Forcing for skew-products on the cylinder

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The problem

We want to study the coexistence and implications between periodic orbits of maps from $\Omega = \mathbb{S}^1 \times I$, of the form:

$$\begin{array}{cccc} & \Omega & \longrightarrow & \Omega \\ F: & \begin{pmatrix} \theta \\ x \end{pmatrix} & \longmapsto & \begin{pmatrix} R(\theta) \\ f(\theta, x) \end{pmatrix} \end{array}$$

where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, $R(\theta) = \theta + \omega \pmod 1$, $\omega \in \mathbb{R} \setminus Q$ and $f(\theta, x) = f_\theta(x)$ is continuous on both variables.

Remark

Instead of $\mathbb{S}^1$ we can take any compact metric space Θ that admits a minimal homeomorphism $R\colon \Theta \longrightarrow \Theta$ such that R^ℓ is minimal $\forall\, \ell > 1$.

We call $\mathcal{S}(\Omega)$ this class of skew products.

The problem

In [FJJK] the Sharkovskii Theorem is extended to the above setting.

[FJJK] R. Fabbri, T. Jäger, R. Johnson and G. Keller, *A Sharkovskii-type theorem for minimally forced interval maps*, Topological Methods in Nonlinear Analysis, Journal of the Juliusz Shauder Center, **26** (2005), 163–188.

Aim of the talk

Extend the Sharkovskii Theorem and the techniques from [FJJK] to study the combinatorial dynamics (*forcing*) and entropy of the skew-products from the class $\mathcal{S}(\Omega)$.

Plan of the talk

1. Short survey on the [FJJK] paper
2. Survey on the forcing relation on the interval
3. Definition of forcing in Ω
4. Characterization of the forcing in Ω
5. Topological entropy and forcing

The notion of a periodic orbit

First we need to introduce what we understand by a *periodic orbit* in this setting.

Idea

Instead of points we use objects that project over the whole $\mathbb{S}^1$.

Strip

is a set $B \subset \Omega$ such that

- B is closed
- $(\{\theta\} \times I) \cap B \neq \emptyset \quad \forall\, \theta \in \mathbb{S}^1$ (B projects on the whole $\mathbb{S}^1$)
- $(\{\theta\} \times I) \cap B$ is an interval $\forall\, \theta$ in a residual set of $\mathbb{S}^1$.

The notion of a periodic orbit (II)

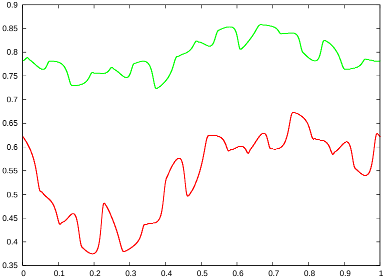
Periodic orbit (of strips) (of period n) for a map $F \in \mathcal{S}(\Omega)$

is a set $B_1, B_2, \dots, B_n$ of pairwise disjoint strips such that

$$\begin{aligned} F(B_i) &\subset B_{i+1} \quad \text{for } i = 1, 2, \dots, n-1 \\ F(B_n) &\subset B_1. \end{aligned}$$

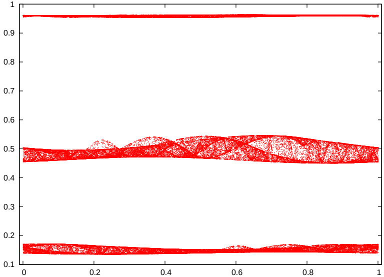
Examples of periodic strips (attractors of F)

In both cases, $R(\theta) = \theta + \frac{\sqrt{5}-1}{2} \pmod 1$ and the map $f(\theta, x)$ is specified below the figure in each case.



$3.28x(1-x) + \frac{4}{100}\cos(2\pi\theta)$

A two periodic orbit of periodic curves.



$3.85x(1-x)(1 + \frac{111}{10^5}\cos(2\pi\theta))$

A three periodic orbit of periodic solid strips.
It corresponds to the three periodic orbit of transitive intervals exhibited by the map $3.85x(1-x)$.

A refinement

The definition of a periodic orbit of strips is too general. It turns out that every periodic orbit of strips contains another (more restrictive) periodic orbit of strips that verifies:

- Every strip is *core*
- The new periodic orbit forms a minimal set
- The new periodic orbit is F-strongly invariant

That is,

for each $i = 1, 2, \dots, n$ there exist a strip $A_i \subset B_i$ such that

- The strips A_i form a *strongly invariant periodic orbit*: $F(A_i) = A_{i+1}$ for $i = 1, 2, \dots, n-1$ and $F(A_n) = A_1$.
- $\bigcup_{i=1}^{n} A_i$ is a minimal set for F.
- Each strip A_i is *core*:

$$A_i = A_i^c := \bigcap_{G \text{ residual}} \overline{A_i \cap (G \times I)}.$$

The *core of A_i* is what remains after "shaving" what is not seen by the closures of A_i inside "fibered residual sets".

There are two kind of strips which are minimal, core and strongly invariant:

The two basic kind of strips: Solid

$B \subset \Omega$ is a *solid strip* if:

- B is closed.
- $B^\theta := B \cap (\{\theta\} \times I)$ is a non-degenerate interval for every $\theta \in \mathbb{S}^1$.
- $\inf_{\theta \in \mathbb{S}^1} \operatorname{diam}(B^\theta) > 0$.

Remark

A solid strip is a strip.

Proposition

A solid strip is connected.

Example

The right figure in Page 6 before.

The two basic kind of strips: pseudo-curves (pinched)

A pseudo-curve is the closure of the graph of (φ, G) where G is a residual set of $\mathbb{S}^1$ and $\varphi \colon G \to I$ is continuous. That is, a pseudo-curve is:

$$\overline{\{(\theta, \varphi(\theta)) : \theta \in G\}}.$$

Example

The left figure in Page 6 before.

The properties of pseudo-curves

- $\{(\theta, \varphi(\theta)) : \theta \in G\}$ is the "pinched set".
- If $\theta \notin G$, the intersection of the pseudo-curve with the fiber $\{\theta\} \times I$ may be a non-degenerate interval.
- A pseudo-curve is either a curve or does not contain any *arc of a curve*.
- Each connected component of the boundary of a solid strip is a pseudo-curve.

Arc of a curve:

is the graph of a continuous function from an arc of $\mathbb{S}^1$ to I.

The Sharkovskiĭ Ordering ${}_{\text{Sh}}\!\geq$

The coexistence of periodic orbits of strips is given by the next theorem. To state it we use the *Sharkovskiĭ Ordering*:

$$
\begin{array}{l}
3 \;{}_{\text{Sh}}\!> 5 \;{}_{\text{Sh}}\!> 7 \;{}_{\text{Sh}}\!> \cdots \;{}_{\text{Sh}}\!> \\
2\cdot 3 \;{}_{\text{Sh}}\!> 2\cdot 5 \;{}_{\text{Sh}}\!> 2\cdot 7 \;{}_{\text{Sh}}\!> \cdots \;{}_{\text{Sh}}\!> \\
4\cdot 3 \;{}_{\text{Sh}}\!> 4\cdot 5 \;{}_{\text{Sh}}\!> 4\cdot 7 \;{}_{\text{Sh}}\!> \cdots \;{}_{\text{Sh}}\!> \\
\qquad\qquad\vdots \\
2^n\cdot 3 \;{}_{\text{Sh}}\!> 2^n\cdot 5 \;{}_{\text{Sh}}\!> 2^n\cdot 7 \;{}_{\text{Sh}}\!> \cdots \;{}_{\text{Sh}}\!> \\
\qquad\qquad\vdots \\
2^\infty \;{}_{\text{Sh}}\!> \cdots \;{}_{\text{Sh}}\!> 2^n \;{}_{\text{Sh}}\!> \cdots \;{}_{\text{Sh}}\!> 16 \;{}_{\text{Sh}}\!> 8 \;{}_{\text{Sh}}\!> 4 \;{}_{\text{Sh}}\!> 2 \;{}_{\text{Sh}}\!> 1.
\end{array}
$$

defined on the set $\mathbb{N}_{\text{Sh}} = \mathbb{N} \cup \{2^\infty\}$ (we have to include the symbol 2^∞ to assure the existence of supremum for certain sets).

In the ordering ${}_{\text{Sh}}\!\geq$ the least element is 1 and the largest is 3. The supremum of the set $\{1, 2, 4, \ldots, 2^n, \ldots\}$ is 2^∞.

Theorem (Fabbri, Jäger, Johnson and Keller)

Let P be a periodic orbit of solid strips or pseudo-curves of period n of $F \in \mathcal{S}(\Omega)$. Then F has a periodic orbit of solid strips or pseudo-curves of period m for every $m \leq_{\text{Sh}} n$.

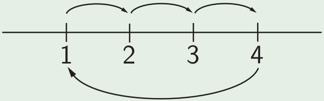
As said, our aim is to extend the above theorem and the techniques from [FJJK] to study the combinatorial dynamics (*forcing*) of periodic orbits of strips for maps from $\mathcal{S}(\Omega)$.

Pattern of a periodic orbit $\longleftrightarrow$ permutation

Definition

Let $p_1 < p_2 < \cdots < p_n$ be a periodic orbit of a map $f \in \mathcal{C}^0(I, I)$. The periodic orbit $\{p_1, p_2, \ldots, p_n\}$ has pattern τ if and only if $f(p_i) = p_{\tau(i)}$ for $i = 1, 2, \ldots, n$.

Example



$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 1$$

$\tau = (1234)$ is the pattern

Forcing in the interval

Definition (forcing)

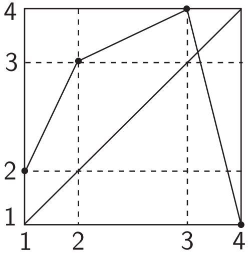
$\tau \Longrightarrow_I \nu$ where τ and ν are patterns if and only if *every* $f \in \mathcal{C}^0(I, I)$ that has a periodic orbit with pattern τ also has a periodic orbit with pattern ν.

Theorem

$\Longrightarrow_I$ *is an ordering relation (partial).*

Given a pattern τ we take the connect-the-dots map f_τ (the τ-linear map):

$$\tau = (1, 2, 3, 4)$$



Theorem (Characterization of forcing)

$\tau \Longrightarrow_I \nu$ if and only if f_τ has a periodic orbit with pattern ν.

Why we are interested in the forcing relation?

Theorem

Every pattern of period n forces a pattern of period m for every $m \leq_{\mathrm{Sh}} n$.

Corollary

The Sharkovskii Theorem for maps from $\mathcal{C}^0(I, I)$ holds.

Patterns and forcing in Ω

Pattern $\longleftrightarrow$ permutation (again)

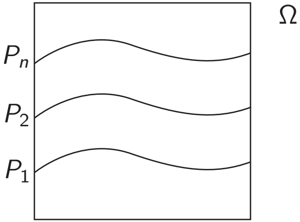
Since a periodic orbit has disjoint strips we can order them from lower to upper

Definition

The periodic orbit $P_1, P_2, \ldots, P_n$ of $F \in \mathcal{S}(\Omega)$ has *pattern τ* if and only if

$$F(P_i) = P_{\tau(i)}$$

for every $i = 1, 2, \ldots, n$.



So, patterns in I and Ω are the same abstract objects.

The quasiperiodic τ-linear map F_τ

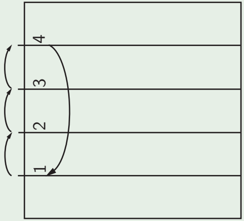
Definition

Given an interval pattern τ we define

$$F_\tau = (R(\theta), f_\tau(x))$$

where $R(\theta) = \theta + \omega \pmod 1$. This map will be called the *quasiperiodic τ-linear map*.

Example



Observation

F_τ has a periodic orbit of strips (curves) with pattern ν if and only if f_τ has a periodic orbit with pattern ν.

Conclusion

Every interval pattern (permutation) occurs as pattern in Ω.

Forcing in Ω

Definition

Let τ, ν be patterns in Ω. $\tau \Longrightarrow_{\Omega} \nu$ if and only if every map $F \in \mathcal{S}(\Omega)$ that has a periodic orbit of strips with pattern τ also has a periodic orbit of strips with pattern ν.

Main result

Theorem

Let τ and ν be patterns (in I and Ω). Then,

$$\tau \Longrightarrow_I \nu \quad \text{if and only if} \quad \tau \Longrightarrow_\Omega \nu.$$

Corollary

Every pattern of period n forces in Ω a pattern of period m for every $m \leq_{\mathrm{Sh}} n$.

Corollary

The Sharkovskii Theorem holds for every $F \in \mathcal{S}(\Omega)$.

Entropy

By using Bowen definition $h(F, I_\theta)$ is defined for every $I_\theta := \{\theta\} \times I$. Then, Bowen Formula gives

$$h(R) + h_{\text{fib}}(F) \geq h(F) \geq \max\{h(R), h_{\text{fib}}(F)\}$$

where

$$h_{\text{fib}}(F) = \sup_{\theta \in \mathbb{S}^1} h(F, I_\theta).$$

Since $h(R) = 0$, $\boxed{h(F) = h_{\text{fib}}(F).}$

In the particular case of the map F_τ, from the definitions and the fact that $F_\tau = (R, f_\tau)$ is uncoupled, we get

Lemma

$h(F_\tau, I_\theta) = h(f_\tau) \quad \forall\, \theta \in \mathbb{S}^1$. *Consequently,*

$$\boxed{h(F_\tau) = h_{\text{fib}}(F_\tau) = h(f_\tau).}$$

Horseshoes in $\mathcal{S}(\Omega)$

Definition

Given $F \in \mathcal{S}(\Omega)$ we define an *s-horseshoe for F* as a pair $(J, \mathcal{D})$ where J is a solid strip and $\mathcal{D}$ is a quasi-partition of J formed by s solid strips such that $F(D) \supset J$, $\forall\ D \in \mathcal{D}$.

Quasi-partition

$J = \bigcup_{D \in \mathcal{D}} D$, and the elements of $\mathcal{D}$ have pairwise disjoint interiors.

Horseshoes and entropy in $\mathcal{S}(\Omega)$

Following the same ideas as in the interval we get

Theorem

Assume that $F \in \mathcal{S}(\Omega)$ has an s-horseshoe. Then

$$h(F) \geq \log s.$$

Theorem

Assume that $F \in \mathcal{S}(\Omega)$ has a periodic orbit of strips with pattern τ. Then

$$h(F) \geq h(f_\tau) = h(F_\tau).$$

Consequences: Entropy of patterns

Definition

$$h(\tau) := \inf \left\{ h(F) \colon \begin{array}{l} F \in \mathcal{S}(\Omega) \text{ and } F \text{ has a periodic orbit} \\ \text{of strips with pattern } \tau \end{array} \right\}.$$

Corollary

$\tau \Longrightarrow_{\Omega} \nu$ implies $h(\tau) \geq h(\nu)$.

Proof.

From the last theorem, $h(\tau) = h(F_\tau)$.
Also, F_τ has a periodic orbit of strips with pattern ν. Hence, by definition,

$$h(F_\tau) \geq h(\nu).$$

□

Consequences: lower bounds of the entropy depending on the set of periods

Corollary

If F has a periodic orbit of strips of period $2^n q$ with $n \geq 0$ and $q \geq 1$ odd then, as in the interval case,

$$h(F) \geq \frac{\log \lambda_n}{2^k}$$

where $\lambda_1 = 1$ and, for each $q \geq 3$ odd, λ_q is the largest root of $x^q - 2x^{q-2} - 1$.

Proof of Main Theorem

- $\tau \Longrightarrow_{\Omega} \nu$ implies $\tau \Longrightarrow_{I} \nu$.
  Trivial: By definition, F_τ has a periodic orbit of strips with pattern ν. Then f_τ has a periodic orbit with pattern ν. Therefore, $\tau \Longrightarrow_{I} \nu$ by the Forcing Characterization Theorem.

- $\tau \Longrightarrow_{I} \nu$ implies $\tau \Longrightarrow_{\Omega} \nu$.

To prove this statement we need to introduce new tools

New tools: Markov Graph of f_τ

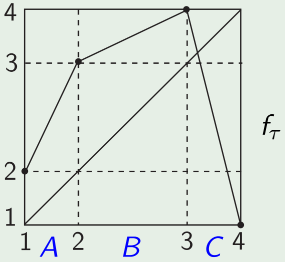
The closure of each connected component of $\langle P \rangle \setminus P$ is a *basic interval*.

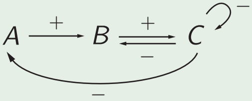
Definition (Signed Markov Graph (Combinatorial))

Vertices: Basic intervals

Signed arrows: $\begin{cases} I \xrightarrow{+} J & \text{iff } f_\tau(I) \supset J;\ f_\tau\big|_I \text{ increases} \\ I \xrightarrow{-} J & \text{iff } f_\tau(I) \supset J;\ f_\tau\big|_I \text{ decreases} \end{cases}$

Example





$$T = \begin{array}{c} \\ A \\ B \\ C \end{array}\!\!\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \end{array}$$

Theorem

$h(f_T) = \max\{0, \log \rho(T)\}$

where $\rho(T)$ denotes the spectral radius of T.

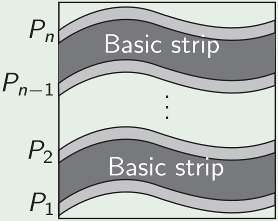
Signed Markov Graph for $F \in \mathcal{S}(\Omega)$ having a periodic orbit of strips: Basic strips

Definition

The closure of the strip between two consecutive strips is called a *basic strip*

Example

Assume that $P_1, P_2, \ldots, P_n$ is a periodic orbit of strips.



Signed Markov Graph for $F \in \mathcal{S}(\Omega)$ having a periodic orbit of strips: Arrows

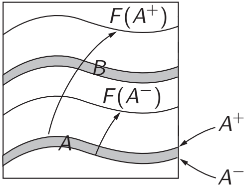
Definition

Given a strip A we set

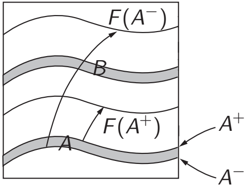
$$\textbf{Top of } A\textbf{:} \quad A^+ := \overline{\left\{(\theta, \max_I((\{\theta\} \times I) \cap A)) : \theta \in \mathbb{S}^1\right\}}$$

$$\textbf{Bottom of } A\textbf{:} \quad A^- := \overline{\left\{(\theta, \min_I((\{\theta\} \times I) \cap A)) : \theta \in \mathbb{S}^1\right\}}$$

$A \xrightarrow{+} B$:



$A \xrightarrow{-} B$:



Properties of signed arrows in $\mathcal{S}(\Omega)$

Lemma

Assume that $A \xrightarrow{\pm} B$. *Then,*

1. $B \subset T(A)$.
2. *If* $D \subset B$ *and* D *is a strip, then* $A \xrightarrow{\pm} D$.

Remark

If A is a pseudo-curve, then B is a pseudo-curve and $T(A) = B$.

Important remark

By continuity, the signed Markov Graph of f_τ is a subgraph of the signed Markov Graph of F with respect to a periodic orbit of strips with pattern τ.

We may assume that $\nu \neq \tau$.

By the Forcing Characterization Theorem, f_τ has periodic orbit $\{q_0, q_1, \cdots, q_{n-1}\}$ with pattern $\nu \neq \tau$ such that $q_0 = \min\{q_0, q_1, \cdots, q_{n-1}\}$.

Consider the loop in the Markov Graph of f_τ associated to q_0. That is,

$$\begin{array}{ccccccccc} I_0 & \xrightarrow{s_0} & I_1 & \xrightarrow{s_1} & \cdots\cdots & \xrightarrow{s_{n-2}} & I_{n-1} & \xrightarrow{s_{n-1}} & I_0 \\ \Cup & & \Cup & & & & \Cup & & \\ q_0 & & f_\tau(q_0) & & & & f_\tau^{n-1}(q_0) & & \end{array}$$

This loop also exists in the Markov Graph of F, replacing the interval I_i by the basic strip $\widetilde{I_i}$. Moreover, since $\nu \neq \tau$ and f_τ is τ-linear, the loop is non-repetitive.

A key lemma

Lemma

There exists a periodic orbit of strips $Q_0, Q_1, \dots, Q_{n-1}$ *of* F *such that* Q_0 *is the lower strip,* $F^n(Q_0) = Q_0$ *and*
$Q_0 \subset \widetilde{I}_0,\ F(Q_0) \subset \widetilde{I}_1,\ \dots,\ F^{n-1}(Q_0) \subset \widetilde{I}_{n-1}.$

Then, by the above lemma we have to see that $\{Q_0, Q_1, \dots, Q_{n-1}\}$ has period n and pattern ν.

- First we prove that $Q_0 < F_i(Q_0)$ for $i = 1, 2, \dots, n-1$ (in particular the period is n).
- Secondly we prove that $\{Q_0, Q_1, \dots, Q_{n-1}\}$ has pattern ν.

$\{Q_0, Q_1, \ldots, Q_{n-1}\}$ has pattern ν

We have:

$$\begin{array}{lcll}
q_0 & \sim & I_0 \longrightarrow I_1 \longrightarrow \ldots \longrightarrow I_{n-1} \longrightarrow I_0 & \sim \quad Q_0 \\
f_\tau(q_0) & \sim & I_1 \longrightarrow I_2 \longrightarrow \ldots \longrightarrow I_{n-1} \longrightarrow I_0 \longrightarrow I_1 & \sim \quad F(Q_0) \\
f_\tau^2(q_0) & \sim & I_2 \longrightarrow I_3 \longrightarrow \ldots \longrightarrow I_{n-1} \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 & \sim \quad F^2(Q_0) \\
 & & \vdots & \\
f_\tau^{n-1}(q_0) & \sim & I_{n-1} \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \ldots \longrightarrow I_{n-1} & \sim \quad F^{n-1}(Q_0)
\end{array}$$

where the symbol $\sim$ means "associated with".

The order of the points $f_\tau^i(q_1)$ induces an order on the shifts of the loop (with the usual lexicographical ordering) that induces the same order on the strips $F^i(Q_0)$. Thus, $\{q_0, q_1, \cdots, q_{n-1}\}$ and $\{Q_0, Q_1, \ldots, Q_{n-1}\}$ have the same pattern.