Maximally transitive semigroups of $n \times n$ matrices

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Introduction

Definition

Let G be a semigroup acting on a topological space X by continuous maps. The action of G on X is called

- hypercyclic, if there exists $x \in G$ such that the G-orbit of x defined by $\{f(x) : f \in G\}$ is dense in X.

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- topologically transitive, if for every pair of nonempty open sets U and V in X, there exists a map $f \in G$ so that $f(U) \cap V \neq \emptyset$.

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- topologically transitive, if for every pair of nonempty open sets U and V in X, there exists a map $f \in G$ so that $f(U) \cap V \neq \emptyset$.
- topologically k-transitive, if the induced action of G on X^k is topologically transitive.

Birkhoff transitivity theorem for semigroup actions

Theorem

Let G be a semigroup acting by continuous maps on a separable complete metric space X without isolated points. If the action of G is topologically transitive, then there exists a G_δ set $W \subseteq X$ so that the G-orbit of every $x \in W$ is dense in X.

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For the action of $GL(n, \mathbb{K})$ on $\mathbb{K}^n$, the action of a subsemigroup is n-transitive if and only if the subsemigroup is dense in $GL(n, \mathbb{K})$.

The one-dimensional case

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- The comlplex case: If $\ln(-a)/\ln(b) < 0$ and the numbers
$$1,\ \frac{\ln(-a)}{\ln b},\ \frac{\arg(c)}{2\pi},$$
are rationally independent, then $\langle a, b\rangle$ is dense in $\mathbb{C}$.

The commutative case

- Feldman: The minimum number of generators for the semigroup of diagoanl matrices is $n+1$.
- Ayadi, Costakis, and Abels-Manoussos: minimum number of generators of an abelian semigroup of matrices with a dense orbit:
  - Real case: $\lfloor (n+3)/2 \rfloor$
  - Complex case: $n+1$
  - Real case traingular non-diagonalizable: $n+1$
  - Complex case triangular non-diagonalizable: $n+2$

The non-commutative case

- *Does there exist a pair of matrices in $GL(n, \mathbb{K})$ that generates a dense subsemigroup of $GL(n, \mathbb{K})$?*
- *Does there exist a pair of matrices in $SL(n, \mathbb{K})$ that generates a dense subsemigroup of $SL(n, \mathbb{K})$?*
- *What is the minimum number of generators of a dense semigroup of lower-triangular matrices?*

A 2-dimensional explicit example

- The semigroup of matrices generated by

$$A = \begin{pmatrix} 1 & 1/2 \\ 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & -8/3 \end{pmatrix}$$

is dense in the set of 2×2 real matrices.

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- The matrices

$$A = \begin{pmatrix} 1/\sqrt{2} & -\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} \sqrt{2/3} & 0 \\ 0 & \sqrt{3/2} \end{pmatrix},$$

  generate a dense subsemigroup of $SL(2, \mathbb{R})$.

n-transitive subsemigroups of matrices

Theorem

There exists a 2-generator semigroup of matrices whose action on the set of $\mathbb{K}^n$ is topologically n-transitive. Equivalently, this semigroup is dense in the set of $n \times n$ matrices with entries in $\mathbb{K}$.

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This is an optimal result because the action of a singly generated subsemigroup is not even hypercyclic, while the action of a subsemigroup of $GL(n, \mathbb{K})$ can never be $(n+1)$-transitive.

Dense subsemigroups of Lie groups

Abels-Vinberg: Connected Lie groups with finite center have 2-generator dense sub(semi)groups.

Sketch of the proof:

- Given a non-central element g, there exists elliptic h so that $\langle g, h \rangle$ is dense.

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- In $SL(n, \mathbb{C})$, choose g and h (of finite order p) so that $\langle g, h \rangle$ is dense in $SL(n, \mathbb{C})$.

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- Choose $a, b \in \mathbb{C}$ so that $\langle a^p, b^p \rangle$ is dense in $\mathbb{C}$. Then $\langle ag, bh \rangle$ is dense in $GL(n, \mathbb{C})$.

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- In $\mathbb{R}$, further care is required.

An alternative approach

Lemma

Let Λ be a closed subsemigroup of $(n+1)\times(n+1)$ matrices with entries in $\mathbb{K}$ such that

$$\forall F \in GL(n,\mathbb{K}): \quad \begin{pmatrix} F & 0 \\ 0 & 1 \end{pmatrix} \in \Lambda.$$

Suppose that there exists

$$K = \begin{pmatrix} F & X \\ Y & \eta \end{pmatrix} \in \Lambda,$$

such that

$$YF^{-1}X \neq 0, \eta.$$

Then Λ contains all $(n+1)\times(n+1)$ matrices with entries in $\mathbb{K}$.

Indutive construction

Theorem

For any $n \geq 1$, there exists a pair of matrices in $\mathcal{M}_{n\times n}(\mathbb{C})$ that generates a dense subsemigroup of $\mathcal{M}_{n\times n}(\mathbb{C})$. Moreover, for $n \geq 2$, we can arrange for one of the matrices to be of the form

$$A = \begin{pmatrix} Z_1 & 0 & \dots & 0 \\ 0 & Z_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & Z_n \end{pmatrix},$$

where $Z_n = 1, Z_1 \neq 0$, and each $Z_i, 1 < i < n$, is a root of unity.

Proof

Induction: Given A and E generating a dense subsemigroup of $GL(n, \mathbb{C})$, let

$$C = \begin{pmatrix} Z_1' & 0 & \dots & 0 \\ 0 & Z_2' & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & Z_n' \end{pmatrix}, \ D = \begin{pmatrix} E & 0 \\ 0 & 1 \end{pmatrix},$$

where $Z_i' = \sqrt{Z_i}$ for $1 \leq i < n$, and

$$Z_n' = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix}.$$

Thank You!

Any Questions?