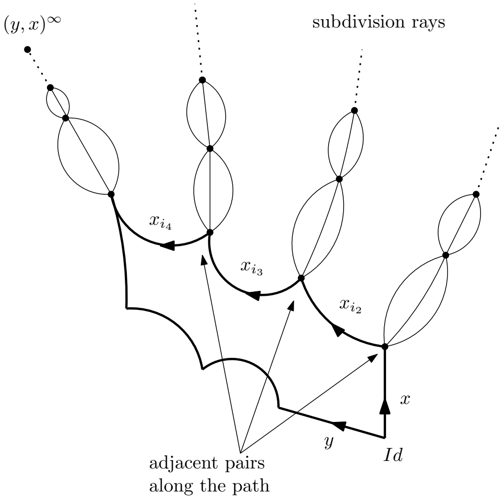
We focus on the "x" side, i.e. on the infinite collection of rays:

$$x \cdot x_{i_2} \cdots x_{i_n} \cdot \beta_v^{(\infty)}[(y,x)^{(1)}],$$

where v is the group element written: $v = x \cdot x_{i_2} \cdots x_{i_n}$. The vertices $v^1 = x$ and $v^j = x \cdot x_{i_2} \cdots x_{i_j}$, for $j = 2, 3, \ldots, n-1$ of $\mathrm{Cay}^1(\Gamma, P)$ belong to γ_r and are ordered along γ_r (this notation is consistent with $v = v_n$).

Basic Definitions: Markov partition for minimal geometric presentations

Bigons of subdivision points



Basic Definitions: Markov partition for minimal geometric presentations

Lemma

If the relation defining $\beta(y, x)$ has even length $2n$ then the collection:

$$\mathcal{R}_L^x := \{x \cdot x_{i_2} \cdots x_{i_j} \cdot \beta_{\nu j}^{(\infty)}[(\overline{x_{i_j}}, x_{i_{j+1}})^{\text{opp}}] : j = 1, \dots, n-1\},$$

is called the *left* (with respect to x) *subdivision rays*. They satisfy the following properties:

1. Each path in the infinite collection $\mathcal{R}_L^x$ is a ray starting at the identity.
2. For a given $j \in \{1, 2, \dots, n-1\}$, all the rays in
$$\mathcal{R}_L^{(x,j)} = x \cdot x_{i_2} \cdots x_{i_j} \cdot \beta_{\nu j}^{(\infty)}[(\overline{x_{i_j}}, x_{i_{j+1}})^{\text{opp}}]$$
converge to the same point $\lambda_x^j \in \partial\Gamma$.
3. For any $j \neq p$, the rays in $\mathcal{R}_L^{(x,j)}$ and in $\mathcal{R}_L^{(x,p)}$ have a common beginning: $x \cdot x_{i_2} \cdots x_{i_\nu}$ where $\nu := \min\{j, p\}$ and are otherwise disjoint.
4. Each λ_x^j, $j \in \{1, 2, \dots, n-1\}$ belongs to the interior of the interval I_x.
5. The limit points λ_x^j are inversely ordered with respect to the index $j \in \{1, 2, \dots, n-1\}$ along $\partial\Gamma$ (that is, $\lambda_x^{n-1} < \lambda_x^{n-2} < \cdots < \lambda_x^2 < \lambda_x^1$).

Basic Definitions: Markov partition for minimal geometric presentations

Bigons of subdivision points

We denote $\mathcal{L}_x = \{\lambda_x^1, \ldots, \lambda_x^{n-1}\}$ this set of *left* (with respect to x) limit points. By the same analysis the adjacent pair (x, z) defines the set of *right* (with respect to x) limit points $\mathcal{R}_x = \{\rho_x^1, \ldots, \rho_x^{n-1}\}$, which are ordered with respect to the superindex.

Consider now the set of all such points:

$$\mathcal{S} = \bigcup_{x \in X} \left(\mathcal{R}_x \cup \mathcal{L}_x \cup \partial I_x\right),$$

called the *subdivision points*.

Basic Definitions: Markov partition for minimal geometric presentations

Lemma

If P is a geometric presentation of a hyperbolic surface group Γ so that all relations have even length, then the set of subdivision points $\mathcal{S}$ is invariant under the map Φ_P and defines a finite Markov partition of $\partial\Gamma$.

The partition of each interval I_x above is given by the points $\mathcal{R}_x \cup \mathcal{L}_x \cup \partial I_x$ which are ordered in the following way:

$$\lambda_x^n := (y,x)^\infty < \lambda_x^{n-1} < \cdots < \lambda_x^2 < \rho_x^1 < \lambda_x^1 < \rho_x^2 < \cdots < \rho_x^n := (x,z)^\infty .$$

Then, we can define a partition of each of the intervals I_x consisting on the following subintervals:

$$L_x^j = \left[\lambda_x^j, \lambda_x^{j-1}\right] \text{ and } R_x^j = \left[\rho_x^{j-1}, \rho_x^j\right], \text{ for } j \in \{3,4,\dots,n\},$$

$$C_x^L = \left[\lambda_x^2, \rho_x^1\right] \text{ and } C_x^R = \left[\lambda_x^1, \rho_x^2\right], \text{ and}$$

$$C_x = \left[\rho_x^1, \lambda_x^1\right].$$

Basic Definitions: Markov partition for minimal geometric presentations

Since the map Φ_P acts, on each interval I_x, as a shift map we obtain:

$$\{\Phi_P(\lambda_x^1)\} = \beta^\infty[(\overline{x}, x_{i_2})^{\text{opp}}], \text{ and}$$

$$\{\Phi_P(\lambda_x^j)\} = x_{i_2} \cdots x_{i_j} \cdot \beta^\infty_{x_{i_2} \cdots x_{i_j}}[(\overline{x_{i_j}}, x_{i_{j+1}})^{\text{opp}}] \text{ for } j \in \{2, 3, \ldots, n\}$$

and there is a similar writing for the points ρ_x^j.

Lemma

If P is a geometric presentation of a surface group with all relations of even length then the image of the central interval $C_x = [\rho_x^1, \lambda_x^1]$ under Φ_P is a single interval I_u, $u \in X$, where u is the generator that is opposite to x^{-1} for the cyclic ordering at the vertex x.

Basic Definitions: Symmetric presentations

We define a particular presentation, which we call *symmetric*.

Definition

Given a surface group $\pi_1(S_g)$ of rank $n = 2g$, where S_g is orientable of genus g, the presentation

$$\left\langle x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1} \middle/ x_1 x_2 \cdots x_n x_1^{-1} x_2^{-1} \cdots x_n^{-1} \right\rangle$$

will be called *symmetric* and denoted by P_n^+.

For the presentation P_n^+ the cyclic ordering at each vertex of $\mathrm{Cay}^1(\Gamma, P)$ is

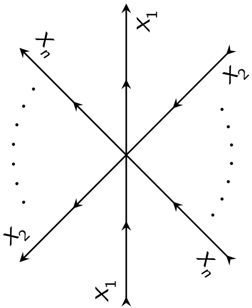
$$x_1 < x_2^{-1} < x_3 < x_4^{-1} < \cdots < x_{n-1} < x_n^{-1} < x_1^{-1} < x_2 < \cdots < x_{n-1}^{-1} < x_n.$$

Proposition

The symmetric presentation P_n^+ is minimal and geometric.

Basic Definitions: Symmetric presentations

The cyclic ordering of a symmetric presentation



The topological entropy of the $\Phi_{P_n^+}$ map–orientable case

Since the surface is orientable and the presentation is geometric and minimal, $\Phi_P\big|_{I_x}$ is an orientation preserving homeomorphism for every $x \in X$ and the set $\mathcal{S}$ defines a Markov partition of Φ_P. Since $\partial I_{x_i} \subset \mathcal{S}$ we also have that Φ_P is a homeomorphism on every $\mathcal{S}-$basic interval. In this situation (see for instance [BGMY]),

$$h_{\text{top}}(\phi) = \log \max\{\rho(M), 1\},$$

where M is the transition matrix of the Markov Graph of Φ_P associated to the invariant set $\mathcal{S}$, and $\rho(M)$ denotes the *spectral radius* of M.

[BGMY] Louis Block, John Guckenheimer, Michał Misiurewicz, and Lai Sang Young.
Periodic points and topological entropy of one-dimensional maps.
In *Global theory of dynamical systems (Proc. Internat. Conf., Northwestern Univ., Evanston, Ill., 1979)*, volume 819 of *Lecture Notes in Math.*, pages 18–34. Springer, Berlin, 1980.

The topological entropy of the map $\Phi_{P_n^+}$–orientable case

We will use the above formula to compute $h_{\text{vol}}(\Gamma, P) = h_{\text{top}}(\Phi_{P_n^+})$.

To this end we first have to compute the Markov matrix of $\mathcal{S}$ that, in what follows, will be denoted by M_n^+.

As we will see, a direct computation of $\rho\big(M_n^+\big)$ is infeasible at a practical level because the size of the matrix grows quadratically with n. So, the computation of $\rho\big(M_n^+\big)$ will be done in two steps by using spectral radius preserving transformations of the matrix M_n^+.

To do this, we need to specify completely the map $\Phi_{P_n^+}$ and then compute its Markov matrix.

The topological entropy of the map $\Phi_{P_n^+}$–orientable case

Computation of $\Phi_{P_n^+}$ in the symmetric case

For the symmetric presentation P_n^+ the cyclic ordering at any vertex is given by:

$$x_1 < x_2^{-1} < \cdots < x_{n-1} < x_n^{-1} < x_1^{-1} < x_2 < \cdots < x_{n-1}^{-1} < x_n < x_1$$

which induces the following ordering of the intervals I_x along the boundary $\partial\Gamma = \mathbb{S}^1$:

$$I_{x_1} < I_{x_2^{-1}} < \cdots < I_{x_n^{-1}} < I_{x_1^{-1}} < I_{x_2} < \cdots < I_{x_n} < I_{x_1}.$$

The fact that makes the symmetric presentation very special and useful is that the edge that is opposite to x at any vertex is simply the edge x^{-1}.

Corollary

Let P_n^+ be the symmetric presentation of an orientable surface group of rank n. Then, $\Phi_{P_n^+}(C_x) = I_x$ for each generator x.

The topological entropy of the map $\Phi_{P_n^+}$–orientable case

Computation of $\Phi_{P_n^+}$ in the symmetric case

Observe that each of the $2n$ intervals I_{y_i} is divided into $2n-1$ intervals

$$L_{y_i}^n < \cdots < L_{y_i}^3 < C_{y_i}^L < C_{y_i} < C_{y_i}^R < R_{y_i}^3 < \cdots < R_{y_i}^n, \qquad (1)$$

where $y_i = x_i^{(-1)^{i+1}}$ for $1 \le i \le n$, and $y_i = x_{i-n}^{(-1)^i}$ for $n+1 \le i \le 2n$. Also, the fact that the edge that is opposite to x at any vertex is the edge x^{-1} now gives $y_i^{-1} = y_{[i+n]_{2n}}$.

Hence, $|\mathcal{S}| = 2n(2n-1)$ and thus, the matrix M_n^+ is $2n(2n-1) \times 2n(2n-1)$.

Then, the images of the points of $\mathcal{S}$ computed above give:

The topological entropy of the map $\Phi_{P_n^+}$–orientable case

Computation of $\Phi_{P_n^+}$ in the symmetric case

$$\Phi_{P_n^+}(L^j_{y_i}) = L^{j-1}_{y_{[i+n+1]_{2n}}} \text{ for } j \in \{4,5,\dots,n\},$$

$$\Phi_{P_n^+}(L^3_{y_i}) = C^L_{y_{[i+n+1]_{2n}}} \cup C_{y_{[i+n+1]_{2n}}},$$

$$\Phi_{P_n^+}(C^L_{y_i}) = C^R_{y_{[i+n+1]_{2n}}} \cup \left(\bigcup_{j=2}^{n} R^j_{y_{[i+n+1]_{2n}}}\right) \cup \left(\bigcup_{k=[i+n+2]_{2n}}^{[i-1]_{2n}} I_{y_k}\right),$$

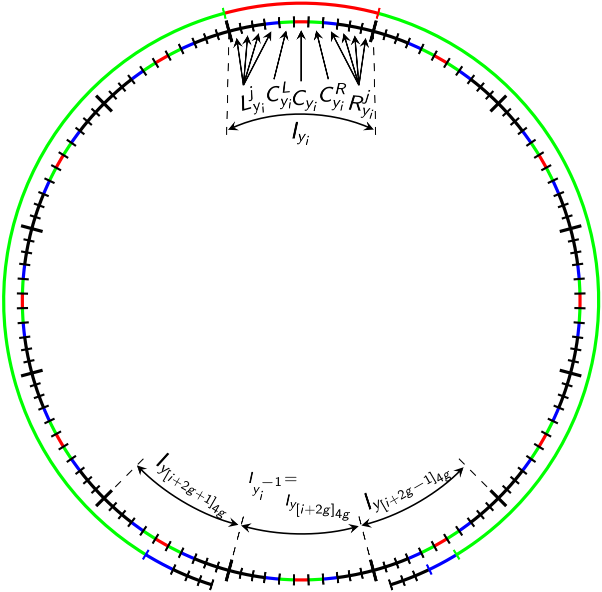
$$\Phi_{P_n^+}(C_{y_i}) = I_{y_i},$$

$$\Phi_{P_n^+}(C^R_{y_i}) = C^L_{y_{[i+n-1]_{2n}}} \cup \left(\bigcup_{j=2}^{n} L^j_{y_{[i+n-1]_{2n}}}\right) \cup \left(\bigcup_{k=[i+1]_{2n}}^{[i+n-2]_{2n}} I_{y_k}\right),$$

$$\Phi_{P_n^+}(R^3_{y_i}) = C_{y_{[i+n-1]_{2n}}} \cup C^R_{y_{[i+n-1]_{2n}}},$$

$$\Phi_{P_n^+}(R^j_{y_i}) = R^{j-1}_{y_{[i+n-1]_{2n}}} \text{ for } j \in \{4,5,\dots,n\}.$$

The topological entropy of the map $\Phi_{P_n^+}$–orientable case



Computation of $\Phi_{P_n^+}$ in the symmetric case: The intervals I_{y_i} in the circle together with the interior intervals.

The outer curve is the image $\Phi_{P_n^+}(I_{y_i})$. The intervals $L^j_{y_i}$, $R^j_{y_i}$ and their images are drawn in black, $L^3_{y_i}$, $R^3_{y_i}$ and their images are drawn in blue, $C^L_{y_i}$, $C^R_{y_i}$ and their images are drawn in green and finally, C_{y_i} and its image are drawn in red

The topological entropy of the map $\Phi_{P_n^+}$–orientable case

Computation of $\Phi_{P_n^+}$ in the symmetric case

The Markov matrix M_n^+ has a structure in blocks, all of size $(2n-1)\times(2n-1)$. So, it is convenient to write the matrix M_n^+ as

$$\begin{pmatrix} M_{11} & M_{12} & \dots & M_{1,2n} \\ M_{21} & M_{22} & \dots & M_{2,2n} \\ \dots & \dots & \dots & \dots \\ M_{n1} & M_{n2} & \dots & M_{n,2n} \\ \dots & \dots & \dots & \dots \\ M_{2n,1} & M_{2n,2} & \dots & M_{2n,2n} \end{pmatrix}$$

where each of the matrices $M_{lt} = (m_{ij}^{lt})_{i,j=1}^{2n-1}$ is of size $(2n-1)\times(2n-1)$.

The next theorem is a first reduction in the effective computation of $h_{\text{top}}(\Phi_{P_n^+})$.

Theorem (First Reduction)

$$h_{top}(\Phi_{P_n^+}) = \log\max\left\{\rho(M_n^+), 1\right\} = \log\max\left\{\rho\left(\sum_{k=1}^{2n} M_{1k}\right), 1\right\}.$$

The topological entropy of the map $\Phi_{P_n^+}$–orientable case

The first three (of the total of eight) block rows of the Markov matrix $M_{P_4^+}$ corresponding to the symmetric presentation of an orientable surface group of rank 4

$$
\left(\begin{array}{c|c|c|c|c|c|c|c}
0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0100000 & 0000000 & 0000000 \\
0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0011000 & 0000000 & 0000000 \\
0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000111 & 1111111 & 1111111 \\
1111111 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 \\
0000000 & 1111111 & 1111111 & 1110000 & 0000000 & 0000000 & 0000000 & 0000000 \\
0000000 & 0000000 & 0000000 & 0001100 & 0000000 & 0000000 & 0000000 & 0000000 \\
0000000 & 0000000 & 0000000 & 0000010 & 0000000 & 0000000 & 0000000 & 0000000 \\
\hline
0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0100000 & 0000000 \\
0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0011000 & 0000000 \\
1111111 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000111 & 1111111 \\
0000000 & 1111111 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 \\
0000000 & 0000000 & 1111111 & 1111111 & 1110000 & 0000000 & 0000000 & 0000000 \\
0000000 & 0000000 & 0000000 & 0000000 & 0001100 & 0000000 & 0000000 & 0000000 \\
0000000 & 0000000 & 0000000 & 0000000 & 0000010 & 0000000 & 0000000 & 0000000 \\
\hline
0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0100000 \\
0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0011000 \\
1111111 & 1111111 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000111 \\
0000000 & 0000000 & 1111111 & 0000000 & 0000000 & 0000000 & 0000000 & 0000000 \\
0000000 & 0000000 & 0000000 & 1111111 & 1111111 & 1110000 & 0000000 & 0000000 \\
0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0001100 & 0000000 & 0000000 \\
0000000 & 0000000 & 0000000 & 0000000 & 0000000 & 0000010 & 0000000 & 0000000
\end{array}\right)
$$

The topological entropy of the map $\Phi_{P_n^+}$–orientable case

A basic tool: block circulant matrix

An *$(r, s)-$block circulant matrix* is a matrix of the form

$$\begin{pmatrix} A_1 & A_2 & A_3 & \dots & A_r \\ A_r & A_1 & A_2 & \dots & A_{r-1} \\ A_{r-1} & A_r & A_1 & \dots & A_{r-2} \\ \dots & \dots & \dots & \dots & \dots \\ A_2 & A_3 & A_4 & \dots & A_1 \end{pmatrix}$$

where each A_i is an $s \times s$ matrix. Notice that a circulant matrix is completely determined by its first block row $(A_1 \; A_2 \; A_3 \; \dots \; A_r)$.

The next lemma will be crucial in effectively computing the spectral radius of M_n^+

Lemma

The Markov matrix M_n^+ is a $(2n, 2n-1)-$block circulant matrix.

The topological entropy of the map $\Phi_{P_n^+}$–orientable case

The spectral radius of a block circulant matrix: proof of Theorem First Reduction

Lemma

Let

$$A = \begin{pmatrix} A_1 & A_2 & A_3 & \dots & A_r \\ A_r & A_1 & A_2 & \dots & A_{r-1} \\ A_{r-1} & A_r & A_1 & \dots & A_{r-2} \\ \dots & \dots & \dots & \dots & \dots \\ A_2 & A_3 & A_4 & \dots & A_1 \end{pmatrix}$$

be a non-negative block circulant matrix. Then

$$\rho(A) = \rho\left(\sum_{i=1}^{r} A_i\right).$$

Remark

The above lemma holds for every matrix for which a given block appears exactly once in every block row and every block column.

The topological entropy of the map $\Phi_{P_n^+}$–orientable case

First Reduction: an explicit formula for the matrix $\sum_{k=1}^{2n} M_{1k}$

- The zero matrix of size $k \times k$ will be denoted by $\mathbf{0}_k$,
- $\mathbf{J}_k$ will denote the $k \times k$ $(0,1)-$matrix with ones in the anti-diagonal,
- $\mathbf{U}_k^i$ will denote the $k \times k$ matrix such that all entries in the $i-$th row are 1 and all other entries are 0, where $i \in \{1, 2, \ldots, k\}$,
- $\mathbf{T}_k = (t_{ij})$ is the $k \times k$ $(0,1)-$matrix such that $t_{ij} = 1$ if and only if
    - $j = i + 1$ and $i \in \{1, 2, \ldots, \widetilde{k} - 3\}$, or
    - $j \in \{\widetilde{k} - 1, \widetilde{k}\}$ and $i = \widetilde{k} - 2$ or
    - $\widetilde{k} + 1 \leq j \leq k$ and $i = \widetilde{k} - 1$,

  where $\widetilde{k} = \frac{k+1}{2}$ and $k \geq 5$ is odd.

The topological entropy of the map $\Phi_{P_n^+}$–orientable case

First Reduction: Examples of the matrices $\mathbf{U}_k^i$, $\mathbf{T}_k$, $\mathbf{J}_k$ and $\mathbf{J}_k\mathbf{T}_k\mathbf{J}_k$ with $k = 7$. Observe that $\mathbf{J}_k\mathbf{T}_k\mathbf{J}_k$ is the matrix obtained from $\mathbf{T}_k$ by a symmetry with respect to the central coordinate $t_{\widetilde{k},\widetilde{k}}$

$$\mathbf{U}_7^3 = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 1&1&1&1&1&1&1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix} \qquad \mathbf{T}_7 = \begin{pmatrix} 0&1&0&0&0&0&0\\ 0&0&1&1&0&0&0\\ 0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}$$

$$\mathbf{J}_7 = \begin{pmatrix} 0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0\\ 0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0\\ 0&1&0&0&0&0&0\\ 1&0&0&0&0&0&0 \end{pmatrix} \qquad \mathbf{J}_7\mathbf{T}_7\mathbf{J}_7 = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 1&1&1&0&0&0&0\\ 0&0&0&1&1&0&0\\ 0&0&0&0&0&1&0 \end{pmatrix}$$

The topological entropy of the map $\Phi_{P_n^+}$–orientable case

First Reduction: an explicit formula for the matrix $\sum_{k=1}^{2n} M_{1k}$; the *compacted matrix of rank n*

Observe that the blocks of the matrix M_n^+ are of one of the four types above. Then, by carefully counting the blocks,

$$\sum_{t=1}^{n} M_{1t} = \mathbf{T}_{2n-1} + \mathbf{J}_{2n-1}\mathbf{T}_{2n-1}\mathbf{J}_{2n-1} + \mathbf{U}_{2n-1}^{n} + (n-2)\left(\mathbf{U}_{2n-1}^{n-1} + \mathbf{U}_{2n-1}^{n+1}\right) = \mathrm{C}_n =$$

$$\begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
n-2 & n-2 & n-2 & \cdots & n-2 & n-2 & n-2 & n-1 & n-1 & \cdots & n-1 & n-1 & n-1 \\
1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
n-1 & n-1 & n-1 & \cdots & n-1 & n-1 & n-2 & n-2 & n-2 & \cdots & n-2 & n-2 & n-2 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0
\end{pmatrix}$$

The topological entropy of the map $\Phi_{P_n^+}$–orientable case

First Reduction: the *compacted matrix of rank n*

Corollary

$$h_{top}(\Phi_{P_n^+}) = \log \max \{\rho(C_n), 1\}.$$

Remark

Note that the map $\Phi_{P_n^+}$ commutes with a rigid rotation R of period $2n$. The quotient space obtained by identifying each orbit of R to a point is a circle. The map induced by $\Phi_{P_n^+}$ on this quotient space is also a Markov map. The matrix C_n is nothing but the Markov matrix of this induced map.

The non-orientable case

We start by extending the definition of *symmetric presentation* to non orientable surface groups.

Definition

Given a surface group $\Gamma = \pi_1(S)$ of rank n, where S is a non orientable surface, the following presentation of Γ will be called *symmetric* and denoted by P_n^-. Its definition depends on the parity of n as follows. For n odd, we define P_n^- as

$$\left\langle x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1} \middle/ x_1 x_2 \cdots x_n x_{n-1} x_{n-2} \cdots x_1 x_n \right\rangle$$

while, for n even, P_n^- is defined as

$$\left\langle x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1} \middle/ x_1 x_2 \cdots x_n x_{n-1} x_{n-2} \cdots x_1 x_n^{-1} \right\rangle.$$

The non-orientable case

Similar arguments to the ones used above yield that the symmetric presentation P_n^- is minimal and geometric.

As in the orientable case, the nomenclature *symmetric* for the presentation P_n^- accounts for the fact that, at each vertex, the cyclic ordering of the generators exhibits the useful property that the edge opposite to x at any vertex is simply the edge x^{-1}. Indeed, one can check that the ordering of the generators at any vertex is

$$x_1 < x_2^{-1} < x_3 < \cdots < x_{n-1} < x_n^{-1} < x_1^{-1} < x_2 < x_3^{-1} < \cdots < x_{n-1}^{-1} < x_n$$

when n is even, and

$$x_1 < x_2^{-1} < x_3 < \cdots < x_{n-1}^{-1} < x_n < x_1^{-1} < x_2 < x_3^{-1} < \cdots < x_{n-1} < x_n^{-1}$$

when n is odd.

The non-orientable case

The fact that the symmetric presentation has associated the above cyclic ordering gives

Corollary

Let P_n^- be the symmetric presentation of a non-orientable surface group of rank n. Then, $\Phi_{P_n^-}(C_x) = I_x$ for each generator x.

Notice that map $\Phi_{P_n^+}$ and the Markov matrix M_n^+ are only defined for n even since the group corresponds to an orientable surface. However, all associated formulae extend to the case n odd. In this sense below we will compare the maps $\Phi_{P_n^+}$ and $\Phi_{P_n^-}$ and the associated Markov matrices M_n^+ and M_n^-, independently on the parity of n.

The non-orientable case

Using the notations introduced in the orientable case one can check that the Markov map $\Phi_{P_n^-}$ behaves essentially as $\Phi_{P_n^+}$ in all intervals I_{y_i} except when $i \in \{n, 2n\}$. In these two intervals the map reverses orientation.

Hence, in a similar way to the previous case it follows that the Markov matrix M_n^- of $\Phi_{P_n^-}$ is as in the orientable case with $M_{i,j}$ replaced by $\mathbf{J}_{2n-1} M_{i,j}$ for $i \in \{n, 2n\}$ and $j = 1, 2, \ldots, 2n$

The non-orientable case

The first three (of the total of six) block rows of the Markov matrix $M_{P_3^-}$ corresponding to the symmetric presentation of a non-orientable surface group of rank 3

$$\left(\begin{array}{ccccc|ccccc|ccccc|ccccc|ccccc|ccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1\\
1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right)$$

The non-orientable case

The spectral radius of $M_{P_n^-}$

An *$(r, s)-$disoriented block circulant matrix* is a matrix of the form

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1r} \\ A_{21} & A_{22} & \dots & A_{2r} \\ \dots & \dots & \dots & \dots \\ A_{r1} & A_{r2} & \dots & A_{rr} \end{pmatrix}$$

where each A_{ij} is a $s \times s$ matrix for which there exists an $(r, s)-$block circulant matrix

$$\widetilde{A} = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1r} \\ \widetilde{A}_{21} & \widetilde{A}_{22} & \dots & \widetilde{A}_{2r} \\ \dots & \dots & \dots & \dots \\ \widetilde{A}_{r1} & \widetilde{A}_{r2} & \dots & \widetilde{A}_{rr} \end{pmatrix}$$

such that given $i \in \{2, \dots, r\}$, either

- $A_{ij} = \widetilde{A}_{ij}$ for every $j = 1, 2, \dots, r$ or
- $A_{ij} = \mathbf{J}_s \widetilde{A}_{ij}$ for every $j = 1, 2, \dots, r$.

That is, every block row of A coincides with the corresponding block row of $\widetilde{A}$ or is obtained from the corresponding block row of $\widetilde{A}$ by pre-multiplying each block by $\mathbf{J}_s$.

The non-orientable case

Disoriented block circulant matrices

This last operation permutes the individual rows of the block row symmetrically with respect to the central horizontal axis.

The matrix $\widetilde{A}$ will be called the *parallelization of A*. The assumption that the first block row of A and $\widetilde{A}$ coincide implies that the parallelization of A is unique.

The non-orientable case

Disoriented block circulant matrices

Lemma

Let

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1r} \\ A_{21} & A_{22} & \dots & A_{2r} \\ \dots & \dots & \dots & \dots \\ A_{r1} & A_{r2} & \dots & A_{rr} \end{pmatrix}$$

be a non-negative disoriented $(r, s)-$*block circulant matrix such that*

$$\left(\sum_{j=1}^{r} A_{1j}\right) \mathbf{J}_s = \mathbf{J}_s \left(\sum_{j=1}^{r} A_{1j}\right) .$$

Then

$$\rho(A) = \rho\left(\sum_{j=1}^{r} A_{1j}\right) .$$

The non-orientable case

Disoriented block circulant matrices

Corollary

$$h_{top}(\Phi_{P_n^-}) = \log \max \{\rho(C_n), 1\}.$$

Second reduction: Super compacting the matrix C_n

To do this reduction we need another intermediate matrix which we obtain from C_n.

The *divided compacted matrix of rank n* of size $2n \times 2n$, denoted by $DC_n = (d_{ij})$, is the matrix

$$
\left(\begin{array}{cccccc|cc|cccccc}
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
n-2 & n-2 & n-2 & \cdots & n-2 & n-2 & n-2 & n-2 & n-1 & n-1 & \cdots & n-1 & n-1 & n-1 \\
\hline
1 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\
\hline
n-1 & n-1 & n-1 & \cdots & n-1 & n-1 & n-2 & n-2 & n-2 & n-2 & \cdots & n-2 & n-2 & n-2 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0
\end{array}\right)
$$

which is, indeed, the Markov matrix of a topological model obtained by subdividing the central interval of the *compacted topological model* at a fixed point (that exists because the central interval covers itself).

Second reduction: Super compacting the matrix C_n

The *super compacted matrix of rank n* is the $n \times n$ matrix SC_n defined as:

$$SC_n = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \dots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \dots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2 \\ 2n-3 & 2n-3 & 2n-3 & 2n-3 & \cdots & 2n-3 & 2n-3 & 2n-4 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{pmatrix}$$

Remark

The *divided compacted topological model* has a fixed point and commutes with the symmetry of degree -1 with respect to this fixed point. The quotient space obtained by identifying each orbit of the symmetry to a point is a closed interval, and the induced map on this quotient space is also a Markov map. The matrix SC_n is in fact the Markov matrix of this quotient map.

Second reduction: From C_n to SC_n

Proposition

For every $n \geq 3$,

$$\max\{\rho(C_n), 1\} = \max\{\rho(DC_n), 1\} = \max\{\rho(SC_n), 1\}.$$

The spectral radius of SC_n and the *Rome Method*

Let $M = (m_{ij})$ be a $k \times k$ matrix. Given a sequence $p = (p_j)_{j=0}^{\ell}$ of elements of $\{1, 2, \dots, k\}$ we define the *width of p*, denoted by $w(p)$, as the number $\prod_{j=1}^{\ell} m_{p_{j-1}p_j}$. If $w(p) \neq 0$ then p is called a *path of length* ℓ. The length of a path p will be denoted by $\ell(p)$. A loop is a path such that $p_\ell = p_0$ i.e. that begins and ends at the same index.

A subset R of $\{1, 2, \dots, k\}$ is called a *rome* if there is no loop outside R, i.e. there is no path $(p_j)_{j=0}^{\ell}$ such that $p_\ell = p_0$ and $\{p_j : 0 \le j \le \ell\}$ is disjoint from R. For a rome R a path $(p_j)_{j=0}^{\ell}$ is called *simple* if $p_i \in R$ for $i = 0, \ell$ and $p_i \notin R$ for $i = 1, 2, \dots, \ell - 1$.

The spectral radius of SC_n and the *Rome Method*

If $R = \{r_1, r_2, \ldots, r_\ell\}$ is a rome of a matrix M then we define an $\ell \times \ell$ matrix-valued real function $M_R(x)$ by setting $M_R(x) = (a_{ij}(x))$, where $a_{ij}(x) = \sum_p w(p) \cdot x^{-\ell(p)}$, where the summation is over all simple paths originating at r_i and terminating at r_j.

Theorem ([BGMY])

If R is a rome of cardinality ℓ of a $k \times k$ matrix M then the characteristic polynomial of M is equal to

$$(-1)^{k-\ell} x^k \det(M_R(x) - \mathbf{I}_\ell).$$

To use this theorem it is helpful to represent the matrix M in form of a combinatorial graph which amounts to draw all paths of length 1 associated to M.
To do this we introduce the following notation.

The spectral radius of SC_n and the *Rome Method*

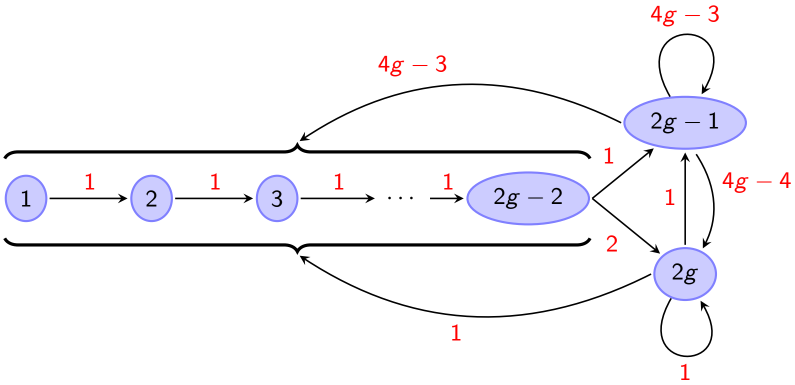
A path (i, j) of length 1 will be written as $i \xrightarrow{w} j$, where w denotes the width of the path. For the matrix M the width w of the path $i \xrightarrow{w} j$, is just the entry $(M)_{i,j} \neq 0$. Observe that, with this notation, a path $p = (p_j)_{j=0}^{k}$ is written as

$$p_0 \xrightarrow{w_0} p_1 \xrightarrow{w_1} \cdots p_{k-1} \xrightarrow{w_{k-1}} p_k$$

and $w(p) = \prod_{i=0}^{k-1} w_i$.

The spectral radius of SC_n and the *Rome Method*

The combinatorial graph associated to SC_n. The arrows ending at braces indicate multiple arrows with the same weight, each one directed towards a node under the brace



Remark

This combinatorial graph is, in fact, the generalized Markov graph of the *super compacted topological model*.

The spectral radius of SC_n and the *Rome Method*

Proposition

The spectral radius of SC_n *is the largest root of the polynomial* $Q_n(x) = x^n - 2(n-1)\sum_{j=1}^{n-1} x^j + 1.$

Lemma

For every $n \geq 3$, $Q_n(x)$ *has a unique real root* λ_n *larger than one. Moreover, for* $n \geq 4$,

$$2n - 1 - \frac{1}{(2n-1)^{n-2}} < \lambda_n < 2n - 1.$$

The spectral radius of SC_n and the *Rome Method*

Proof of the proposition

Clearly $R = \{n-1, n\}$ is a rome of SC_n. Hence,

$$M_R(x) = \begin{pmatrix} \beta\big(x^{-1} + z(x)\big) & (\beta-1)x^{-1} + 2\beta z(x) \\ x^{-1} + z(x) & x^{-1} + 2z(x) \end{pmatrix}$$

where $\beta = 2n - 3$, $z(x) := \sum_{\ell=2}^{n-1} x^{-\ell}$.

By [BGMY] Theorem, the characteristic polynomial of SC_n is

$$(-1)^{n-2}x^n \begin{vmatrix} \beta\big(x^{-1} + z(x)\big) - 1 & 2\beta\big(x^{-1} + z(x)\big) - (\beta+1)x^{-1} \\ x^{-1} + z(x) & 2\big(x^{-1} + z(x)\big) - x^{-1} - 1 \end{vmatrix}$$

$$= x^n - (2n-2)\sum_{j=1}^{n-1} x^j + 1.$$