# Chain Transitivity and Variations of the Shadowing Property

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# Outline

1. Preliminaries

2. Shadowing and Chain Transitiivity

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1. Preliminaries
   - Definitions
   - Variations on Shadowing
2. Shadowing and Chain Transitiivity

# Basic Terminology

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- An *orbit* for $f$ is a sequence of the form $\langle f^i(x)\rangle_{i\in\mathbb{N}}$ for some $x \in X$.
- For $\delta > 0$, a $\delta$-pseudo-orbit is a sequence $\langle z_i\rangle_{i\in\mathbb{N}}$ in $X$ satisfying $d(z_{i+1}, f(z_i)) < \delta$ for $i \in \mathbb{N}$.

# Shadowing

- A map $f$ has shadowing provided that for all $\epsilon > 0$ there exists a $\delta > 0$ such that for every $\delta$-pseudo-orbit $\langle z_i \rangle$ there exists $x \in X$ such that $d(z_i, f^i(x)) < \epsilon$ for all $i \in \mathbb{N}$.

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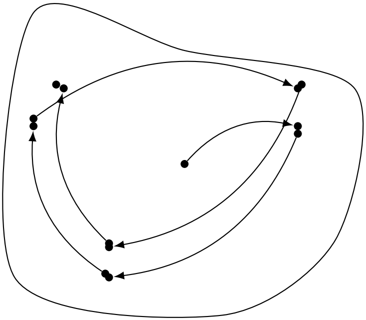
- The point $x$ is said to $\epsilon$-shadow the pseudo-orbit $\langle z_i \rangle$.

# Shadowing



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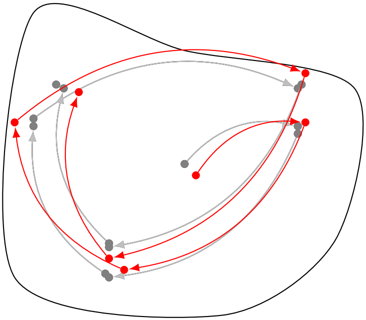

# Chain Transitivity

- A $\delta$-chain from $x$ to $y$ is a sequence $x = z_0, z_1, \ldots z_n = y$ in $X$ which satisfies $d(z_{i+1}, f(z_i)) < \delta$ for $i < n$.

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- A map $f$ is *chain transitive* provided that for all $\delta > 0$ and all $x, y \in X$, there exists a $\delta$-chain from $x$ to $y$.

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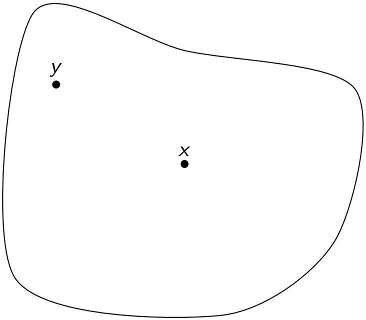

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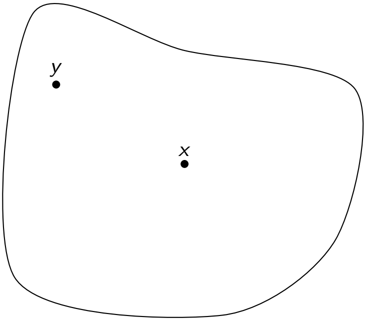

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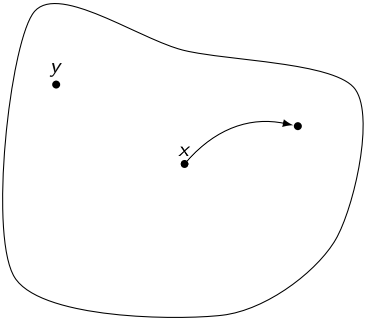

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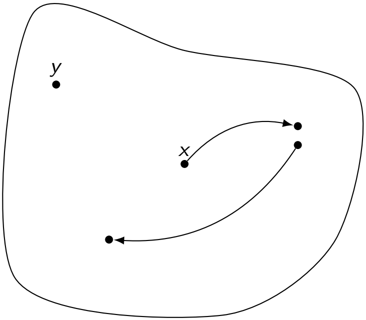

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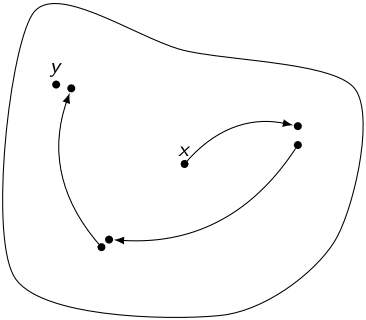

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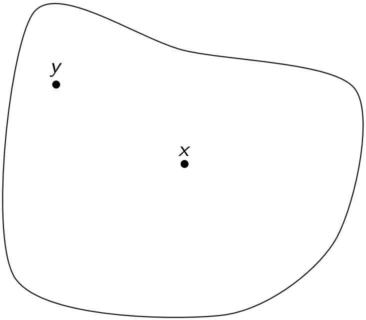

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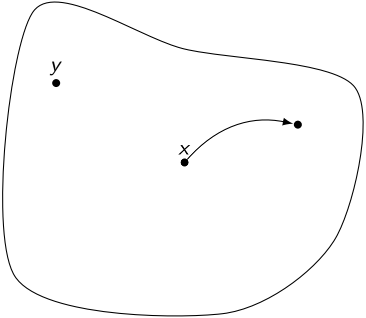

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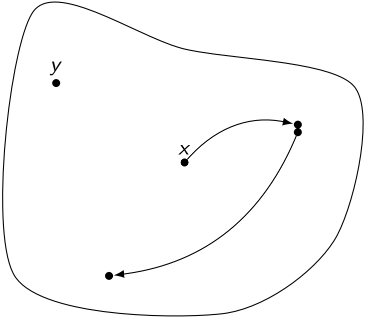

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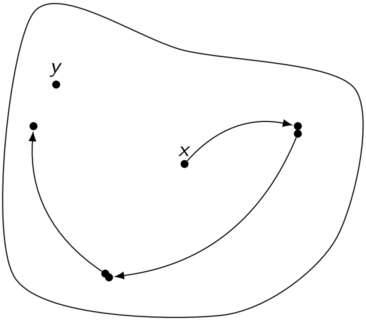

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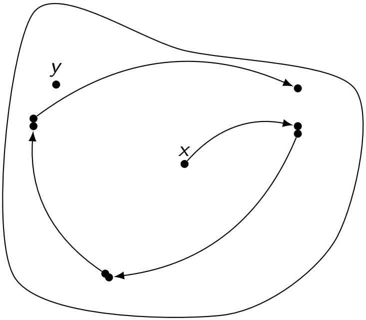

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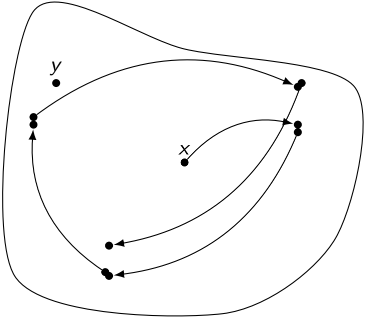

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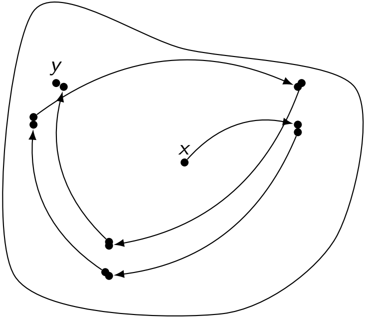

# Terminology

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- A point $x \in X$ $\epsilon$*-shadows* $\langle z_i \rangle$ *on* $B$ provided that $B \subseteq \{i \in \mathbb{N} : d(z_i, f^i(x)) < \epsilon\}$.

# $(\mathcal{F}, \mathcal{G})$-shadowing

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- For families $\mathcal{F}$ and $\mathcal{G}$, a map $f$ has $(\mathcal{F}, \mathcal{G})$-shadowing provided that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $\langle z_i \rangle$ is a $\delta$-pseudo-orbit on a set $A \in \mathcal{F}$ then there exists a point $x \in X$ which $\epsilon$-shadows $\langle z_i \rangle$ on a set $B \in \mathcal{G}$.

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**Theorem [BMR]**

Suppose that $\mathcal{F} \supseteq \mathcal{F}'$ and that $\mathcal{G} \subseteq \mathcal{G}'$. Then every space with $(\mathcal{F}, \mathcal{G})$-shadowing has $(\mathcal{F}', \mathcal{G}')$-shadowing.

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- Let $\mathcal{T}$ denote the family of thick subsets of $\mathbb{N}$, i.e. those sets $A \subseteq \mathbb{N}$ containing arbitrarily long intervals.
- Let $\mathcal{D}$ denote the family of subsets of $\mathbb{N}$ with lower density equal to 1.

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- Several other shadowing subtypes fit this framework (though not all.)

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1. Preliminaries

2. Shadowing and Chain Transitiivity
   - Lemmas
   - Theorem

# Chain transitivity

**Lemma [Richeson, Wiseman 2008]**

Let $f : X \to X$ be chain transitive and let $\delta > 0$. Then there exists $k_\delta \in \mathbb{N}$ such that for any $x \in X$, $k_\delta$ is te greatest common denominator of the lengths of $\delta$-chains from $x$ to $x$.

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- Define the relation $\sim_\delta$ on $x$ by $x \sim_\delta y$ provided that there is a $\delta$-chain from $x$ to $y$ of length a multiple of $k_\delta$.

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- Define the relation $\sim_\delta$ on $x$ by $x \sim_\delta y$ provided that there is a $\delta$-chain from $x$ to $y$ of length a multiple of $k_\delta$.
- The are precisely $k_\delta$ many equivalence classes of $\sim_\delta$ which are clopen and are permuted cyclicly by $f$.

# Chain Lengths

**Lemma [BMR]**

Let $f : X \to X$ be chain transitive. For each $\delta > 0$ there exists $M \in \mathbb{N}$ such that for any $m \geq M$, and any $x, y \in X$ with $x \sim_\delta y$, there is a $\delta$-chain from $x$ to $y$ of length exactly $mk_\delta$.

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- This is a straightforward application of the fact that $\delta$-chains can be concatenated and Schur's Theorem.

# Main Theorem

## Theorem [BMR]

For a chain transitive dynamical system, the following are equivalent:

1. shadowing, i.e. $(\{\mathbb{N}\},\{\mathbb{N}\})$-shadowing,
2. $(\mathcal{T},\mathcal{T})$-shadowing,
3. thick shadowing, i.e. $(\mathcal{D},\mathcal{T})$-shadowing, and
4. $(\{\mathbb{N}\},\mathcal{T})$-shadowing.

# Sketch of Proof

- First, note that $\{\mathbb{N}\} \subset \mathcal{D} \subset \mathcal{T}$ so as an application of the earlier theorem, (2) implies (3) and (3) implies (4).

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- So, we need only establish that (1) implies (2) and (4) implies (1).

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- It is sufficient to show that for any $\epsilon > 0$ we can find $\delta > 0$ such that any $\delta$-chain in $X$ can be $\epsilon$-shadowed.

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- Let $\epsilon > 0$ and let $\delta > 0$ be given by $(\{\mathbb{N}\}, \mathcal{T})$-shadowing.
- Fix a $\delta$-chain $z_0, z_1, \ldots z_n$. Since $f$ is chain transitive we can find a $\delta$-chain $z_n, y_1, y_2, \ldots y_m, z_0$ from $z_n$ to $z_0$.

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- Then $z_0, z_1, \ldots z_n, y_1, \ldots y_m, z_0, \ldots z_n, y_1, \ldots y_m, \ldots$ is a $\delta$-pseudo-orbit.

# (4) implies (1)

- Let $x \in X$ shadow $z_0, z_1, \dots z_n, y_1, \dots y_m, z_0, \dots z_n, y_1, \dots y_m, \dots$ on a set $A \in \mathcal{T}$.

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- In particular, one long enough to gaurantee that $x$ shadows the pseudo-orbit on some segment coinciding with $z_0, z_1, \ldots z_n$.
- Then, the approriate iterate of $x$ shadows the $\delta$-chain $z_0, z_1, \ldots z_n$.

# (1) implies (2)

- We must show that for any $\epsilon > 0$ we can find $\delta > 0$ such that for any $\delta$-pseudo-orbit $\langle z_i \rangle$ on a set $A \in \mathcal{T}$, there is an $x \in X$ that $\epsilon$-shadows it on a set $B \in \mathcal{T}$.

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- Our strategy is to construct a proper $\delta$-pseudo-orbit $\langle q_i \rangle$ which agrees with $\langle z_i \rangle$ on a thick set and then find a point $x$ that shadows this modified pseudo-orbit.

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- The point $x$ will then shadow the original pseudo-orbit on a thick set as desired.

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- Define for each $i \in \mathbb{N}$ the number $m(i) \in \mathbb{Z}_K$ to be the element of $\mathbb{Z}_K$ such that $z_i \in X_{i+m(i)}$.

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- Define for each $i \in \mathbb{N}$ the number $m(i) \in \mathbb{Z}_K$ to be the element of $\mathbb{Z}_K$ such that $z_i \in X_{i+m(i)}$.
- If $d(z_{i+1}, f(z_i)) < \delta$ it follows that $m(i) = m(i+1)$.
- Let $A = \{i \in \mathbb{N} : m(i) = m(i+1)\}$, and notice that this contains $T$ and is hence thick.

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- By previous lemma, fix $M \in \mathbb{N}$ such that for all $m \geq M$, and any $x, y \in X_0$ there is a $\delta$-chain of length $mK$ from $x$ to $y$.

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- We can then leverage the thickness to replace segments of $\langle z_i \rangle$ for which $m(i) \neq 0$ (and some parts where $m(i) = 0$ as well) with $\delta$-chains of lengths $mK$.

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- We can then leverage the thickness to replace segments of $\langle z_i \rangle$ for which $m(i) \neq 0$ (and some parts where $m(i) = 0$ as well) with $\delta$-chains of lengths $mK$.
- In particular, do this in such a way that we retain subintervals of $A_0$ of arbitrary length.
- The modified sequence $\langle q_i \rangle$ is now a proper $\delta$-pseudo-orbit and agrees with $\langle z_i \rangle$ on a thick set.

# Thank you

Thank you!