Maximum topological entropy of permutations and cycles

Deborah King

University of Melbourne

Australia

A review of results including joint work with John Strantzen, Lluis Alseda, David Juher.

A finite, fully invariant set of a continuous map of a compact interval to itself induces a permutation in a natural way. If the invariant set is a periodic orbit the permutation is cyclic. We can calculate the topological entropy of any permutation θ, and it is well known that this gives a lower bound for the topological entropy of any continuous self map of the interval which exhibits a permutation of type θ.

In the paper *Combinatorial Patterns for Maps of the Interval*, Misiurewicz and Nitecki asked, which permutations and cycles achieve the maximum topological entropy amongst all permutations or cycles of the same cardinality?

I first learned of this problem in the mid 90's, when Andrew Coppel visited LaTrobe University. At that time, the answer was known in some cases, but the problem is still not completely solved.

In this talk I will review the known results, and discuss the final stages of this classification problem.

1. Preliminaries and Notation

1. The map f is a continuous map of a closed, bounded interval I into itself.

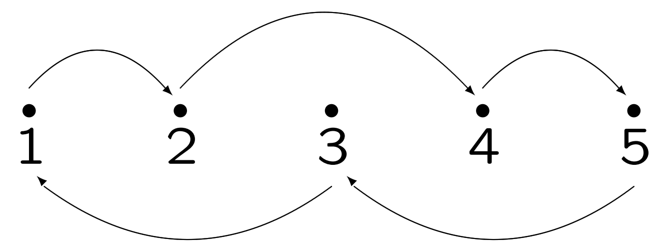
2. P is a finite, fully invariant set of f, (that is, $f(P) = P$) labelled $p_1 < p_2 < \cdots < p_n$.

3. The action of f on P induces a bijection

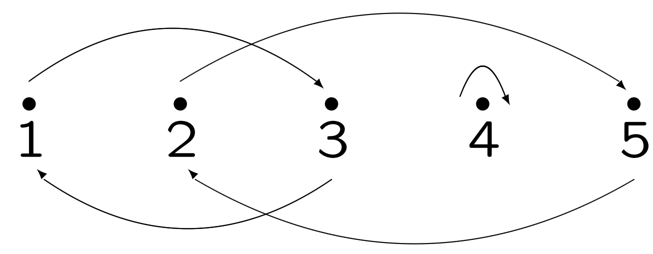
$$\theta\colon \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\} : \theta(i) = j \quad \Longleftrightarrow \quad f(p_i) = p_j.$$

The permutation θ is called the *type* of P. If P is a periodic orbit then θ is a cyclic permutation, otherwise P is a union of periodic orbits and θ is a non-cyclic permutation.

So here are two 5-permutations of different type, the first is cyclic, the second is not.



$\theta(1) = 2,\ \theta(2) = 4,\ \theta(4) = 5,\ \theta(5) = 3,\ \theta(3) = 1.$



$\theta(1) = 3,\ \theta(3) = 1,\ \theta(2) = 5,\ \theta(5) = 2,\ \theta(4) = 4.$

How can we compare the complexity (in terms of the dynamics) of permutations of different type?

4. The topological entropy of a permutation θ, which will be denoted by $h(\theta)$, is defined as follows:

$$h(\theta) := \inf\{h(f)\colon\ f \text{ has an invariant set of type } \theta\},$$

where the topological entropy $h(f)$ of a map f is a topological invariant which measures the dynamical complexity of f.

Typically, computing the entropy of a map is difficult. However, the computation of the entropy of a permutation can be easily done by using the following algebraic tools:

5. If P (with $|P| = n$) is a finite, fully invariant set for f of type θ, then there is a unique map $f_\theta\colon [1, n] \longrightarrow [1, n]$ which satisfies

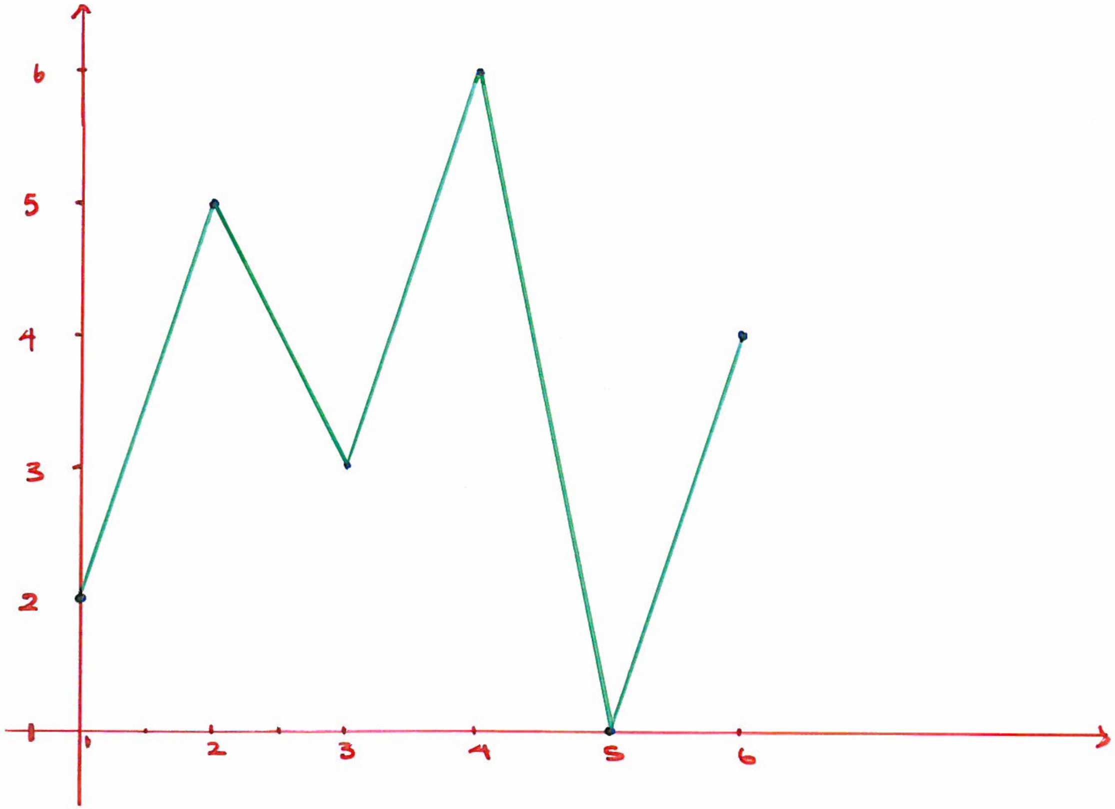
   (i) $f_\theta(i) = \theta(i), \quad$ for $i \in \{1, \ldots, n\}$,

   (ii) f_θ is monotone on each interval $I_i = \{x \in \mathbb{R} : i \leq x \leq i + 1\}$ for each $i \in \{1, \ldots, n - 1\}$.

   The map f_θ is known as the “connect-the-dots” map and clearly it has an invariant set of type θ.

Example : $\theta : \{1, \dots, 6\} \to \{1, \dots, 6\}$

θ	1	2	3	4	5	6
	2	5	3	6	1	4



The graph of f_θ.

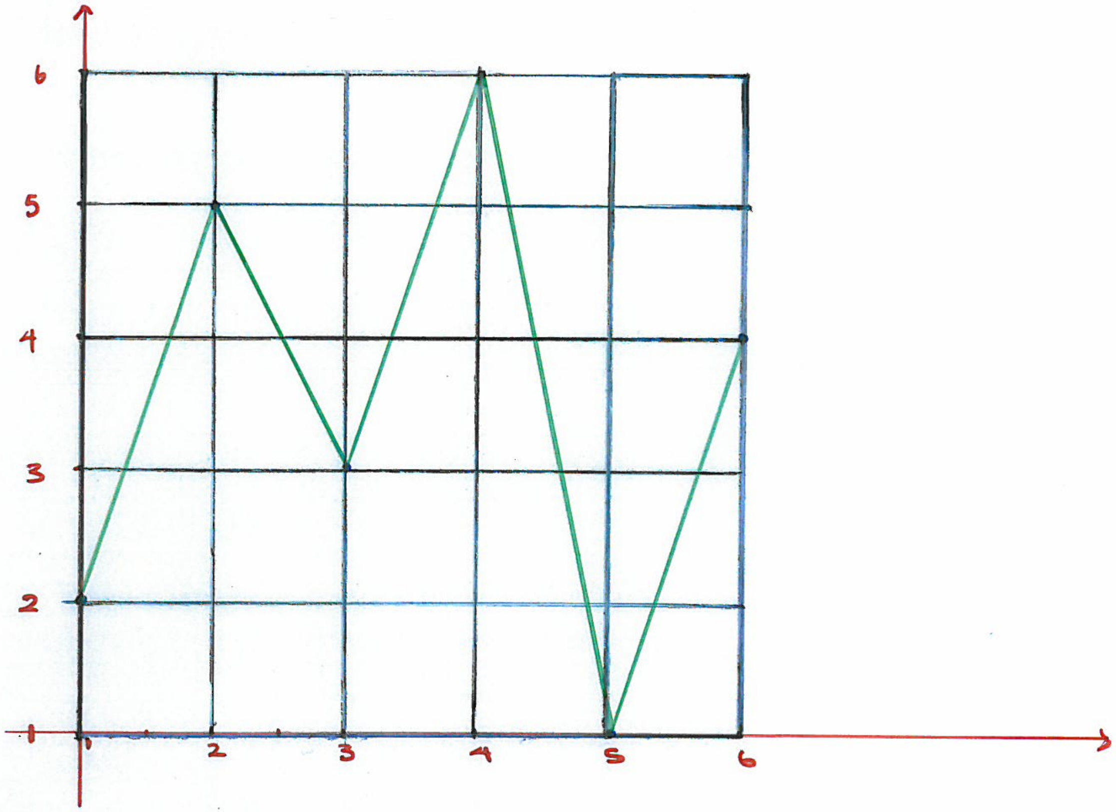
6. From this map we can construct a matrix $M(\theta)$ with $i\,j$th entry given by:

$$m_{ij} = \begin{cases} 1, & \text{if } f_\theta(I_i) \supset I_j \\ 0, & \text{otherwise} \end{cases}$$

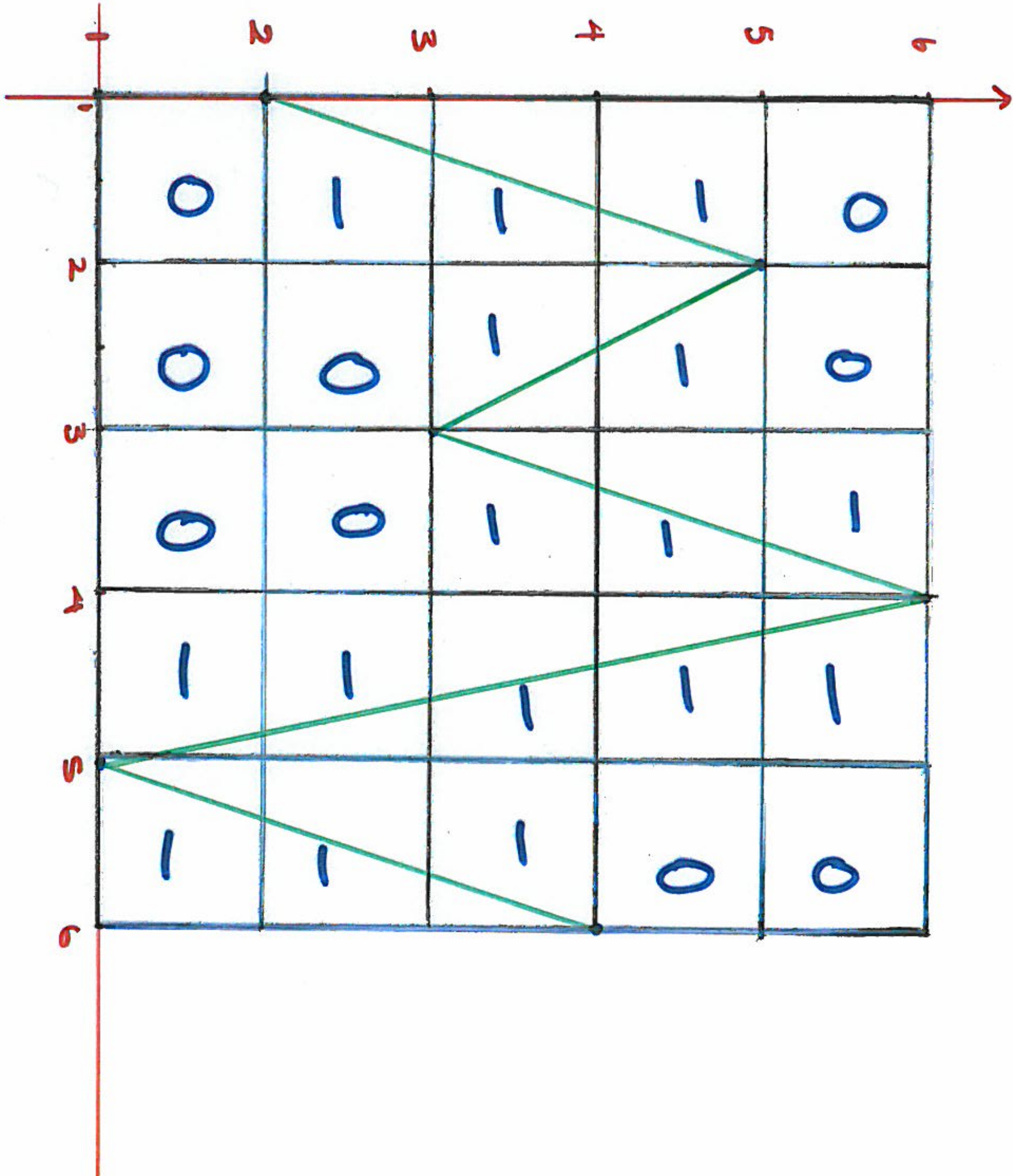
for $i,\ j \in \{1, \ldots, n-1\}$.

Example : $\theta : \{1, \dots, 6\} \to \{1, \dots, 6\}$

θ	1	2	3	4	5	6
	2	5	3	6	1	4



The graph of f_θ.



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0	1	1	1	0
0	0	1	1	0
0	0	1	1	1
1	1	1	1	1
1	1	1	0	0

The induced matrix f_θ

It is well known that

$$h(\theta) = \mathsf{max}\{\mathsf{log}(\rho(M(\theta))), \mathsf{0}\},$$

where $\rho(M(\theta))$ is the spectral radius of $M(\theta)$.

In the early 90's MN obtained an asymptotic result which shows that the maximum entropy for n-cycles and n-permutations approaches $\log(2n/\pi)$ as $n \to \infty$. To prove this result, they constructed a "diamond shaped" family of *cyclic* permutations of period $n \equiv 1 \pmod 4$ which had the required asymptotic growth rate.

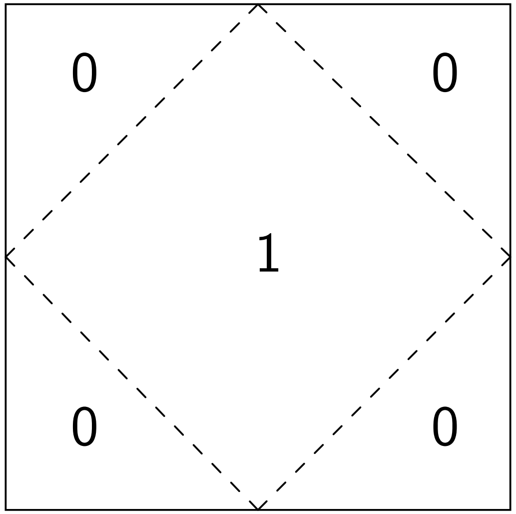


Figure: An approximate shape of the matrix for a cycle with maximal entropy.

2. Review of Results

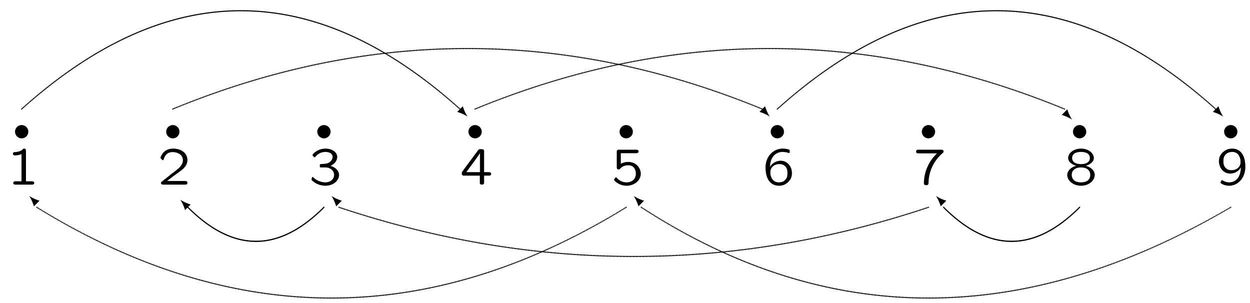
In the early 1990's Geller and Tolosa proved that a particular family of permutations (originally defined by Misiurewicz and Nitecki) actually attained the maximum in P_n (the set of all n-permutations).

Let $n \in \mathbb{N}$ be odd and let $l = \lfloor \frac{n-1}{4} \rfloor$. The permutation θ_n is defined by

$$\theta_n : j \to \begin{cases} n - 2l - j, & \text{if } j \in O[1, n-2l-2] \\ j - n + 2l + 1, & \text{if } j \in O[n-2l, n] \\ n - 2l + j - 1, & \text{if } j \in E[2, 2l] \\ n + 2l - j + 2, & \text{if } j \in E[2l+2, n-1]. \end{cases}$$

Here $O[1, k]$ (respectively $E[1, k]$) is the set of odd (resp. even) integers in $[1, k]$.

It is easy to verify that this is actually a family of cyclic permutations, hence the maximal elements in C_n (the set of all n-cycles) were also found.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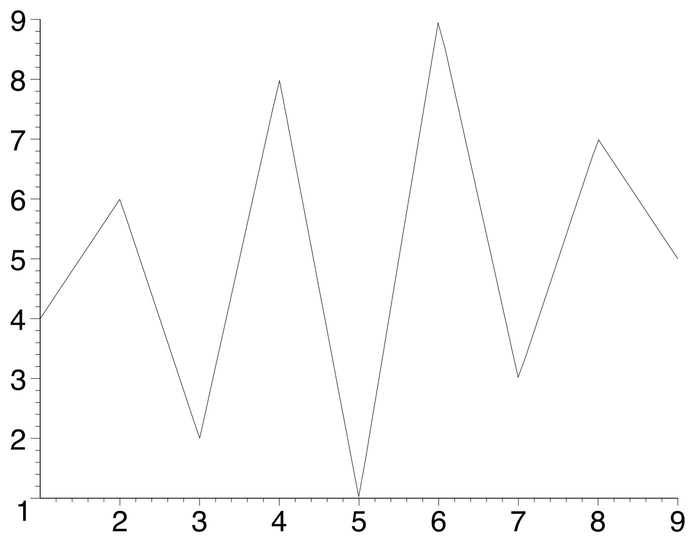


$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

The graph of f_{θ_9} and the induced matrix $M(\theta_9)$.

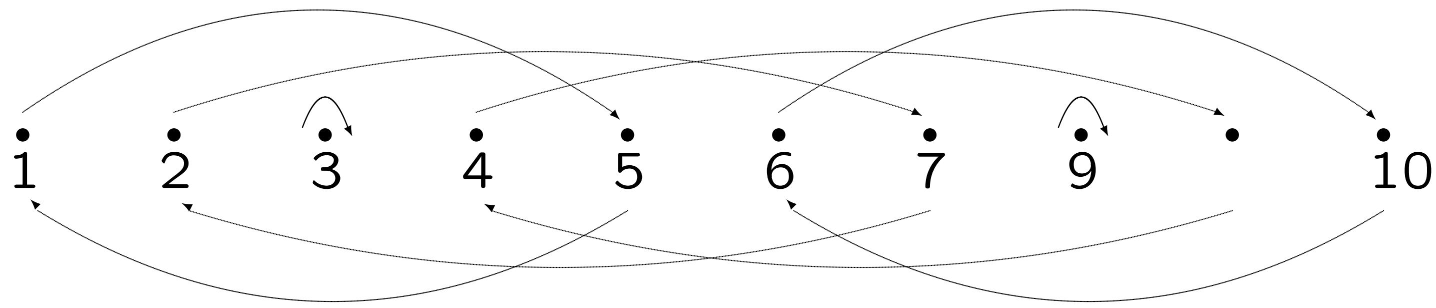
The map f_θ is maximodal (each integer point is a turning point), and has all local minima less than all local maxima.

Is this the only maximal element? The answer is “almost” - the dual permutation has the same entropy (this is always true) where dual means, reverse the orientation (this was shown by Geller and Weiss).

That was the state of play at the time Coppel visited, so I started to look at the case where n is even. A straightforward generalization of the Misiurewicz-Nitecki orbit types to n even yields a family of n-permutations which has maximum topological entropy. The main difference in this case is that these permutations are not cyclic so that they are maximal in P_n but not in C_n.

Let n be even and $k = n/2$. The family of noncyclic n-permutations ϑ_n is defined by:

$$\vartheta_n : j \to \begin{cases} k - j + 1, & \text{if } j \in O[1, k] \\ j - k, & \text{if } j \in O[k+1, n-1] \\ k + j, & \text{if } j \in E[2, k] \\ 3k - j + 1, & \text{if } j \in E[k+1, n]. \end{cases}$$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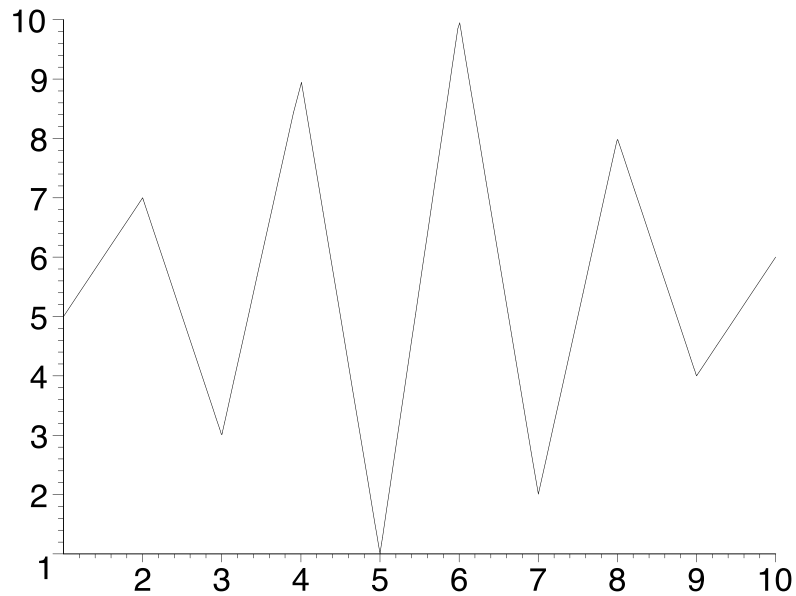
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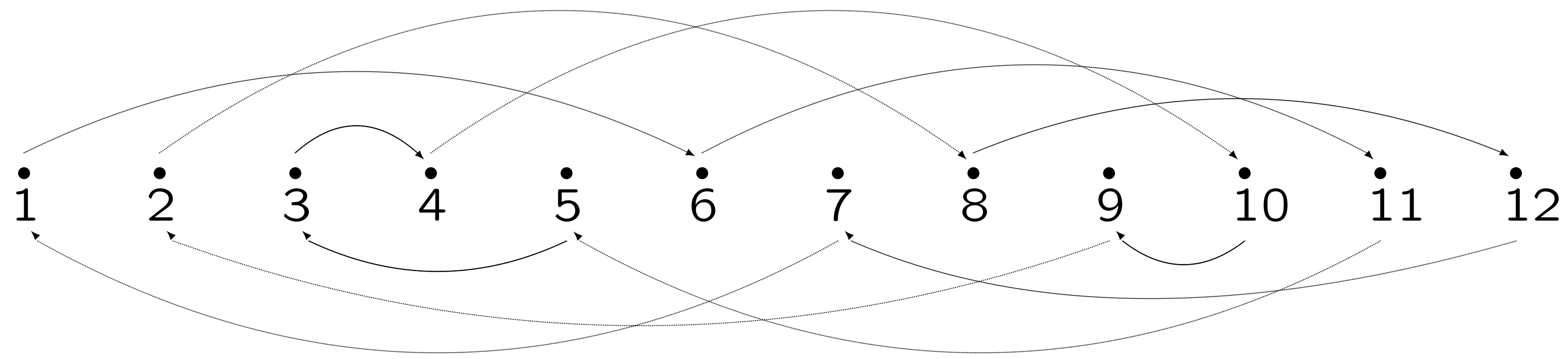
The features are similar to the previous case, but these permutations are self-dual (look the same when you reverse the orientation), but in this case there is another permutation with the same entropy. We called it the *reverse* permutation, and it is also self-dual. So for the even case, we have two different maximal families.

These results were found indepenently by Geller and Zhang.

Having identified all maximum elements of P_n it is a natural question to consider the maximum elements of C_n. But at this point life gets a bit harder. The methods of proof used, rely on the induced matrices, but now we have to try to distinguish matrices of non-cyclic permutations from cyclic ones.

Let $n = 4k$ for $k \in \mathbb{N} \setminus \{1\}$. We define the cyclic n-permutation ψ_n as follows:

$$\psi_n : j \to \begin{cases} 2k - j + 1, & \text{if } j \in O[1, k+1] \\ 2k - j + 2, & \text{if } j \in O[k+2, 2k+1] \\ j - 2k - 1, & \text{if } j \in O[2k+3, 3k] \\ j - 2k, & \text{if } j \in O[3k+1, n-1] \\ 2k + j, & \text{if } j \in E[2, k+1] \\ 2k + j - 1, & \text{if } j \in E[k+2, 2k] \\ 6k - j + 2, & \text{if } j \in E[2k+2, 3k] \\ 6k - j + 1, & \text{if } j \in E[3k+1, n]. \end{cases}$$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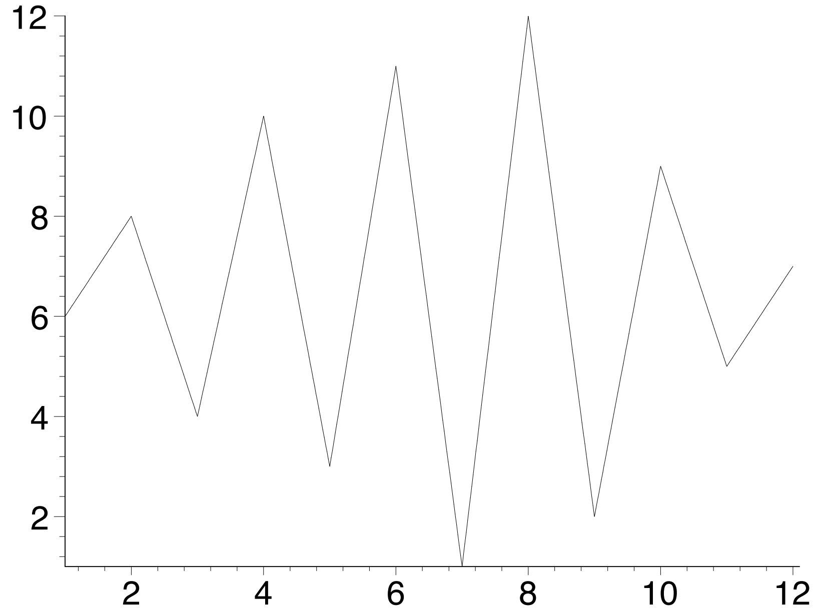
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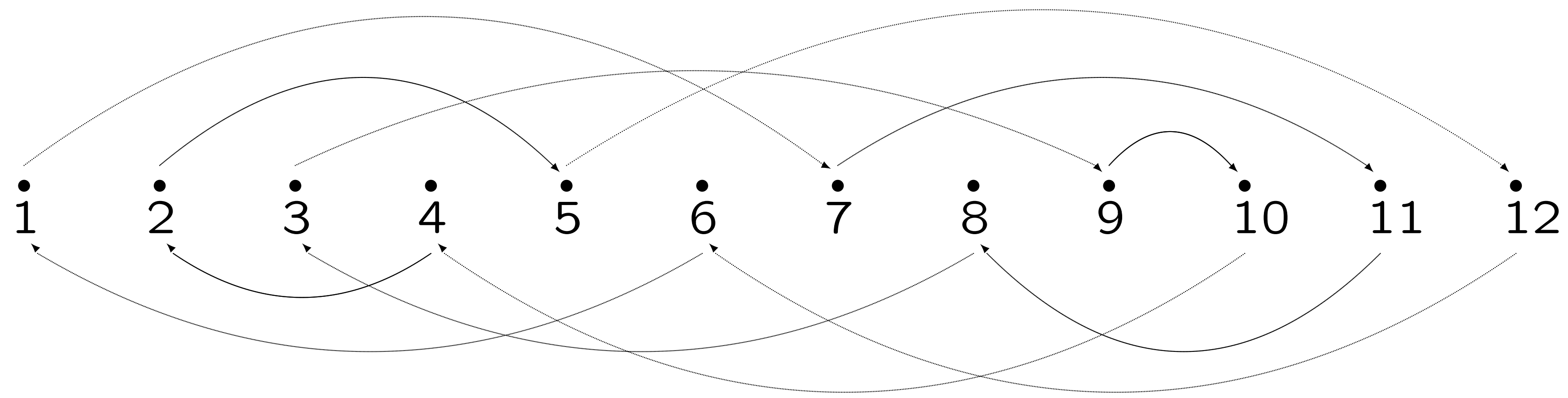
These cycles aren't self-dual, and the reverse cycle also has the same entropy.



This is joint work with John Strantzen, the "uniqueness" paper is not yet published, but I hope that recent results (Alseda, Juher and Manosas) will help to make that paper more accessible.

3. Work in Progress

Recently (Alseda and Juher) started to work on finding a description of maximal $4k+2$-cycles. It is well known that the complexity of these types of combinatorial problems grows factorially, so to carry out a computer investigation, we needed to find some way of restricting the number of cycles to be considered.

1. C_n is endowed with a partial order (called the *forcing relation*, Jungries, MN) and it has been shown that topological entropy respects this partial order (that is, if θ forces ϕ then $h(\theta) \geq h(\phi)$), so maximum entropy cycles must be forcing maximal.

2. According to Jungreis, forcing maximal cycles are maximodal and either

   (a) All maximum values of θ are above all minimum values of θ (C_n^0), or

3. Exactly one maximum value is less than some minimum value, and exactly one minimum value is greater than some maximum value (C_n^1).

To have an idea of the computational complexity of this task, see the following table:

n	(C_n^0)	(C_n^1)	Total
4	2	0	2
5	2	1	3
6	7	5	12
7	24	15	39
8	72	105	177
9	288	561	849
10	1452	3228	4680
11	8640	20548	29188
12	43320	145572	188892
13	259200	1084512	1343712
14	1814760	8486268	10301028
15	14515200	73104480	87619680
16	101606400	636109560	737715960
17	812851200	5937577920	6750429120

Machinery to study C_n^0: the cross product

Definition of $\otimes$

$\alpha \otimes \beta := \alpha \oplus \delta(\beta)$, where

$$(a_1, a_2, \ldots, a_p) \oplus (b_1, b_2, \ldots, b_p) = (a_1, b_1, a_2, b_2, \ldots, a_p, b_p)$$

and

$$\delta(a_1, a_2, \ldots, a_p) = (n - a_p, n - a_{p-1}, \ldots, n - a_2, n - a_1),$$

with $n := 2p + 1$.

Example

$$(7,4,1,9,2,3,5,6,8) \otimes (7,4,2,3,1,9,5,6,8) =$$
$$(7,11,4,13,1,14,9,10,2,18,3,16,5,17,6,15,8,12)$$

Machinery to study C_n^0: the dot product

Definition of $\odot$

$\alpha \odot \beta := \sigma^-(\delta(\alpha)) \oplus \sigma^+(\beta)$, where

- $\sigma^+\big(a_1, a_2, \ldots, a_p\big) := \big(\widehat{\sigma}^+(a_1), \widehat{\sigma}^+(a_2), \ldots, \widehat{\sigma}^+(a_p)\big);$
  with $\widehat{\sigma}^+(a) := \begin{cases} a+1 & \text{when } a < p \\ 1 & \text{when } a = p. \end{cases}$

- $\sigma^-\big(a_1, a_2, \ldots, a_p\big) := \big(\widehat{\sigma}^-(a_1), \widehat{\sigma}^-(a_2), \ldots, \widehat{\sigma}^-(a_p)\big);$
  with $\widehat{\sigma}^-(a) := \begin{cases} a-1 & \text{when } a > p+1 \\ 2p & \text{when } a = p+1. \end{cases}$

Example

$$(7,4,1,9,2,3,5,6,8) \odot (7,4,2,3,1,9,5,6,8) =$$
$$(10,8,12,5,13,3,15,4,16,2,18,1,17,6,14,7,11,9)$$

It is very easy to see that

- $d(\alpha \oplus \beta) = \delta(\beta) \oplus \delta(\alpha)$.

where $d(\alpha)$ is the dual permutation.

Then,

Proposition Properties of the cross and dot products.

1. $d(\alpha \otimes \beta) = \beta \otimes \alpha$ and $d(\alpha \odot \beta) = \beta \odot \alpha$.
2. If $\theta \in C_n^{0,m}$ then $\theta = \theta_1 \otimes \theta_2$ for some $\theta_1, \theta_2 \in P_{n/2}$, where $C_n^{0,m}$ denotes the subset of C_n^0 which contains all cycles for which $f_\theta(1)$ is a minimum.
3. If $\theta \in C_n^{0,M}$ then $\theta = \theta_1 \odot \theta_2$ for some $\theta_1, \theta_2 \in P_{n/2}$, where $C_n^{0,M}$ denotes the subset of C_n^0 which contains all cycles for which $f_\theta(1)$ is a maximum.

Using these restrictions, we have calculated maximum entropy cycles for $n = 6, 10$ and 14. For $n = 18$ we needed to develop some more tools to efficiently generate the list of cycles, but even for $n = 22$, this problem is currently beyond computational capabilities ($n = 17$ took 38 hours). But we have formulated a conjecture about what these families might look like, and so these serve as a lower bound.

Let $n = 4k + 2$ for $k \in \mathbb{N} \setminus \{1\}$. We denote by θ_n the element of C_n^0 that is given as follows:

If $k = 2p$ with p odd, then

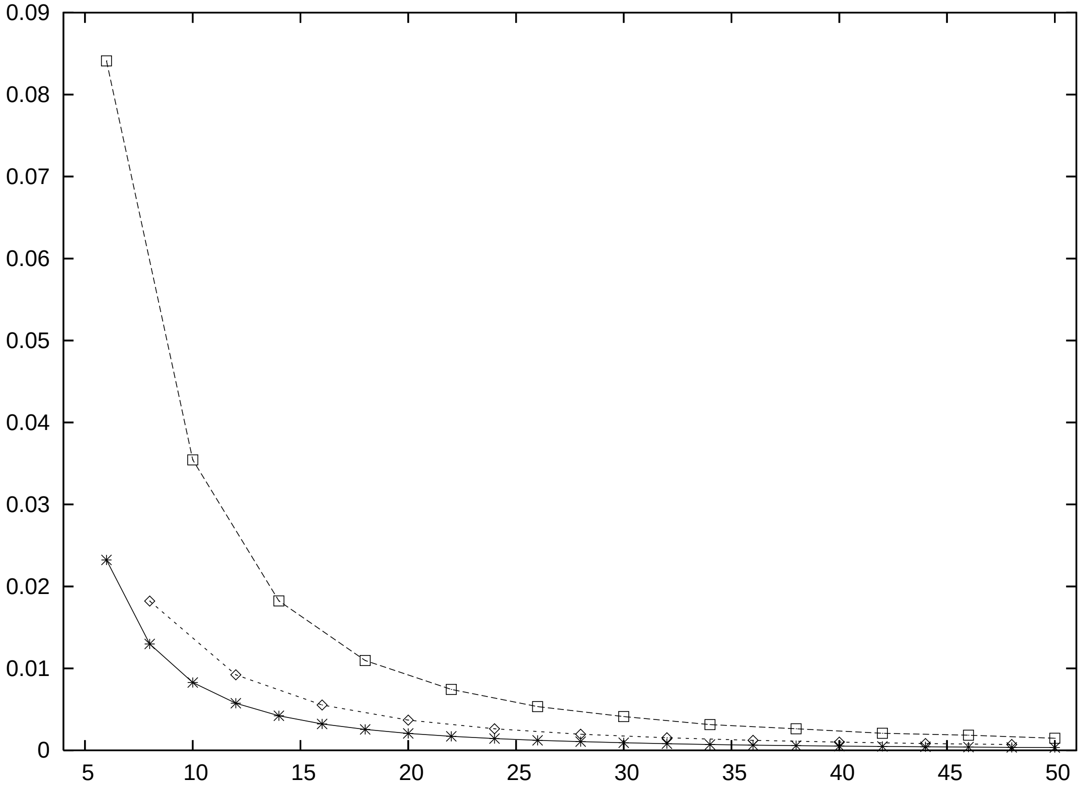
$$\theta_n : j \to \begin{cases} 4p+1+j, & \text{if } j \in O[1,p] \\ 4p+2+j, & \text{if } j \in O[p+2,3p] \\ 4p-1+j, & \text{if } j \in O[3p+2,4p+3] \\ 12p+6-j, & \text{if } j \in O[4p+5,5p+2] \\ 12p+3-j, & \text{if } j \in O[5p+4,7p] \\ 12p+4-j, & \text{if } j \in O[7p+2,n-1] \\ 4p+2-j, & \text{if } j \in E[2,p+1] \\ 4p+3-j, & \text{if } j \in E[p+3,3p+1] \\ 4p+4-j, & \text{if } j \in E[3p+3,4p+2] \\ j-4p-3, & \text{if } j \in E[4p+4,5p+3] \\ j-4p-2, & \text{if } j \in E[5p+5,7p+1] \\ j-4p-1, & \text{if } j \in E[7p+3,n]. \end{cases}$$

Definition for $k = 2p$, p even

$$\theta_n : j \to \begin{cases} 4p+1+j, & \text{if } j \in O[1, p+1] \\ 4p+j, & \text{if } j \in O[p+3, 3p+1] \\ 4p-1+j, & \text{if } j \in O[3p+3, 4p+3] \\ 12p+6-j, & \text{if } j \in O[4p+5, 5p+3] \\ 12p+5-j, & \text{if } j \in O[5p+5, 7p+1] \\ 12p+4-j, & \text{if } j \in O[7p+3, n-1] \\ 4p+2-j, & \text{if } j \in E[2, p] \\ 4p+1-j, & \text{if } j \in E[p+2, 3p] \\ 4p+4-j, & \text{if } j \in E[3p+2, 4p+2] \\ j-4p-3, & \text{if } j \in E[4p+4, 5p+2] \\ j-4p, & \text{if } j \in E[5p+4, 7p] \\ j-4p-1, & \text{if } j \in E[7p+2, n]. \end{cases}$$

Definition for $k \geq 3$, odd

$$\theta_n : j \to \begin{cases} 2k - j + 2, & \text{if } j \in O[1, k-2] \\ k + 1, & \text{if } j = k \\ 2k - j, & \text{if } j \in O[k+2, 2k-1] \\ j - 2k + 1, & \text{if } j \in O[2k+1, 3k-2] \\ j - 2k, & \text{if } j \in O[3k, 3k+2] \\ j - 2k - 1, & \text{if } j \in O[3k+4, n-1] \\ 2k + 1 + j, & \text{if } j \in E[2, k-1] \\ 3k + 1, & \text{if } j = k+1 \\ 2k + 3 + j, & \text{if } j \in E[k+3, 2k-2] \\ 6k + 2 - j, & \text{if } j \in E[2k, 3k-1] \\ 6k + 5 - j, & \text{if } j \in E[3k+1, 3k+3] \\ 6k + 4 - j, & \text{if } j \in E[3k+5, n]. \end{cases}$$



The figure shows the asymptotic behaviour of the entropies of the cycles in the conjectured families. The three curves above represent the difference between the Misiurewicz-Nitecki bound, $\log(2n/\pi)$, and the entropies of (i) the maximum entropy n-permutation, for $n \in E[6, 50]$ (lower curve), (ii) the maximum entropy $4k$-cycle, for $k \in [2, 12]$ (centre curve), (iii) the cycle θ_{4k+2}, for $k \in [1, 12]$ (upper curve).

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