This lecture is dedicated to the memory of Robert Ellis, who died in December 2013. Bob was the leading researcher in the abstract theory of topological dynamics and much of what I will say today has been inspired by his work.

The theme of my talk is the "reduction" of a flow to a distal or equicontinuous flow, as explained below.

Let (X,T) be a flow, that is the jointly continuous action of the group T on the compact Hausdorff space X, $(t,x) \to tx$.

If (X,T) and (Y,T) are flows, a homomorphism is a continuous surjective equivariant map $\pi : X \to Y$, $\pi(tx) = t\pi(x)$

Recall that (X,T) is said to be equicontinuous if the maps defined by the action of T form an equicontinuous family. (If X is a metric space this is the "$\varepsilon\ \delta$" condition: if $\varepsilon > 0$ there is a $\delta > 0$ such that if $d(x,x') < \delta$ then $d(tx,tx') < \varepsilon$ for all $t \in T$.)

The main "obstruction" to a flow being equicontinuous is the existence of regionally proximal pairs.

We say that x and y are regionally proximal, $(x,y) \in RP$ if there are nets $\{x_n\}$ and $\{y_n\}$ in X and $\{t_n\}$ in T such that $(x_n,y_n) \to (x,y)$ and $t_n(x_n,y_n) \to \Delta$. RP is symmetric, reflexive, closed, and T invariant. It is not in general transitive, although as we'll see there are some important cases where it is in fact transitive (so an equivalence relation).

It is easy to see that (X,T) is equicontinuous if and only if $RP = \Delta$.

If $x_n = x$ and $y_n = y$ we obtain the proximal relation P. By definition the flow (X,T) is distal if $P = \Delta$. Clearly $P \subset RP$ and an equicontinuous flow is distal.

Regional proximality plays an important role in work of Host and Kra on the structure of nil flows.

A flow (X,T) has a maximal distal and a maximal equicontinuous factor, or what is the same thing, there are closed invariant equivalence relations (S_{eq} and S_d such that the factor flows X/S_{eq} and X/S$_d$ are, respectively, equicontinuous and distal.

Clearly to obtain S_d, it is necessary to collapse the proximal pairs, and for S_{eq} the regionally proximal pairs.

There are a couple of problems here. In general, neither P nor RP is an equivalence relation (although as we'll see later in fact RP is in some important cases). And even when factors out the closed invariant equivalence relation containing P or RP it isn't clear that the resulting quotient flow is distal or, respectively, equicontinuous.

This question is settled by our two "folk theorems"

Theorem. *Let (X,T) and (Y,T) be flows, and let $\pi : X \to Y$ be a homomorphism. Suppose $\pi(P_X) = \Delta$. Then (Y,T) is distal.*

Theorem. *Let (X,T) and (Y,T) be flows, and let $\pi : X \to Y$ be a homomorphism. Suppose $\pi(RP_X) = \Delta$. Then (Y,T) is equicontinuous.*

While these theorems are "well known" I have not seen them explicitly stated in the literature, nor are the proofs entirely trivial.

Note that if it were the case that whenever $\pi : X \to Y$ is a flow homomorphism $\pi(P_X) = P_Y$ and/or $\pi(RP_X) = RP_Y$ then in fact the proofs of our folk theorems would be immediate. But there are easy counterexamples.

As immediate corollaries we obtain the classic theorems of Gottschalk and Ellis: the equicontinuous structure relation S_{eq} is the closed T invariant equivalence relation generated by RP, as well as the corresponding theorem on the distal structure relation

Another (non-obvious) corollary is that a factor of an equicontinuous flow is equicontinuous, and the same for distal.

Regarding the proofs: The proof for P, so (Y,T) is distal depends on properties of the enveloping semigroup (one of Ellis' contributions).

The enveloping semigroup $E(X,T)$ of the flow (X,T) is the closure of T in X^X (the set of all maps from X to itself, provided with the product topology). Thus if $\{t_n\}$ is a net in T, $t_n \to p$ if $t_n x \to px$ for all $x \in X$.

The enveloping semigroup has a rich algebraic structure, which is correlated with dynamical properties of the flow. It is particularly useful in the study of proximality: $(x, y) \in P$ if and only if there is an $\eta \in E(X,T)$ such that $\eta x = \eta y$. The set of such η contains an idempotent u ($u^2 = u$). Note that if $x \in X$, $(ux, x) \in P$, since $u(ux) = ux$.

So suppose $\pi(P_X) = \Delta_Y$. Let $(y, y') \in P_Y$ and let $\pi(x, x') = (y, y')$. Let u be an idempotent such that $uy = uy'$. Then $y = \pi(x) = \pi(ux) = uy = uy' = \pi(ux') = \pi(x') = y'$. That is $P_Y = \Delta$ so (Y,T) is distal.

Everyone should know the enveloping semigroup!

On to equicontinuity:

The classical proof of the equicontinuity theorem depends on a deep theorem of Ellis, namely that a flow is equicontinuous if and only if its enveloping semigroup is a group of homeomorphisms.

I was searching for a more direct proof. Thereby hangs a tale, in fact two tales.

That is, I found two proofs, each of which illuminates certain aspects. The first involves the "star" operation, which is a map using the enveloping semigroup from X to 2^X (the space of closed subsets of X with the Hausdorff topology. If $p \in E(X)$ and $t_n \to p$ then $x' \in p^* x$ if there are $x_n \to x$ such that $t_n x_n \to x'$.

Using the star operation, there is a proof which imitates the "distal" proof. This gives rise to the vague idea that regional proximality is just proximality on "some other" space.

But in fact there is even a short direct proof:

Let $y_n \to y$, $t_n(y, y_n) \to (y', y")$. For equicontinuity of (Y,T) it's sufficient to show that $y' = y"$.

Let $\pi(x_n) = y_n$, $x_n \to x$ so $\pi(x) = y$. Let $t_n(x, x_n) \to (x', x")$ so $\pi(x', x") = (y', y")$. But $(x', x") \in RP$ so $\pi(x') = \pi(x")$. That is $y' = y"$.

Now we restrict our attention to minimal flows.

In this case, it is a somewhat surprising fact the "frequently" RP is an equivalence relation, and therefore coincides with S_{eq} the equicontinuous structure relation. (This is the case when the acting group T is abelian, and in fact more generally when the flow has an invariant measure.) There have been a number of proofs, under various hypotheses (all including an acting abelian group), starting with Ellis and Keynes in 1971.

There are examples of minimal actions of the free group on two generators for which RP is not an equivalence relation.

We remark that in contrast P is "rarely" an equivalence relation, even for minimal $\mathbb{Z}$ actions.

In a 1968 paper, Veech introduced a relation (we call it V) defined by $(x, y) \in V$ if there is a net $\{t_n\}$ in T and a point $z \in X$ such that $t_n x \to z$ and $t_n^{-1} z \to y$. It is easy to see that $V \subset RP$ ($(x, t_n^{-1} z) \to (x, y)$ and $t_n(x, t_n^{-1} z) \to (z, z)$.)

In fact, Veech proved that if (X, T) is minimal with X metric and T abelian, then $V = S_{eq}$ (so $V = RP$).

Thus Veech proved (but didn't state!) that in this case RP is an equivalence relation (and of course he showed a lot more).

As Hillel Furstenberg pointed out, it's not even clear that V is symmetric.

Veech's proof, which was in the spirit of harmonic analysis, was lengthy and complicated. There was no mention of regional proximality. A challenge I set myself was to show that $RP \subset V$ (so $RP = V$).

I first was able to show that $P \subset V$, if (X, T) is minimal, with X metric (but with no restriction on the acting group T).

Let (X, T) be minimal with X a compact metric space.

Let $(x, y) \in P$, and let V_j be a decreasing sequence of neighborhoods of y whose intersection is $\{y\}$.

We will define W_j a decreasing sequence of open sets.

Let W_1 be a non-empty open set. Since $(x, y) \in P$ there is a $t_1 \in T$ such that $t_1 x$ and $t_1 y$ are in W_1. Therefore $W_1 \cap t_1 V_1 \neq \emptyset$. Let W_2 be non-empty open with $\overline{W}_2 \subset W_1 \cap t_1 V_1$ and let $t_2 \in T$ with $t_2 x$ and $t_2 y$ in W_2. Then $W_2 \cap t_2 V_2 \neq \emptyset$.

Inductively, choose W_{j+1} non-empty open with $\overline{W}_{j+1} \subset W_j \cap t_j V_j$ and let t_{j+1} in T with $t_{j+1} x$ and $t_{j+1} y$ in W_{j+1}. We can suppose that the diameters of the W_j approach 0 so there is a $z \in X$ with $\cap W_j = \{z\}$.

Then we have $t_j x \in W_j$ so $t_j x \to z$ and $z \in t_j V_j$. Then $t_j^{-1} z \in V_j$ so $t_j^{-1} z \to y$. That is $(x, y) \in V$ the Veech relation.

The proof that $RP \subset V$ depends on still another characterization of RP

This is in terms of a subset RP^c of RP, $RP^c = \{(x, y)|$ there are nets $\{y_i\}$ in X and $\{t_i\}$ in T such that $y_i \to y$ and $t_i(x, y_i) \to (z, z)$ for some $z \in X$. That is RP^c consists of those $(x, y) \in RP$ for which one of the nets can be held constant. It turns out that often for minimal flows $RP = RP^c$. This is the case when the minimal flow has an invariant measure (McMahon) and also when the minimal flow satisfies a certain "Ellis group" condition (important recent work of Dave Ellis, Bob's son).

Both of these include the case that the acting group T is abelian, as well as (X, T) distal minimal.

In this case a slight modification of the above proof (that $P \subset V$) works. The details are in a joint paper with Gernot Greschonig and Anima Nagar., as well as the forthcoming paper of Dave Ellis.