Title

Weak product recurrence and related properties

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Recurrence

1. X - compact,
2. $f : X \to X$ - continuous
3. $x \in X$ is recurrent if $x \in \omega(x, f)$.
   - or in other words, $N(x, U, f) \neq \emptyset$ for any neighborhoods U of x,
   - where $N(x, U, f) = \{i > 0 \ : \ f^i(x) \in U\}$.
4. $x \in X$ is uniformly recurrent (or minimal) if it is recurrent and $\omega(x, f)$ is a minimal set.
   - or equivalently $N(x, U, f)$ is syndetic (has bounded gaps between its elements, i.e. any sufficiently long block of consecutive integers intersects it).

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Product recurrence

1. $x \in X$ is (uniformly) recurrent if $x \in \omega(x, f)$ (and it is a minimal set).
2. $x \in X$ is product recurrent if
   1. given any recurrent point y in any dynamical system g
   2. and any neighborhoods U of x and V of y,
   3. $N(x, U, f) \cap N(y, V, g) \neq \emptyset$.

   where $N(x, U, f) = \{i > 0 \ : \ f^i(x) \in U\}$.
3. $x, z \in X$ are proximal if $\liminf_{n\to\infty} d(f^n(x), f^n(z)) = 0$
4. x is distal if it is not proximal to any point in its orbit closure other than itself.

Theorem (Furstenberg)

A point x is product recurrent if and only if it is (uniformly recurrent) distal point.

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Weak product recurrence

1. $x \in X$ is **weakly** product recurrent if
   1. given any **uniformly** recurrent (=almost periodic) point y in any dynamical system g
   2. and any neighborhoods U of x and V of y,
   3. $N(x, U, f) \cap N(y, V, g) \neq \emptyset$.

Question

,,Another question (even for $\mathbb{Z}$ or $\mathbb{N}$ actions): If (x, y) is recurrent for all almost periodic points y, is x necessarily a distal point?"
[J. Auslander and H. Furstenberg, *Product recurrence and distal points*, Trans. Amer. Math. Soc., **343** (1994) 221–232.]

2. It was first by Haddad and Ott that product recurrence and weak product recurrence are not equivalent (Answer NO to the above).

[*Recurrence in pairs*, Ergod. Th. & Dynam. Sys. **28** (2008) 1135–1143]

Haddad and Ott example

Theorem

A point $x \in X$ is weakly product recurrent if it has the following property:

- *for every neighborhood V of x there exists n such that if $S \subset \mathbb{N}$ is any finite set satisfying $|s - t| > n$ for all distinct $s, t \in S$, then there exists $l \in \mathbb{N}$ such that $l + s \in N(x, V, f)$ for every $s \in S$.*

1. the above conditions are satisfied by many points/systems (e.g. point with dense orbit in full shift on 2 symbols)
2. dynamical system satisfying above must be at least mixing
3. dynamical system satisfying above cannot be minimal

Disjointness

1. We a closed set $\emptyset \neq J \subset X \times Y$ is a joining of (X, f) and (Y, g) if it is invariant (for the product map $f \times g$) and its projections on first and second coordinate are X and Y respectively.
2. If $X \times Y$ is the only joining of f and g then we say that they are disjoint.

Question

How to characterize systems disjoint from any distal or minimal system?
[H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory, **1** (1967), 1–49]

Theorem (Petersen, 1970)

A system is disjoint with every distal system iff it is weakly mixing and minimal.

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Disjointness and product recurrence

1. Only partial answers are known when a system is disjoint with all minimal systems.

Theorem (Furstenberg, 1967)

If f is weakly mixing with dense periodic points then it is disjoint from every minimal systems.

Theorem (Huang & Ye; Oprocha)

If (X, f) is disjoint from every minimal system then every transitive point in (X, f) is weakly product recurrent.

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Disjointness and product recurrence (cont.)

Remark

The class of weak product recurrent points is much wider than can detected by disjointness theorems, e.g.

- If $([0,1], f)$ is mixing and (S^1, R) is irrational rotation then for any $z \in S^1$ there is a residual set in $[0,1] \times \{z\} \subset (S^1, R)$ in dynamical system $([0,1] \times S^1, f \times R)$ consisting of weakly product recurrent points.
- But $([0,1] \times S^1, f \times R)$ is not disjoint with (S^1, R).

Product recurrence in terms of Furstenberg families (Dong, Shao, Ye)

1. $\mathcal{F}$ - upward hereditary set of subsets of $\mathbb{N}$ = Furstenberg family
2. $x \in X$ is $\mathcal{F}$-recurrent if $N(x, U, f) \in \mathcal{F}$ for any open neighborhood U of x,
3. recurrence = $\mathcal{F}_{inf}$-recurrence ($\mathcal{F}_{inf}$ = infinite subsets of $\mathbb{N}$)
4. $x \in X$ is $\mathcal{F}$-product recurrent ($\mathcal{F}$-PR for short) if for any dynamical system (Y, g) and any $\mathcal{F}$-recurrent point $y \in Y$ the pair (x, y) is recurrent for $(X \times Y, f \times g)$.
5. $\mathcal{F}$-PR$_0$ = $\mathcal{F}$-PR but only with (Y, g) of topological entropy zero.
6. $\mathcal{F}_{inf}$-PR = product recurrence (as introduced by Furstenberg)
7. $\mathcal{F}_s$-PR = weak product recurrence (where $\mathcal{F}_s$ is the family of syndetic sets)

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Further results on product recurrence

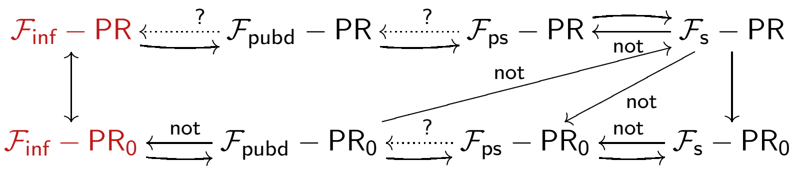


Figure: Product recurrence and product recurrence with zero entropy systems (Dong, Shao & Ye)

- $\mathcal{F}_{ps}$ = piecewise syndetic, i.e. intersections of syndetic and thick set
- $\mathcal{F}_{pubd}$ = sets with positive upper Banach denisty

$$0 < D(A) = \limsup_{n \to \infty} \frac{1}{n} \sup_{i \geq 0} \#(A \cap [i, i+n))$$

Results on PR obtained by results on disjointness

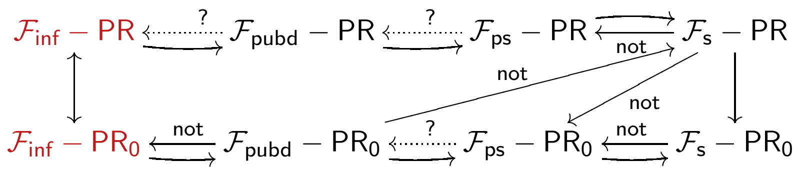
1. If (X, f) is a minimal flow (i.e. homeomorphism) such that any of its invariant measures is a K-measure, then it is disjoint from any transitive zero entropy E-system.
   - If (X, f) is a strictly ergodic flow with its unique invariant measure being a K-measure, then every point $x \in X$ is $\mathcal{F}_{\mathrm{pubd}}$-$\mathrm{PR}_0$.
   - But it has positive topological entropy, so also asymptotic pairs...
   - So there are points in X which are not recurrent in pair with minimal points.
   - Hence we have an example $\mathcal{F}_{pubd} - \mathrm{PR}_0 \not\Longrightarrow \mathcal{F}_s - \mathrm{PR}$.

W. Huang, K. K. Park, and X. Ye, *Topological disjointness from entropy zero systems*, Bull. Soc. Math. France **135** (2007), no. 2, 259–282.

2. If x is $\mathcal{F}_{\mathrm{ps}}$-$\mathrm{PR}_0$ then it is a minimal point.
   - Hence we have an example $\mathcal{F}_s - \mathrm{PR} \not\Longrightarrow \mathcal{F}_{ps} - \mathrm{PR}_0$.

P. Dong, S. Shao, and X. Ye, *Product recurrent properties, disjointness and weak disjointness*, Israel J. Math. **188** (2010), 463–507.

Further results on product recurrence (cont.)



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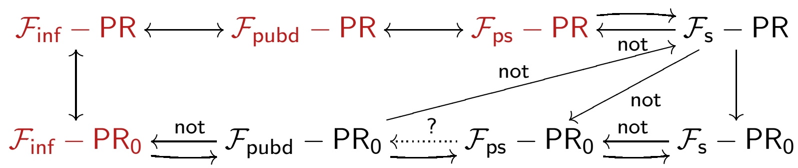

Figure: Product recurrence and product recurrence with zero entropy systems (Dong, Shao & Ye) + work of Oprocha and G.H. Zhang

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Results with Guo Hua Zhang

Theorem

If x is $\mathcal{F}_{ps}$-PR then it is distal.

Theorem

The following statements are equivalent:

1. *x is distal,*
2. *(x, y) is recurrent for any recurrent point y of any system (Y, g),*
3. *(x, y) is $\mathcal{F}_{pubd}$-recurrent for any $\mathcal{F}_{pubd}$-recurrent point y of any system (Y, g),*
4. *(x, y) is $\mathcal{F}_{ps}$-recurrent for any $\mathcal{F}_{ps}$-recurrent point y of any system (Y, g),*
5. *(x, y) is minimal for any minimal point y of any system (Y, g).*

Open problems

1. $\mathcal{F}_{ps} - \text{PR}_0 \Longrightarrow \mathcal{F}_{pubd} - \text{PR}_0$?
2. $\mathcal{F}_s - \text{PR} + \text{minimal} \Longrightarrow$ distal?