Representation of Markov chains

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Markov chain model

We consider $f : M \to M$ to be C^r for $r \geq 0$ and a small perturbation parameter $\varepsilon > 0$.
The Markov chain model is a family $\{p_\varepsilon(\,\cdot\,|x)\}$ of Borel probability measures.

- Every $p_\varepsilon(\,\cdot\,|x)$ is supported inside an ε-neighbourhood of $f(x)$.
- Random orbit: $\{x_j\}$ where each x_{j+1} has distribution $p_\varepsilon(\,\cdot\,|x_j)$.
- Jumps $x_j \mapsto f(x_j)$ and disperses with distribution $p_\varepsilon(\,\cdot\,|x_j)$.
- $x_j \mapsto p_\varepsilon(\,\cdot\,|x_j)$ continuous w.r.t. weak* topology in compact spaces $\Rightarrow$ existence of invariant measures:

$$\mu_\varepsilon(E) = \int p_\varepsilon(E|x)d\mu_\varepsilon(x)$$

for every Borel set $E \subset U$.

Iteration of random maps

We consider $f : M \to M$ to be C^r for $r \geq 0$ and a small perturbation parameter $\varepsilon > 0$. The random iteration of maps is given by

- Assuming the existence of a family of probability distributions $\{\nu_\varepsilon\}$ on the space of C^r-maps.
- Support of ν_ε is in a ε-neighbourhood of $f(x)$.
- Random orbit: $x_j = f_{\omega_j} \circ \cdots \circ f_{\omega_1}(x_0)$, where f_{ω_j} are i.i.d. random variables with distribution ν_ε.
- The random orbits generated by the random maps indeed give rise to a discrete time Markov chain.

For continuous maps invariant measures exists:

$$\mu_\varepsilon(E) = \int \mu_\varepsilon(f_\omega^{-1}(E))d\nu_\varepsilon(f_\omega)$$

for every Borel $E \subset U$.

Stochastic stability

Physical measures: μ is *physical* if for a set of x with positive Lebesgue measure

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n-1} \varphi(f^j(x)) = \int \varphi d\mu,$$

for every continuous function $\varphi : M \to \mathbb{R}$.

The randomly perturbed dynamics: supposing existence of a unique μ_ε for every small $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n-1} \varphi(f^j_\omega(x)) \to \int \varphi d\mu_\varepsilon$$

for almost every random orbit and every $\varphi : M \to \mathbb{R}$.

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Stochastic stability

A system (f, μ) is stochastically stable under the perturbation scheme $\{p_\varepsilon(\cdot\,|x)\}$ or $\{\nu_\varepsilon : \varepsilon > 0\}$ if

$$\lim_{\varepsilon \to 0} \int \varphi d\mu_\varepsilon = \int \varphi d\mu \quad \text{for every continuous } \varphi : U \to \mathbb{R}.$$

- Several contributions proving stochastic stability of different systems: Sinai, Kifer, L.-S. Young, Keller, Araújo, Alves, Viana, etc.
- Arguments: assume existence of probability in the space of maps, control of distortion, hyperbolic times, thermodynamics formalism, etc.
- Questions: dependence of the probability distributions of the Markov chains, relation with structural properties, shadowing, etc.

Representation of Markov chains

The sequence of random maps is a representation of the Markov chain if for any Borel U

$$p_\varepsilon(U|x) = \nu_\varepsilon(\{f_\omega : f_\omega(x) \in U\}).$$

- Some contributions: Blumenthal and Corso '70, Kifer '86, Quas '91, Araújo '00, Benedicks and Viana '06, ...
- We tackled the general case in terms of a transportation problem.

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Representation of Markov chains

Theorem (Jost, Kell, R.)

Let M and N be compact Riemannian C^r-manifolds without boundary, and let m be the normalised volume measure on N. Let $\{p_\varepsilon(\cdot|x)\}$ for $x \in M$ be a continuous family of probability measures on N such that each $p_\varepsilon(\cdot|x)$ is absolutely continuous with respect to m, has positive density and convex support. Suppose that there is a C^r-diffeomorphism $f: M \to N$, for $r \geq 1$, such that for each x, the support of $p_\varepsilon(\cdot|x)$ is contained in a small neighbourhood of $f(x)$. Then $\{p_\varepsilon(\cdot|x)\}$ can be represented by a family $(f_\omega)_{\omega\in\Omega}$ of C^r-random diffeomorphisms.

Theorems (Jost, Kell, R.)

Measurable (continuous) abs cont probability measures can be represented by measurable (continuous) random maps.

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On optimal transport

- Basic problem (G. Monge, 1781): moving a distribution like a pile of sand from a place to another at minimum cost.
- In probability terms: M, N are probability spaces, $\mu \in \mathcal{P}(M)$, $\nu \in \mathcal{P}(N)$, we seek a coupling connecting the measures.

  Example: a transport map (measurable) $T : M \to N$ s.t. $\forall$ Borel $E \subset N$, one has $\mu(T^{-1}(E)) = \nu(E)$ (deterministic coupling).

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- Alternatively: weak solutions (Kantorovich): $\gamma \in \mathcal{P}(M \times N)$, with $\pi_{\mathcal{P}(M)*}\gamma = \mu$ and $\pi_{\mathcal{P}(N)*}\gamma = \nu$,

  Minimisation problem:

$$C(\mu, \nu) = \inf_{\gamma \in \mathcal{P}(M \times N)} \int_{M \times N} c(x, y) d\gamma(x, y),$$

$c : M \times N \to [0, +\infty]$.

Using optimal transport

Monge problem: find deterministic optimal couplings minimising

$$C(\mu,\nu) = \inf_{\gamma \in \mathcal{P}(M\times N)} \int_{M\times N} c(x,y)d\gamma(x,y),$$

$c : M \times N \to [0,+\infty]$.

1. Translate the problem in terms of Monge problem.
   - Existence and stability results.
   - Regularity theory on $\mathbb{R}^n$ (Loeper '09).

Lifting measures

2. Measures on bundles.

   The map $x \mapsto p_\varepsilon(\cdot|x) \in \mathcal{P}(N)$ implicitly lifts locally to $x \mapsto q_\varepsilon(\cdot|x) \in \mathcal{P}(T_{f(x)}N)$, where $f : M \to N$ is the centre of mass, via exponential map

$$(exp_{f(x)}^{-1})_* p_\varepsilon(\cdot|x) = q_\varepsilon(\cdot|x).$$

   - For parallelizable tangent bundles $TN \cong N \times \mathbb{R}^n$ we consider $x \mapsto q_{\varepsilon,x}$ as a pair

$$x \mapsto (f(x), q_{\varepsilon,x}) \in N \times \mathbb{R}^n.$$

   then

$$f_\omega(x) = \exp_{f(x)}(X_\omega(x)).$$

Finally...

2. Measures on bundles
   - General case: lift the measures to the tangent bundles and construct fiber bundles using isometric embeddings.
3. Perturbation in the space of diffeomorphisms.
   - $\mathrm{Diff}^r(M, N)$ of diffeomorphisms is open in $C^r(M, N)$, for $r \geq 1$.
   - Using regularity theory to control the distributions on the fiber bundles and projections lead to the result.

Thanks for your attention!

Reference: Pre-print [arXiv:1207.5003]